

Linear-Length Mixed Identities for Finite Projective Unitary Groups

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Abstract

Let $q = p^e$ be a prime power and $n \geq 2$. We construct a nontrivial singular mixed identity for $\text{PSU}_n(q)$ of variable length at most $16q + 8p + 12$. The bound is uniform in the dimension and includes characteristic two without an exception. A product of two unitary transvections has order dividing one of $q - 1$, $q + 1$ or $2p$. A nested commutator combines these alternatives, while separate free conjugators ensure formal nontriviality. Together with the lower bound of Bradford, Schneider and Thom, this gives $\Theta(q)$ for the length of a shortest mixed identity in the projective special unitary family. The construction does not address shortest nonsingular identities or all remaining Lie-type families.

1 The unitary length bound

A mixed word for a group G is an element of the free product $G * F_r$, where F_r is free on r variables. It is a *mixed identity* if it is nonidentity in this free product but evaluates to 1_G for every substitution of its variables by elements of G . Its length $\ell(w)$ counts variable occurrences and their inverses, not constants. Let $\varepsilon : G * F_r \rightarrow F_r$ be the homomorphism deleting all constants. A mixed word is *singular* if $\varepsilon(w) = 1$.

Throughout, $\text{SU}_n(q)$ acts on a nondegenerate Hermitian space of dimension n over $\mathbb{F}_{q^2}$, with involution $a \mapsto a^q$, and $\text{PSU}_n(q)$ is its projective image. Thus q denotes the size of the fixed field, not the size of the matrix coefficient field.

Theorem 1. *For every prime power $q = p^e$ and $n \geq 2$, the group $\text{PSU}_n(q)$ has a nontrivial singular mixed identity in three variables of length at most*

$$16q + 8p + 12 \leq 24q + 12.$$

The same word with a lifted constant is a mixed identity for $\text{SU}_n(q)$.

Bradford, Schneider and Thom prove a dimension-uniform lower bound $\Omega(q)$ for $\text{PSU}_n(q)$, even when constants from $\text{PGL}_n(q^2)$ are allowed [1, Theorem 5, Lemmas 8.2 and 8.5]. Consequently Theorem 1 gives the following.

Corollary 2. *The length of a shortest mixed identity for $\text{PSU}_n(q)$ is $\Theta(q)$, with implied constants independent of n .*

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The upper bound in [1, Theorem 5] is $O(q^2)$, with a polylogarithmic factor in one characteristic-two case. Its proof uses low-support elements and linear groups over $\mathbb{F}_{q^2}$. We keep the Hermitian structure on a two-dimensional invariant subspace, so that the relevant semisimple orders divide $q-1$ or $q+1$. The separate conjugators used below also avoid an involution exception in the formal-word argument. The transvection method and commutator combination principle are classical; the result here is the explicit uniform linear bound.

Schneider's Oberwolfach discussion states the earlier unitary estimate immediately before questions on the remaining Lie-type families and nonsingular identities [2, p. 732]. Corollary 2 sharpens that unitary estimate; it does not answer both broader questions. The later structural results of Bradford, Ersoy, Schneider and Thom [3] concern, among other things, lower bounds and mixed identities over bounded fields.

2 A product of two transvections

Let $K = \mathbb{F}_{q^2}$, write $\bar{a} = a^q$, and let h be linear in its first argument and conjugate-linear in its second. For a nonzero isotropic vector v and a nonzero $\lambda \in K$ satisfying $\lambda + \bar{\lambda} = 0$, define

$$T_{v,\lambda}(u) = u + \lambda h(u, v)v.$$

Such choices exist for every $n \geq 2$. Indeed, the norm map $K^\times \rightarrow \mathbb{F}_q^\times$ is surjective, so an orthogonal two-dimensional subspace contains an isotropic vector. The trace-zero subspace of K has q elements.

Write $T_{v,\lambda} = I + N$. Then $N \neq 0$ and $N^2 = 0$. The cross terms in $h(Tu, Tw)$ cancel because $\lambda + \bar{\lambda} = 0$, and the remaining quadratic term contains $h(v, v) = 0$. Thus $T_{v,\lambda}$ preserves h , has determinant one and has order exactly p . It is nonscalar, and its projective image also has order p .

Lemma 3. *For any two unitary transvections a, b of this form, at least one of*

$$(ab)^{q-1} = I, \quad (ab)^{q+1} = I, \quad (ab)^{2p} = I$$

holds.

Proof. Write $a = T_{v,\lambda}$ and $b = T_{w,\mu}$. If $h(v, w) = 0$, then the nilpotent parts satisfy $N_v N_w = N_w N_v = 0$. Therefore

$$ab = I + N_v + N_w, \quad (N_v + N_w)^2 = 0,$$

and $(ab)^p = I$. This case includes linearly dependent directions.

Suppose instead that $h(v, w) \neq 0$. The two directions are independent, and the Gram matrix on $W = \langle v, w \rangle_K$ has determinant

$$-h(v, w)h(w, v) \neq 0.$$

Hence $V = W \perp W^\perp$. Both transvections preserve W and fix $W^\perp$ pointwise. Their restrictions to W have determinant one. Put $B = (ab)|_W$.

For a unitary matrix, conjugating the trace gives the trace of the inverse. Since B is two-dimensional with determinant one,

$$\overline{\text{tr } B} = \text{tr}(B^{-1}) = \text{tr } B.$$

Thus its characteristic polynomial $X^2 - (\text{tr } B)X + 1$ belongs to $\mathbb{F}_q[X]$. If its roots are distinct, either both lie in $\mathbb{F}_q^\times$, or Frobenius interchanges the reciprocal roots. In the first case $B^{q-1} = I$; in the second $B^{q+1} = I$.

If the roots coincide, the repeated root α satisfies $\alpha^2 = 1$. We may write $B = \alpha(I + M)$ with $M^2 = 0$, and hence $B^{2p} = I$. This retains the negative-unipotent case in odd characteristic. The action on $W^\perp$ is the identity, so each conclusion extends from W to V . The argument also applies in characteristic two. $\square$

3 The mixed word and its nontriviality

Fix one transvection $t \in \mathrm{SU}_n(q)$. In $\mathrm{SU}_n(q) * F(x, z, r)$ put

$$\begin{aligned} Y &= tx^{-1}tx, & A &= Y^{q-1}, \\ B &= z^{-1}Y^{q+1}z, & C &= r^{-1}Y^{2p}r, \end{aligned}$$

and define

$$w = [[A, B], C], \quad [a, b] = a^{-1}b^{-1}ab.$$

For every substitution $x = g$, the element $Y(g) = t(g^{-1}tg)$ is a product of two unitary transvections. Indeed,

$$g^{-1}T_{v,\lambda}g = T_{g^{-1}v,\lambda}.$$

Lemma 3 makes at least one of $Y(g)^{q-1}$, $Y(g)^{q+1}$ and $Y(g)^{2p}$ equal to I . Conjugating by arbitrary values of z, r preserves that equality. If $A = I$ or $B = I$, the inner commutator vanishes; if $C = I$, the outer one does. Thus w evaluates to I for every triple of substitutions in $\mathrm{SU}_n(q)$.

Every projective substitution has a lift to $\mathrm{SU}_n(q)$. The word with constant $\bar{t}$ therefore vanishes under every substitution in $\mathrm{PSU}_n(q)$. We next check separately that it is nontrivial as a formal word in both groups.

Lemma 4. *Let t be a nonidentity element of any group, and let a, b, c be positive integers. With $Y = tx^{-1}tx$, the word*

$$[[Y^a, z^{-1}Y^b z], r^{-1}Y^c r]$$

is nonidentity in the free product of that group with $F(x, z, r)$.

Proof. The subgroup $H = \langle t \rangle * \langle x \rangle$ embeds in the free product. The word Y is cyclically reduced of syllable length four in H . It has infinite order, even when t is an involution.

For any nonidentity $u, v \in H$, the expression

$$[u, z^{-1}vz] = u^{-1}z^{-1}v^{-1}zuz^{-1}vz$$

is a reduced alternating word in $H * \langle z \rangle$. Hence $U = [Y^a, z^{-1}Y^b z] \neq 1$. Applying the same observation in $(H * \langle z \rangle) * \langle r \rangle$, with $u = U$ and $v = Y^c$, proves the assertion. $\square$

Apply Lemma 4 with $(a, b, c) = (q-1, q+1, 2p)$. It proves nontriviality with either t or $\bar{t}$, because both constants are nonidentity. Merely showing nontriviality upstairs would not suffice: a general mixed identity need not descend nontrivially to a quotient. Here the normal-form argument applies separately downstairs.

Deleting all constants makes $Y = x^{-1}x = 1$. The displayed identity is therefore singular. Without the separate conjugators, the proposed commutator of powers of Y would already be the identity as a formal word and would not be useful.

4 Length and one-variable minimum

The word Y has two variable occurrences. Thus

$$\ell(A) \leq 2(q-1), \quad \ell(B) \leq 2(q+1) + 2, \quad \ell(C) \leq 4p + 2.$$

Writing $U = [A, B]$, we obtain $\ell(U) \leq 8q + 4$ and

$$\ell(w) \leq 2\ell(U) + 2\ell(C) \leq 16q + 8p + 12.$$

Free reductions can only lower this count. This completes the proof of Theorem 1.

The definition in [1] permits any finite number of variables. Its Lemma 2.2 and Corollary 2.4 show that the minimum can be attained with a single variable. Consequently our upper bound also bounds that one-variable minimum; we do not need to specify the auxiliary constants in that reduction. Combining it with the cited lower bound proves Corollary 2. No lower bound for arbitrary almost simple overgroups is inferred, since allowing further constants can shorten mixed identities.

5 Exact diagnostics and scope

The source archive includes `check_unitary_identity.py`, a standalone program using only Python’s standard library. From its `reproducibility` directory, run

```
python3 -B check_unitary_identity.py \
  --output normal.json
python3 -O -B check_unitary_identity.py \
  --output optimized.json
```

It constructs finite fields by polynomial arithmetic and checks their operation tables, transvection determinants, Hermitian preservation, product orders, full-word evaluations and free-product normal forms. Both modes pass 7851 transvection-pair tests and 480 full-word substitutions for $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$ in dimensions between two and five. The report distinguishes exhaustive pair enumerations from tests with one factor fixed. Ten formal-word cases include characteristic two and $q = 101$. The unseparated-commutator negative control correctly reduces to the empty word, and witnesses exercise all three order alternatives. No assertion is used, so optimization does not disable tests.

These finite checks are diagnostics, not a proof of the infinite family; the universal proof is the two-plane argument and free-product normal form above. The theorem concerns the unitary family and singular identities, not a full solution of the general questions in [2]. The bounded literature review in the source package is current through 20 October 2026 and does not certify exhaustive novelty or absolute priority. AI tools assisted the author with exploration, drafting and checker development. The manuscript is self-audited and unrefereed, without independent expert review or proof-assistant verification.

References

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