

A Signed-Weight Counterexample to a Grushin Hardy–Rellich Converse

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Abstract

We construct a smooth signed radial weight for which a Grushin Hardy–Rellich inequality holds for every compactly supported smooth test function on the punctured unit gauge ball, but the associated Bessel equation has no positive solution. The principal weight is the constant one and the homogeneous dimension is five. The second weight is specified by the lowest eigenvalue of $-d^2/dr^2 + r^{-4}$ on $(0, 1)$. A quantitative spectral estimate on a codimension-one subspace yields an explicit positive remainder, and the Grushin spherical decomposition extends the inequality from radial tests to all tests. The construction concerns the remainder-free inequality with an unrestricted signed second weight; it does not contradict the forward Bessel-pair theorem or establish failure under nonnegative-weight assumptions.

1 Statement and scope

Let $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, and set

$$\mathcal{L}_G = \Delta_x + |x|^2 \partial_t^2, \quad \nabla_G = (\nabla_x, |x| \partial_t), \quad \rho = (|x|^4 + 4t^2)^{1/4}, \quad \psi = |\nabla_G \rho|^2 = \frac{|x|^2}{\rho^2}.$$

The homogeneous dimension is $Q = 5$. Write $B_1 = \{(x, t) : \rho < 1\}$ and $\mathcal{D} = C_c^\infty(B_1 \setminus \{(0, 0)\}; \mathbb{R})$. All spatial integrals below use Lebesgue measure $dx dt$. For a real smooth function W on $(0, 1)$, the five-dimensional Bessel equation for the pair $(1, W)$ is

$$(r^4 f')' + r^4 W f = 0 \quad (0 < r < 1). \quad (1)$$

There is no imposed boundary value at either endpoint.

Roychowdhury’s workshop question [3, p. 606] asks about converses to the Grushin Hardy and Hardy–Rellich inequalities. Ganguly, Jotsaroop and Roychowdhury [1, Theorem 1.3] establish a forward Hardy–Rellich inequality from a Bessel pair. Their definition permits real C^1 second weights, and their structural hypothesis on a positive principal weight V is

$$(Q - 5) \frac{V}{r^2} + 3 \frac{V'}{r} - V'' \geq 0.$$

It is satisfied with equality by $Q = 5$ and $V = 1$. Dropping the solution-dependent nonnegative remainder from that forward theorem gives the inequality considered here. We show that this

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remainder-free inequality does not characterize Bessel pairs when unrestricted signed weights are allowed.

Define

$$\lambda_1 = \inf_{\substack{y \in X \\ y \neq 0}} \frac{\int_0^1 (y'^2 + r^{-4}y^2) dr}{\int_0^1 y^2 dr}, \quad X = \left\{ y \in H_0^1(0,1) : \int_0^1 r^{-4}y^2 dr < \infty \right\}. \quad (2)$$

Thus λ_1 is an exact variationally specified constant, not a numerically fitted parameter.

Theorem 1.1. *Let*

$$W(r) = \lambda_1 + \frac{1}{100} + \frac{2}{r^2} - \frac{1}{r^4}. \quad (3)$$

Then $W \in C^\infty(0,1)$ changes sign, and for every $u \in \mathcal{D}$,

$$\int_{B_1} \frac{(\mathcal{L}_G u)^2}{\psi} - \int_{B_1} \left(W(\rho) + \frac{4}{\rho^2} \right) |\nabla_G u|^2 \geq \frac{51}{8900} \int_{B_1} |\nabla_G u|^2. \quad (4)$$

Nevertheless, (1) has no positive solution on $(0,1)$.

The quotient in the first integral is defined almost everywhere. It is integrable for every $u \in \mathcal{D}$: on its support, ρ is bounded above and away from zero, and $\psi^{-1} = \rho^2/|x|^2$ is locally integrable in the three x variables. Both right-hand integrals are finite as well. Complex tests are covered by applying the real inequality to their real and imaginary parts, with squared absolute values. The constant 51/8900 is only a convenient explicit margin; no sharpness claim is made.

2 A quantitative constrained spectral gap

On the real Hilbert space $L^2(0,1)$ consider the closed form

$$a[y] = \int_0^1 (y'^2 + r^{-4}y^2) dr, \quad y \in X.$$

Its form norm is equivalent to $a[y]^{1/2}$, since $r^{-4} \geq 1$. Completeness follows from completeness of H_0^1 and of the weighted L^2 space. The embedding $X \hookrightarrow L^2(0,1)$ is compact, because the form norm bounds the H_0^1 norm. The associated Friedrichs operator has discrete eigenvalues, listed with multiplicity, $\lambda_1 \leq \lambda_2 \leq \dots$.

For clarity, $C_c^\infty(0,1)$ is a form core. At zero, use a cutoff with derivative bounded by C/δ on $[\delta, 2\delta]$. Its extra derivative contribution is bounded by

$$C\delta^{-2} \int_\delta^{2\delta} y^2 dr \leq C'\delta^2 \int_\delta^{2\delta} r^{-4}y^2 dr \longrightarrow 0.$$

The discarded derivative and weighted-potential integrals also tend to zero. At one the potential is bounded, and the usual H_0^1 cutoff approximation applies. For example, $|y(r)|^2 \leq (1-r) \int_r^1 |y'|^2$ controls its derivative error. Mollification on the remaining compact interval completes the core approximation.

Choose a nonnegative normalized minimizer g for λ_1 . Existence follows from compactness and lower semicontinuity. The absolute-value operation does not change its form energy. Interior regularity gives

$$-g'' + r^{-4}g = \lambda_1 g$$

classically on $(0, 1)$. If g vanished at an interior point, nonnegativity would give $g' = 0$ there, and ODE uniqueness would force $g = 0$. Consequently $g > 0$ throughout $(0, 1)$. The estimates below in particular give $\lambda_1 < \lambda_2$.

Lemma 2.1. *The spectral data satisfy*

$$\pi^2 + 1 \leq \lambda_1 \leq 30, \quad \lambda_2 \geq 4\pi^2 + 1 > 37.$$

For the continuous linear functional

$$\ell(y) = \int_0^1 r^{-2} y(r) dr \quad (y \in X)$$

one has $|\ell(y)|^2 \leq a[y]$ and $\ell(g)^2 > 1/12$.

Proof. The potential is at least one. Comparison of Rayleigh quotients with the Dirichlet Laplacian, followed by the min–max principle, gives $\lambda_j \geq j^2\pi^2 + 1$ for $j = 1, 2$. The trial function $y = r^3(1 - r)$ belongs to X and has

$$\int_0^1 y^2 dr = \frac{1}{252}, \quad \int_0^1 y'^2 dr = \frac{3}{35}, \quad \int_0^1 r^{-4} y^2 dr = \frac{1}{30}.$$

Its Rayleigh quotient is 30. Cauchy–Schwarz on an interval of length one gives $|\ell(y)|^2 \leq \int r^{-4} y^2 \leq a[y]$. Since $\|g\|_2 = 1$ and $\|g'\|_2^2 \leq \lambda_1$,

$$\|g\|_\infty^2 \leq 2\|g\|_2\|g'\|_2 \leq 2\sqrt{30} < 12.$$

Also $1 = \int g^2 \leq \|g\|_\infty \int g$, and $\ell(g) \geq \int g$ because $g > 0$ and $r^{-2} \geq 1$. Therefore $\ell(g)^2 > 1/12$. $\square$

Lemma 2.2. *If $y \in X$ and $\ell(y) = 0$, then*

$$a[y] - \lambda_1 \|y\|_2^2 \geq \frac{7}{445} \|y\|_2^2. \quad (5)$$

Proof. Write $y = \alpha g + z$, where $z \perp g$, and put

$$T = a[z] - \lambda_1 \|z\|_2^2, \quad t_0 = \ell(g)^2.$$

Orthogonality for the form follows from the weak eigenvalue equation. Thus $a[y] - \lambda_1 \|y\|_2^2 = T$. The spectral bound on $g^\perp$ yields

$$\|z\|_2^2 \leq \frac{T}{\lambda_2 - \lambda_1}, \quad a[z] \leq \frac{\lambda_2}{\lambda_2 - \lambda_1} T.$$

The constraint gives $\alpha \ell(g) = -\ell(z)$, so

$$\alpha^2 t_0 \leq a[z], \quad \|y\|_2^2 \leq \frac{\lambda_2 + t_0}{(\lambda_2 - \lambda_1)t_0} T.$$

Consequently

$$T \geq \frac{(\lambda_2 - \lambda_1)t_0}{\lambda_2 + t_0} \|y\|_2^2 \geq \frac{(37 - 30)/12}{37 + 1/12} \|y\|_2^2 = \frac{7}{445} \|y\|_2^2.$$

Here the fraction decreases in λ_1 and increases in λ_2 and t_0 in the positive range at issue. $\square$

3 The scalar form and absence of a Bessel solution

Set

$$W_0(r) = \lambda_1 + 2r^{-2} - r^{-4}, \quad \mathcal{B}_0(v) = \int_0^1 r^4 (v'^2 - W_0 v^2) dr.$$

For $v \in C_c^\infty(0, 1)$ the substitution $y = r^2 v$ and an integration by parts give

$$\mathcal{B}_0(v) = a[y] - \lambda_1 \|y\|_2^2. \quad (6)$$

At the integrand level the difference of the two sides is the derivative of $-2y^2/r$. Thus $\mathcal{B}_0(v) \geq 0$. Moreover, $\ell(y) = \int_0^1 v dr$, so Lemma 2.2 implies

$$\int_0^1 v dr = 0 \implies \mathcal{B}_0(v) \geq \frac{7}{445} \int_0^1 r^4 v^2 dr. \quad (7)$$

Let $\epsilon = 1/100$, so $W = W_0 + \epsilon$, and define $\mathcal{B}_W$ with W in place of W_0 . Choose $y_j \in C_c^\infty(0, 1)$ tending to g in the form norm, and set $v_j = r^{-2} y_j$. By (6),

$$\mathcal{B}_W(v_j) = a[y_j] - (\lambda_1 + \epsilon) \|y_j\|_2^2 \longrightarrow -\epsilon < 0.$$

If a positive solution f of (1) existed, the ground-state identity would give, for every compact test v ,

$$\mathcal{B}_W(v) = \int_0^1 r^4 f^2 \left(\frac{v}{f} \right)^2 dr \geq 0. \quad (8)$$

This follows by expansion and integration by parts, using (1); all boundary terms vanish. Taking $v = v_j$ for large j is a contradiction. It also follows that the corresponding first-order Hardy inequality fails on radial tests, unlike (4).

4 The inequality in every angular mode

We use the Grushin polar decomposition and spherical harmonics as in [1, Section 2]. Writing $\sigma = (\phi, \omega) \in (0, \pi) \times S^2$, one has

$$dx dt = \frac{r^4}{2\psi} dr d\Omega, \quad d\Omega = (\sin \phi)^{3/2} d\phi d\omega, \quad \mathcal{L}_G = \psi \left(\partial_r^2 + \frac{4}{r} \partial_r + \frac{4}{r^2} \mathcal{L}_\sigma \right).$$

There is a complete orthonormal family $\Phi_{k,j}$ in $L^2(\Omega, d\Omega)$ with

$$-4\mathcal{L}_\sigma \Phi_{k,j} = c_k \Phi_{k,j}, \quad c_k = k(k+3), \quad k = 0, 1, \dots$$

The index j runs through the finite multiplicity at fixed k . For a normalized mode $U(r)\Phi_{k,j}(\sigma)$, twice the deficit on the left of (4) is

$$D_{c,W}(U) = \int_0^1 r^4 \left(U'' + \frac{4}{r} U' - \frac{c}{r^2} U \right)^2 dr - \int_0^1 r^4 \left(W + \frac{4}{r^2} \right) \left(U'^2 + \frac{c}{r^2} U^2 \right) dr. \quad (9)$$

Lemma 4.1. *For any $U \in C_c^\infty(0, 1)$ and $c = c_k$,*

$$D_{c,W}(U) \geq \frac{51}{8900} \int_0^1 r^4 \left(U'^2 + \frac{c}{r^2} U^2 \right) dr.$$

Proof. A direct integration by parts gives the exact identity

$$D_{c,W_0}(U) = \mathcal{B}_0(U') + c\mathcal{B}_0(U/r) + c \int_0^1 r^2 U'^2 dr + c(c-4) \int_0^1 U^2 dr. \quad (10)$$

For explicit verification, the difference between the integrands on the two sides is

$$\frac{d}{dr} \left(4r^3 U'^2 - 2cr^2 U U' - cr U^2 \right).$$

All its endpoint values vanish. Because $\int_0^1 U' dr = 0$, (7) applies to U' . Also $\mathcal{B}_0(U/r) \geq 0$. The elementary weighted Hardy identity

$$\int_0^1 r^2 U'^2 dr - \frac{1}{4} \int_0^1 U^2 dr = \int_0^1 (rU' + U/2)^2 dr$$

shows that $\int r^2 U'^2 \geq \frac{1}{4} \int r^2 U^2$, since $r < 1$. Finally, $c_k = 0$ or $c_k \geq 4$, so $c_k(c_k - 4) \geq 0$. Subtracting the change in (9) from W_0 to $W_0 + \epsilon$ therefore gives

$$\begin{aligned} D_{c,W}(U) &\geq \left(\frac{7}{445} - \epsilon \right) \int_0^1 r^4 U'^2 dr \\ &\quad + c \left(\frac{1}{4} - \epsilon \right) \int_0^1 r^2 U^2 dr + c(c-4) \int_0^1 U^2 dr. \end{aligned}$$

Here $7/445 - \epsilon = 51/8900 > 0$ and $1/4 - \epsilon = 6/25 > 51/8900$, proving the assertion. $\square$

Completion of Theorem 1.1. Expand $u \in \mathcal{D}$ as $u = \sum_{k,j} U_{k,j}(r) \Phi_{k,j}(\sigma)$. Every radial coefficient is smooth and is supported in the same compact subinterval $[a, b] \subset (0, 1)$ as u . Parseval's identity for $\mathcal{L}_G u / \psi$, with measure $r^4 dr d\Omega / 2$, gives

$$\int_{B_1} \frac{(\mathcal{L}_G u)^2}{\psi} = \frac{1}{2} \sum_{k,j} \int_0^1 r^4 \left(U_{k,j}'' + \frac{4}{r} U_{k,j}' - \frac{c_k}{r^2} U_{k,j} \right)^2 dr.$$

The angular gradient form and radial Parseval identity likewise give

$$\int_{B_1} |\nabla_G u|^2 = \frac{1}{2} \sum_{k,j} \int_0^1 r^4 \left(U_{k,j}'^2 + \frac{c_k}{r^2} U_{k,j}^2 \right) dr.$$

The same latter identity holds after inserting any bounded radial multiplier on $[a, b]$, in particular $W + 4/r^2$. Absolute convergence follows by bounding its absolute value by its supremum times the unweighted gradient energy. The Laplacian integral is finite by the local integrability noted after the theorem. Thus the modal identities extend to the full expansion, not merely finite angular sums. Summing Lemma 4.1 proves (4). The absence of a positive Bessel solution was proved above. Finally, $W(r) \rightarrow -\infty$ as $r \downarrow 0$, while $W(r) \rightarrow \lambda_1 + 1 + \epsilon > 0$ as $r \uparrow 1$. $\square$

5 What the example does and does not show

For radial $u = U(\rho)$ the Hardy–Rellich deficit is a positive constant times $\mathcal{B}_W(U')$. The derivative U' has zero integral, and every compactly supported smooth zero-integral function arises this way. The scalar Bessel equation, by contrast, tests the full quadratic form. Our singular negative part makes the missing scalar functional continuous in the form norm and leaves a strict gap on its kernel. A

small spectral shift then separates the two tests. The angular identity (10) is what makes this more than a radial counterexample.

The traditional positive-weight Bessel framework [2] and the unrestricted signed formulation must be distinguished. Here $W_-(r)/r^{-2}$ is unbounded at zero, where $W_- = \max\{-W, 0\}$. Our result says nothing adverse about a converse with a nonnegative second weight, a controlled negative part, or additional endpoint assumptions. It does not refute the forward theorem in [1]. Nor does it assert a converse or a counterexample for an inequality retaining a prescribed solution-dependent remainder. The broader workshop entry also mentions the first-order Hardy inequality; this note establishes a complete negative answer only to the unrestricted signed, remainder-free Hardy–Rellich implication stated explicitly here.

The accompanying standard-library Python script checks the three total-derivative identities with exact multivariate Laurent arithmetic and verifies the rational constants. It does not establish spectral existence, density, or angular completeness; those are the analytic arguments in the text. AI assistance was used in developing and checking the argument and preparing the manuscript. The author is responsible for the final statements. This preprint has not been independently refereed or formally verified, and makes no absolute priority claim.

References

- [1] D. Ganguly, K. Jotsaroop and P. Roychowdhury, *Hardy and Rellich identities and inequalities for Baouendi–Grushin operators via spherical vector fields*, arXiv:2404.05510v3 (2026), <https://arxiv.org/abs/2404.05510v3>.
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