

A Negative Answer to a Question of Basu and Perrucci on Two Multi-Affine Polynomials

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Abstract

A real polynomial is multi-affine if it has degree at most one in each variable. Basu and Perrucci proved that the real zero set of one multi-affine polynomial of degree d in $\mathbb{R}^n$ has at most 2^{d-1} connected components, independently of n , and they gave three multi-affine polynomials of degree at most 4 whose common real zero set has $\binom{n}{k}$ components. They asked whether the number of connected components of the common real zero set of *two* multi-affine polynomials of degree at most d is bounded in terms of d alone; the question was posed again in the Oberwolfach Report 9/2025. We show that the answer is no. For $m \geq 2$ the multi-affine polynomials $P = x_0 - \sum_i x_i y_i$ and $Q = x_0 \sum_i x_i y_i - 2 \sum_{i < j} x_i y_i x_j y_j - 2 \sum_i x_i y_i + m$ in $2m+1$ variables have degrees 2 and 4 and satisfy the identity $Q = (\sum_i x_i y_i)P + \sum_i (x_i y_i - 1)^2$. Hence their common real zero set is $\{x_0 = m\}$ times a product of m hyperbolas, and it has exactly 2^m connected components. The construction is a two-polynomial analogue of Basu and Perrucci's three-polynomial example: a single equation introducing an auxiliary variable takes the place of the two equations that fix power sums. More generally, k multi-affine polynomials in disjoint sets of variables combine in the same way into a pair whose zero set is homeomorphic to the product of their zero sets. With k copies of Basu and Perrucci's extremal hypersurface of degree d this gives degrees d and $2d$ and $2^{(d-1)k}$ components in $dk+1$ variables. The negative answer persists inside boxes and, as a lower bound, inside any sets with nonempty interior. Thus no bound in terms of the degrees exists once the degree bound is at least 4; for degree bounds 2 and 3 the question remains open, and we show only that the mechanism used here yields at most two components in that range. This is an unrefereed note.

1 Introduction

For a finite set $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_n]$ let $\mathcal{Z}(\mathcal{P})$ be the set of common real zeros of $\mathcal{P}$ in $\mathbb{R}^n$, and let $b_0(S)$ be the number of connected components of a set $S \subseteq \mathbb{R}^n$. A polynomial P is *multi-affine* if $\deg_{X_i} P \leq 1$ for every i . By the theorem of Petrovskiĭ–Oleĭnik, Thom and Milnor [PO49, Tho65, Mil64], a real algebraic set in $\mathbb{R}^n$ defined by polynomials of degree at most d has at most $d(2d-1)^{n-1}$ connected components, and exponential growth in n cannot be avoided in general. Basu and Perrucci [BP23, Theorem 2] proved that for a single multi-affine polynomial P of degree d ,

$$b_0(\mathcal{Z}(P)) \leq 2^{d-1},$$

independently of n , and that $P = X_1 \cdots X_d - 1$ attains the bound [BP23, Example 2.1]. They point out that the usual reduction of a system of equations to a single one by a sum of squares does not work well for multi-affine polynomials, because the square of a multi-affine polynomial is in general not multi-affine. In [BP23, Example 2.2] they give three multi-affine polynomials of degree at most 4 in n

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variables whose common real zero set consists of the $\binom{n}{k}$ points of $\{0, 1\}^n$ with exactly k coordinates equal to 1. So for three polynomials no bound depending only on the degrees exists. Right after this example they state that they do not know whether such a bound exists for two polynomials, and they list the question as Problem 2 in [BP23, Section 4]. Basu posed it again, as joint work with Perrucci, in the problem session of the MFO–RIMS tandem workshop of February 2025 [ORSS25, pp. 445–446]. It is record OWR-14299088-013 of the UnsolvedMath dataset [ula26]. In our notation the question reads as follows.

Problem 1.1 ([BP23, Section 4, Problem 2], [ORSS25, p. 446]). *Is there a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $b_0(\mathcal{Z}(P, Q)) \leq f(d)$ for all n and all multi-affine $P, Q \in \mathbb{R}[X_1, \dots, X_n]$ of degree at most d ?*

We show that the answer is no. Numbered statements of [BP23] are cited as in its arXiv version.

Theorem 1.2. *Let $m \geq 1$ and $n = 2m + 1$, and use coordinates $(x_0, x_1, \dots, x_m, y_1, \dots, y_m)$ on $\mathbb{R}^n$. Put $t_i = x_i y_i$, $T = t_1 + \dots + t_m$ and $e_2 = \sum_{1 \leq i < j \leq m} t_i t_j$, and let*

$$P_m = x_0 - T, \quad Q_m = x_0 T - 2e_2 - 2T + m.$$

- (a) P_m and Q_m are multi-affine, $\deg P_m = 2$, and $\deg Q_m = 4$ for $m \geq 2$ (while $\deg Q_1 = 3$).
- (b) The following identity holds in $\mathbb{R}[x_0, x, y]$:

$$Q_m = T \cdot P_m + \sum_{i=1}^m (x_i y_i - 1)^2. \tag{1}$$

- (c) $\mathcal{Z}(P_m, Q_m) = \{x_0 = m, x_1 y_1 = \dots = x_m y_m = 1\}$. This set has exactly 2^m connected components, namely the sets $C_s = \{(m, s_1 a_1, \dots, s_m a_m, s_1/a_1, \dots, s_m/a_m) : a \in (0, \infty)^m\}$ for $s \in \{\pm 1\}^m$, and each of them is homeomorphic to $\mathbb{R}^m$.

Corollary 1.3. *For every $n \geq 5$ there are multi-affine $P, Q \in \mathbb{R}[X_1, \dots, X_n]$ of degrees 2 and 4 with $b_0(\mathcal{Z}(P, Q)) = 2^{\lfloor (n-1)/2 \rfloor}$. Consequently Problem 1.1 has a negative answer: no bound $f(d)$ exists for any $d \geq 4$.*

Proof. For odd $n = 2m + 1$ take P_m, Q_m with $m \geq 2$. For even $n = 2m + 2$ take the same polynomials, which do not involve X_n ; their zero set is $\mathcal{Z}(P_m, Q_m) \times \mathbb{R}$, which again has $2^m = 2^{\lfloor (n-1)/2 \rfloor}$ components. $\square$

For $d = 1$ a bound exists trivially, namely $f(1) = 1$, since two affine functions have an affine subspace or the empty set as common zero set. So the problem remains open exactly for degree bounds $d = 2$ and $d = 3$. Section 5 shows that the mechanism behind Theorem 1.2 cannot give more than two components in that range.

The identity (1) is the whole point. On $\mathcal{Z}(P_m)$ we have $x_0 = T$, so the multi-affine polynomial $x_0 T = \sum_i x_0 t_i$ acts as a proxy for the square T^2 , which is not multi-affine; the cross terms $2t_i t_j$ of T^2 are multi-affine and are cancelled explicitly. This is the two-polynomial analogue of [BP23, Example 2.2] (see Remark 3.4). The same device works for any multi-affine polynomials in disjoint sets of variables (Lemma 3.1). It yields pairs of degrees d and $2d$ whose zero set has $2^{(d-1)k}$ components (Corollary 3.3). The answer stays negative for the bounded version of the question (Proposition 4.1).

2 Preliminaries

We write $X_I = \prod_{i \in I} X_i$ for $I \subseteq \{1, \dots, n\}$. A polynomial is multi-affine if and only if it is a linear combination of the monomials X_I .

Lemma 2.1. (i) If f and g are multi-affine and involve disjoint sets of variables, then fg is multi-affine.

(ii) Let $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\varphi(X)_i = a_i X_i + b_i$, with all $a_i \neq 0$. If f is multi-affine of degree d , then so is $f \circ \varphi$.

Proof. (i) A product $X_I X_J$ with $I \cap J = \emptyset$ equals $X_{I \cup J}$. (ii) The monomial X_I becomes $\prod_{i \in I} (a_i X_i + b_i)$. This is multi-affine, and its part of degree $|I|$ is $(\prod_{i \in I} a_i) X_I$. Hence $f \circ \varphi$ is multi-affine, and its part of degree d is obtained from that of f by multiplying the coefficient of each X_I by $\prod_{i \in I} a_i \neq 0$. $\square$

The following lemma is [BP23, Example 2.1]; we include the short proof, because the count of components below rests on it.

Lemma 2.2. For $d \geq 1$ let $W_d = \{z \in \mathbb{R}^d : z_1 z_2 \cdots z_d = 1\}$. For $s \in \{\pm 1\}^d$ let $W_d^s = \{z \in W_d : \text{sgn } z_i = s_i \text{ for all } i\}$. Then W_d^s is nonempty if and only if $s_1 \cdots s_d = 1$, and the nonempty sets W_d^s are the 2^{d-1} connected components of W_d . Each of them is homeomorphic to $\mathbb{R}^{d-1}$.

Proof. No coordinate vanishes on W_d , so W_d is the disjoint union of the sets W_d^s . Each of them is open in W_d , being cut out by the strict inequalities $s_i z_i > 0$, and hence each is also closed in W_d . If $z \in W_d^s$ then $s_1 \cdots s_d = \text{sgn}(z_1 \cdots z_d) = 1$. Conversely, if $s_1 \cdots s_d = 1$, the map $(a_1, \dots, a_{d-1}) \mapsto (s_1 a_1, \dots, s_{d-1} a_{d-1}, s_d / (a_1 \cdots a_{d-1}))$ is a homeomorphism from $(0, \infty)^{d-1} \cong \mathbb{R}^{d-1}$ onto W_d^s , with inverse $z \mapsto (|z_1|, \dots, |z_{d-1}|)$. So the nonempty W_d^s are connected, open and closed in W_d , and there are 2^{d-1} of them. $\square$

3 The construction

Lemma 3.1. Let $f_1, \dots, f_k$ be multi-affine polynomials of degree at most d ($d \geq 1$) in pairwise disjoint sets of variables, none of which contains the variable x_0 . Put $S = f_1 + \cdots + f_k$ and

$$P = x_0 - S, \quad Q = x_0 S - 2 \sum_{1 \leq j < l \leq k} f_j f_l.$$

(a) P and Q are multi-affine, $\deg P \leq d$ and $\deg Q \leq 2d$.

(b) $Q = S \cdot P + \sum_{j=1}^k f_j^2$.

(c) Let V_j be the set of variables of f_j , let r be the number of variables other than x_0 outside $V_1 \cup \cdots \cup V_k$, and let $\mathcal{Z}(f_j) \subseteq \mathbb{R}^{V_j}$. Then $\mathcal{Z}(P, Q)$ is the set where $x_0 = 0$ and $f_1 = \cdots = f_k = 0$. It is homeomorphic to $\mathcal{Z}(f_1) \times \cdots \times \mathcal{Z}(f_k) \times \mathbb{R}^r$, and $b_0(\mathcal{Z}(P, Q)) = \prod_{j=1}^k b_0(\mathcal{Z}(f_j))$.

Proof. (a) S is multi-affine of degree at most d and does not involve x_0 , so P and $x_0 S$ are multi-affine. For $j \neq l$ the product $f_j f_l$ is multi-affine by Lemma 2.1(i) and has degree at most $2d$. Also $\deg(x_0 S) \leq d + 1 \leq 2d$. (b) $S \cdot P = x_0 S - S^2 = x_0 S - \sum_j f_j^2 - 2 \sum_{j < l} f_j f_l$. (c) By (b), a common zero of P and Q satisfies $\sum_j f_j^2 = 0$, hence $f_1 = \cdots = f_k = 0$, hence $S = 0$ and $x_0 = S = 0$. Conversely every such point is a common zero. Forgetting the coordinate $x_0 = 0$ and reordering the others gives the homeomorphism. The number of components of a finite product of spaces is the product of the numbers of components, and the factor $\mathbb{R}^r$ is connected. $\square$

Proof of Theorem 1.2. (a) P_m is x_0 minus a sum of the monomials $x_i y_i$. The monomials of Q_m are $x_0 x_i y_i$, $x_i y_i x_j y_j$ with $i < j$, $x_i y_i$ and 1. Each is a product of distinct variables, so P_m and Q_m are multi-affine, and $\deg P_m = 2$. For $m \geq 2$ the monomial $x_1 y_1 x_2 y_2$ occurs in Q_m with coefficient -2 , and all other monomials of Q_m have degree at most 3; so $\deg Q_m = 4$. For $m = 1$, $Q_1 = x_0 x_1 y_1 - 2x_1 y_1 + 1$ has degree 3.

(b) Since $T^2 = \sum_i t_i^2 + 2e_2$,

$$T \cdot P_m + \sum_i (t_i - 1)^2 = x_0 T - \sum_i t_i^2 - 2e_2 + \sum_i t_i^2 - 2T + m = Q_m.$$

(c) If P_m and Q_m vanish at a point, then by (1) $\sum_i (t_i - 1)^2 = 0$ there, so $t_i = 1$ for every i , $T = m$ and $x_0 = T = m$. Conversely, at every point with $x_0 = m$ and all $t_i = 1$ we have $P_m = 0$ and, by (1), $Q_m = 0$. Hence $\mathcal{Z}(P_m, Q_m) = \{m\} \times H^m$, up to the order of the coordinates, where $H = \{(x, y) \in \mathbb{R}^2 : xy = 1\} = W_2$. By Lemma 2.2, H has the two components $\{x > 0\}$ and $\{x < 0\}$, each homeomorphic to $\mathbb{R}$ via $a \mapsto (\pm a, \pm 1/a)$, $a > 0$. So $\mathcal{Z}(P_m, Q_m)$ has exactly 2^m components, the sets C_s , and each is homeomorphic to $(0, \infty)^m \cong \mathbb{R}^m$. $\square$

Remark 3.2. Theorem 1.2 is an instance of Lemma 3.1. Take $f_i = x_i y_i - 1$ ($1 \leq i \leq m$), so that $S = T - m$, and write P', Q' for the pair of Lemma 3.1. Replacing x_0 by $x_0 - m$ turns P' into P_m and Q' into $Q_m - m P_m$. This polynomial has the same common zeros with P_m as Q_m . So Theorem 1.2 is Lemma 3.1 up to a shift of x_0 and the addition of a multiple of P_m .

Corollary 3.3. *Let $d \geq 2$ and $k \geq 2$, and let $f_j = z_{j,1} z_{j,2} \cdots z_{j,d} - 1$ for $1 \leq j \leq k$, in dk distinct variables $z_{j,l}$. The pair (P, Q) of Lemma 3.1 consists of multi-affine polynomials of degrees exactly d and $2d$ in $n = dk + 1$ variables, and*

$$b_0(\mathcal{Z}(P, Q)) = 2^{(d-1)k} = 2^{(d-1)(n-1)/d}.$$

Proof. Each f_j has degree d , so $\deg P = d$. In Q the monomial $z_{1,1} \cdots z_{1,d} z_{2,1} \cdots z_{2,d}$ has coefficient -2 , and every other monomial of degree $2d$ is of the same form for another pair $j < l$; the terms of $x_0 S$ have degree at most $d + 1 < 2d$. Hence $\deg Q = 2d$. By Lemma 2.2, $b_0(\mathcal{Z}(f_j)) = 2^{d-1}$, and Lemma 3.1(c) gives the product. $\square$

For $d = 2$ Corollary 3.3 gives the same counts as Theorem 1.2. The blocks f_j are copies of the extremal hypersurfaces of [BP23, Example 2.1]. For a fixed degree bound $D = 2d$ the number of components grows like $(2^{1-2/D})^{n-1}$. The upper bound of Petrovskii–Oleńnik, Thom and Milnor for degree D is $D(2D - 1)^{n-1}$.

Remark 3.4 (Comparison with [BP23, Example 2.2]). In Example 2.2 of [BP23] the first two equations $P_1 = \sigma_1 - k$ and $P_2 = \sigma_2 - \binom{k}{2}$ fix the power sums $N_1 = \sum_i X_i$ and $N_2 = \sum_i X_i^2$. By the Newton identities, modulo the ideal (P_1, P_2) , the sum of squares $\sum_i X_i^2 (X_i - 1)^2 = N_4 - 2N_3 + N_2$ is congruent to a multi-affine polynomial P_3 of degree 4, a linear combination of σ_3, σ_4 and 1. This polynomial is the third equation. Our construction follows the same pattern with two equations. A single equation $P_m = x_0 - T$ introduces an auxiliary variable, and modulo (P_m) the sum of squares $\sum_i (x_i y_i - 1)^2$ is congruent to the multi-affine polynomial Q_m . This is the content of the certificate (1), and of $Q = S \cdot P + \sum_j f_j^2$ in Lemma 3.1. The price of saving one equation is that the zero set is no longer finite: its components are copies of $\mathbb{R}^m$.

4 Bounded versions

Basu and Perrucci also consider the number of components of the zero set inside a sequence of closed semi-algebraic sets $B_n \subseteq \mathbb{R}^n$, such as balls or boxes [BP23, Definition 1.1]. The negative answer persists in this setting.

Proposition 4.1. *Let $m \geq 1$, $n = 2m + 1$, and let P_m, Q_m be as in Theorem 1.2.*

- (a) *If $R \geq m$, then $\mathcal{Z}(P_m, Q_m) \cap [-R, R]^n$ has exactly 2^m connected components.*
- (b) *Let $B \subseteq \mathbb{R}^n$ be any set with nonempty interior. Then there is a map φ as in Lemma 2.1(ii) such that the multi-affine polynomials $\tilde{P} = P_m \circ \varphi^{-1}$ and $\tilde{Q} = Q_m \circ \varphi^{-1}$ have the same degrees as P_m, Q_m and satisfy $b_0(\mathcal{Z}(\tilde{P}, \tilde{Q}) \cap B) \geq 2^m$.*

Proof. Let $Z = \mathcal{Z}(P_m, Q_m)$. By Theorem 1.2(c), Z is the disjoint union of the 2^m sets C_s , each open and closed in Z .

(a) A point $(m, s_1 a_1, \dots, s_m a_m, s_1/a_1, \dots, s_m/a_m)$ of C_s lies in $[-R, R]^n$ if and only if $1/R \leq a_i \leq R$ for all i ; here we use $m \leq R$. So $C_s \cap [-R, R]^n$ is homeomorphic to $[1/R, R]^m$. It is nonempty because $R \geq m \geq 1$, and it is connected. These 2^m sets are open and closed in $Z \cap [-R, R]^n$, so they are its components.

(b) Choose $c \in \mathbb{R}^n$ and $r > 0$ with $c + [-r, r]^n \subseteq B$, and let $\varphi(X) = c + (r/m)X$. Its inverse $\varphi^{-1}(Y) = (m/r)(Y - c)$ is again a map of the form in Lemma 2.1(ii), with all $a_i = m/r \neq 0$. Applying Lemma 2.1(ii) to φ^{-1} shows that $\tilde{P} = P_m \circ \varphi^{-1}$ and $\tilde{Q} = Q_m \circ \varphi^{-1}$ are multi-affine with the degrees of P_m and Q_m . Moreover $\mathcal{Z}(\tilde{P}, \tilde{Q}) = \varphi(Z)$. The point $p_s = (m, s_1, \dots, s_m, s_1, \dots, s_m)$ lies in $C_s \cap [-m, m]^n$, so $\varphi(p_s) \in \varphi(C_s) \cap B$. The 2^m sets $\varphi(C_s) \cap B$ are nonempty, pairwise disjoint, open and closed in $\varphi(Z) \cap B$, and they cover it. Hence $b_0(\varphi(Z) \cap B) \geq 2^m$. $\square$

Remark 4.2. When every B_n is a compact convex semi-algebraic set of dimension n , part (b) also follows from [BP23, Proposition 1.1]. Its proof applies to pairs of polynomials, because pairs of multi-affine polynomials of degree at most d are stable under translations and scalings of the variables. The direct argument above needs only that B has nonempty interior.

Theorem 1.2, Corollary 1.3 and the results of Sections 2 to 4 remain true over any real closed field K , the setting of [BP23], in the following form: $\mathbb{R}$ is replaced by K , homeomorphisms by semi-algebraic homeomorphisms, connected components by semi-algebraically connected components, and in part (b) the set B is assumed to be semi-algebraic. The proofs carry over, for the following reasons. The identities (1) and Lemma 3.1(b) and the statements on multi-affinity and degrees are algebraic, and a sum of squares in the ordered field K vanishes only if each term vanishes. All maps in the proofs are continuous and semi-algebraic; in place of $(0, \infty) \cong \mathbb{R}$ one uses the semi-algebraic homeomorphism $a \mapsto a - 1/a$ from $(0, \infty)$ onto K . Semi-algebraically connected components are defined only for semi-algebraic sets, which is why B must be semi-algebraic; then all sets that the proofs show to be open and closed are semi-algebraic. Finally, intervals in K are semi-algebraically connected, and so are finite products of semi-algebraically connected semi-algebraic sets [BCR98, Sections 2.4 and 2.5]. This gives Lemma 2.2 over K , and with it all the component counts above.

5 Limits of the method in degree at most three

We do not know whether two multi-affine polynomials of degree at most 3, or at most 2, have a number of components bounded independently of n . Here we show only that the mechanism of Section 3 cannot give a negative answer in that range.

Definition 5.1. Let $u = (u_1, \dots, u_N)$ and let x_0 be one further variable. A pair (P, Q) of multi-affine polynomials in $\mathbb{R}[x_0, u]$ is of *elimination type* if

$$P = x_0 - T, \quad Q = C \cdot P + G, \quad G = \sum_{j=1}^r h_j^2,$$

with $T, C, h_1, \dots, h_r \in \mathbb{R}[u]$. In that case T and C are multi-affine, and so is $D = Q - x_0 C = G - TC$, the part of Q free of x_0 .

The pairs of Theorem 1.2 and Lemma 3.1 are of elimination type, with $C = T$ and $h_i = x_i y_i - 1$, respectively $T = C = S$ and $h_j = f_j$. For a pair of elimination type the projection $(x_0, u) \mapsto u$ maps $\mathcal{Z}(P, Q)$ homeomorphically onto $\mathcal{Z}(G) = \mathcal{Z}(h_1, \dots, h_r)$; its inverse is $u \mapsto (T(u), u)$.

For $\alpha \in \mathbb{N}^N$ write u^α for the monomial with exponent α , and $\text{supp}(h)$ for the set of exponents occurring in h . The following is the Newton polytope argument of Reznick [Rez78], in the form we need.

Lemma 5.2. *Let $h_1, \dots, h_r \in \mathbb{R}[u]$, not all zero, let $G = \sum_j h_j^2$, and let $\ell: \mathbb{R}^N \rightarrow \mathbb{R}$ be linear. Let M be the maximum of $\ell(\alpha)$ over all j and all $\alpha \in \text{supp}(h_j)$. Then $2M = \max\{\ell(\gamma) : \gamma \in \text{supp}(G)\}$. In particular, $\deg G$ is twice the maximum of the $\deg h_j$, and the homogeneous part of G of top degree is $\sum_j (h_j^*)^2$. Here h_j^* is the homogeneous part of h_j of degree $\frac{1}{2} \deg G$, which may be zero.*

Proof. Let h_j^* be the sum of the terms cu^α of h_j with $\ell(\alpha) = M$. In the expansion of $\sum_j h_j^2$ every product of two terms has ℓ -value at most $2M$, with equality exactly for products of two terms of the h_j^* . Hence the terms of G with ℓ -value $2M$ add up to $\sum_j (h_j^*)^2$, and G has no terms of larger ℓ -value. Some h_j^* is nonzero, so it does not vanish at some point of $\mathbb{R}^N$. There $\sum_j (h_j^*)^2 > 0$, so this sum is not the zero polynomial. For the last statement take $\ell(\alpha) = \alpha_1 + \dots + \alpha_N$. $\square$

Lemma 5.3. *Let q be an indefinite real quadratic form in u . If $a \in \mathbb{R}[u]$ vanishes at every real zero of q , then q divides a in $\mathbb{R}[u]$.*

Proof. An invertible linear change of variables preserves both the hypothesis and the conclusion. By Sylvester's law of inertia we may therefore assume that $q = w_1^2 - \phi(w')$, where $w' = (w_2, \dots, w_N)$ and $\phi(w') = -\sum_{2 \leq i \leq p} w_i^2 + \sum_{p < i \leq \rho} w_i^2$ with $1 \leq p < \rho \leq N$. Here $p \geq 1$ and $\rho > p$ because q is indefinite. Since q is monic of degree 2 in w_1 , division with remainder gives $a = qb + a_1(w')w_1 + a_0(w')$. The set $U = \{w' : \phi(w') > 0\}$ is open and nonempty; it contains the point with $w_{p+1} = 1$ and all other coordinates 0. For $w' \in U$ both points $(\pm \sqrt{\phi(w')}, w')$ are real zeros of q . So $a_0(w') \pm a_1(w')\sqrt{\phi(w')} = 0$, hence $a_0 = a_1 = 0$ on U . A polynomial that vanishes on a nonempty open set is zero, so $a = qb$. $\square$

Lemma 5.4. *If $g \in \mathbb{R}[y_1, \dots, y_k]$ has degree at most 2, then $\mathcal{Z}(g)$ has at most two connected components.*

Proof. If $\deg g \leq 1$, then $\mathcal{Z}(g)$ is empty, a hyperplane or $\mathbb{R}^k$. Let $\deg g = 2$. After an orthogonal change of coordinates and a translation, $g = \sum_{i \leq r} \lambda_i y_i^2 + \beta y_{r+1} + \delta$ with all $\lambda_i \neq 0$, where $\beta = 0$ if $r = k$. If $\beta \neq 0$, then $\mathcal{Z}(g)$ is the graph of a continuous function of the coordinates other than y_{r+1} , hence homeomorphic to $\mathbb{R}^{k-1}$. Let $\beta = 0$. After rescaling the coordinates and possibly replacing g by $-g$, $\mathcal{Z}(g) = \{|y'|^2 - |y''|^2 = c\}$ with $c \in \{0, 1\}$, where $y' \in \mathbb{R}^p$ and $y'' \in \mathbb{R}^{r-p}$ collect the first r coordinates; the remaining $k - r$ coordinates are free. If $c = 0$, then $\mathcal{Z}(g)$ contains the origin and is invariant under $y \mapsto \tau y$ for $\tau \in [0, 1]$, so it is connected. If $c = 1$, then $(y', y'', y''') \mapsto (y'/|y'|, y'', y''')$ is a homeomorphism from $\mathcal{Z}(g)$ onto $S^{p-1} \times \mathbb{R}^{r-p} \times \mathbb{R}^{k-r}$, with inverse $(\theta, y'', y''') \mapsto (\sqrt{1 + |y''|^2} \theta, y'', y''')$. This set is empty for $p = 0$, has two components for $p = 1$, and is connected for $p \geq 2$. $\square$

Proposition 5.5. *Let (P, Q) be a pair of elimination type, with T, C, G as in Definition 5.1.*

(a) *If $\deg T \leq 1$ or $\deg C \leq 1$, then $\mathcal{Z}(P, Q)$ is empty or homeomorphic to an affine space; so $b_0(\mathcal{Z}(P, Q)) \leq 1$. This applies in particular whenever $\deg P \leq 1$ or $\deg Q \leq 2$, whatever the degree of the other polynomial.*

(b) *If $\deg P \leq 3$ and $\deg Q \leq 3$, then $b_0(\mathcal{Z}(P, Q)) \leq 2$.*

Proof. Recall that $\mathcal{Z}(P, Q) \cong \mathcal{Z}(G)$. Since D does not involve x_0 , we have $\deg Q \geq 1 + \deg C$ whenever $C \neq 0$; and $\deg P \geq \deg T$. This gives the last sentence of (a).

(a) Every monomial of TC has the form $u^{\beta+\beta'}$ with β, β' in $\{0, 1\}^N$, one of them of weight at most 1. So all its exponents are at most 1, except for at most one exponent equal to 2. The monomials of D have all exponents at most 1. Hence every $\gamma \in \text{supp}(G)$ satisfies $\gamma_a \leq 2$ and $\gamma_a + \gamma_b \leq 3$ for all $a \neq b$. If all h_j vanish, then $\mathcal{Z}(G) = \mathbb{R}^N$. Otherwise Lemma 5.2, applied with $\ell(\alpha) = \alpha_a$ and with $\ell(\alpha) = \alpha_a + \alpha_b$, shows that every $\alpha \in \text{supp}(h_j)$ satisfies $\alpha_a \leq 1$ and $\alpha_a + \alpha_b \leq 3/2$. As α is integral, it has at most one nonzero entry, and that entry equals 1. So every h_j is affine. Hence $\mathcal{Z}(G) = \bigcap_j \mathcal{Z}(h_j)$ is empty or an affine subspace.

(b) Now $\deg T \leq 3$, $\deg C \leq 2$ and $\deg D \leq 3$. By (a) we may assume $\deg C = 2$ and $\deg T \in \{2, 3\}$. The top-degree part of TC is the product of the top-degree parts of T and C , which is nonzero. Its degree $\deg T + 2 \geq 4$ exceeds $\deg D$, so $G \neq 0$ and $\deg G = \deg T + 2$. By Lemma 5.2, $\deg G$ is even. So $\deg T = 2$, $\deg G = 4$, every h_j has degree at most 2, and

$$T_2 C_2 = \sum_j (h_j^{(2)})^2,$$

where T_2 , C_2 and $h_j^{(2)}$ are the homogeneous quadratic parts of T , C and h_j . The form T_2 is nonzero and has no square terms, because T is multi-affine. Let $\epsilon_1, \dots, \epsilon_N$ be the standard basis of $\mathbb{R}^N$. If $cu_a u_b$ ($a \neq b$, $c \neq 0$) is one of the terms of T_2 , then $T_2(\epsilon_a + \epsilon_b) = c$ and $T_2(\epsilon_a - \epsilon_b) = -c$. So T_2 is indefinite. At every real zero of T_2 the right-hand side vanishes, so each $h_j^{(2)}$ vanishes there. By Lemma 5.3, T_2 divides $h_j^{(2)}$, that is, $h_j^{(2)} = \mu_j T_2$ with $\mu_j \in \mathbb{R}$. The vector $\mu = (\mu_j)$ is nonzero, since $T_2 C_2 \neq 0$. Let O be an orthogonal $r \times r$ matrix whose first row is $\mu/|\mu|$, and put $h' = Oh$. Then $\sum_j h_j'^2 = \sum_j h_j^2 = G$. The quadratic part of h_j' is $(O\mu)_j T_2$, which is $|\mu| T_2$ for $j = 1$ and 0 for $j \geq 2$. Hence $\mathcal{Z}(G) = \mathcal{Z}(h_1') \cap V$, where $V = \bigcap_{j \geq 2} \mathcal{Z}(h_j')$ is empty or an affine subspace. If $V \neq \emptyset$, an affine parametrization $\mathbb{R}^k \cong V$ turns $h_1'|_V$ into a polynomial of degree at most 2 on $\mathbb{R}^k$. By Lemma 5.4, $\mathcal{Z}(G)$ has at most two components. $\square$

Remark 5.6. The bound in Proposition 5.5(b) is attained. The case $m = 1$ of Theorem 1.2 is of elimination type, has degrees $(2, 3)$ and has two components. In Theorem 1.2 with $m \geq 2$ the degree 4 of Q_m comes from $D = -2e_2 - 2T + m$, which cancels the cross terms of T^2 . Proposition 5.5 concerns only pairs of elimination type. It says nothing about general pairs of multi-affine polynomials of degree at most 3, for which Problem 1.1 remains open.

6 Concluding remarks

Theorem 1.2 settles Problem 1.1 in the negative. Two multi-affine polynomials of degrees 2 and 4 in $2m+1$ variables can have exactly 2^m connected components of common real zeros. So the gap between one polynomial, where the bound 2^{d-1} holds [BP23], and three polynomials [BP23, Example 2.2] closes on the unbounded side. The following questions remain open.

1. Is there a bound in terms of d alone for two multi-affine polynomials of degree at most $d \in \{2, 3\}$?
By Proposition 5.5, a negative answer would need a mechanism other than elimination of one variable followed by a sum of squares.
2. For $d \geq 4$, what is the largest possible growth rate in n of $b_0(\mathcal{Z}(P, Q))$ for multi-affine P, Q of degree at most d ? Let $d' = \lfloor d/2 \rfloor$. For $n = d'k + 1$ with $k \geq 2$, Corollary 3.3 with blocks of degree d' gives $2^{(d'-1)k} = (2^{1-1/d'})^{n-1}$ components. For even d this is $(2^{1-2/d})^{n-1}$, and for $d = 4$ it is $2^{(n-1)/2}$.

Verification. The claims were checked as follows.

1. All proofs are elementary and were checked by hand. The only algebraic input is the pair of identities (1) and Lemma 3.1(b), and each can be checked in two lines.
2. The finder's program `verify_construction.py` uses the standard library only, with exact rational arithmetic. For $m \leq 12$ it checks Theorem 1.2(a) and the identity $Q_m|_{x_0=T} = \sum_i (x_i y_i - 1)^2$. This identity is equivalent to (1), because the two sides of (1) have the same coefficient T of x_0 . It also checks Lemma 3.1(a),(b) for $f_j = z_{j,1} \cdots z_{j,d} - 1$ with $2 \leq d \leq 4$ and $1 \leq k \leq 6$, namely multi-affinity, the degrees and the identity $Q|_{x_0=S} = \sum_j f_j^2$, which is equivalent to Lemma 3.1(b) for the same reason. Finally, it exhibits exact rational points of $\mathcal{Z}(P_m, Q_m)$ in all 2^m sign classes for $m \leq 10$.
3. An independent referee wrote a separate program, `indep_check.py`, from scratch. It checks the following.
 - The certificate (1), as an exact polynomial identity, for $m \leq 10$.
 - For each $m \in \{11, 15, 20, 25, 30, 40\}$, the certificate, by exact integer evaluation at 200 random points.
 - The certificate of Lemma 3.1(b), for $f_j = z_{j,1} \cdots z_{j,d} - 1$ with $2 \leq d \leq 5$ and $k \leq 5$.
 - As a numerical refutation attempt, least-squares minimisation of (P_m, Q_m) from random starts for $m = 2, 3, 4$. Every computed zero satisfied $x_i y_i = 1$ and $x_0 = m$ up to $2 \cdot 10^{-5}$, and all 2^m sign patterns occurred. This part is numerical evidence only.
4. A lead program, `lead_check.py`, was written for this paper and uses exact arithmetic. It checks (1) for $m \leq 16$ and the identity of Remark 3.2 for $m \leq 12$. It checks Lemma 3.1(a),(b) on 200 random instances with integer coefficients. For $2 \leq d \leq 5$ and $2 \leq k \leq 5$ it checks the certificate and the exact degrees $(d, 2d)$ of Corollary 3.3; the component count is not computed. It also checks the rescaling step of Proposition 4.1(b) for $m \leq 5$.
5. Two independent referees checked by hand the arguments of the underlying write-up, including the degree-three analysis on which Section 5 is based. The second referee also re-ran the finder's program and checked (1) with its own exact code for $m \leq 7$; that code is not included in the package. The proofs of Lemmas 5.2 to 5.4 were written out in self-contained form for this paper.
6. A further independent referee re-derived all proofs of this paper by hand, including those of Lemmas 5.2 to 5.4, and re-ran the programs of items 2–4, which reproduced their recorded outputs. Before reading any of them, it wrote its own exact program, `r2_check.py`, which uses the standard library only. This program checks the following.
 - Theorem 1.2(a),(b) for $m \leq 16$, and an exact common zero in each of the 2^m sign classes for $m \leq 10$.
 - Lemma 3.1(a),(b) on 250 random instances. For $2 \leq d \leq 5$ and $2 \leq k \leq 5$ with $dk + 1 \leq 21$, the exact degrees $(d, 2d)$ of Corollary 3.3, and an exact common zero in each of the $2^{(d-1)k}$ sign classes whenever $2^{(d-1)k} \leq 4096$.
 - Remark 3.2 for $m \leq 12$, and the Newton-identity computation of Remark 3.4 for $k \leq 11$.
 - The box criterion in the proof of Proposition 4.1(a) for $m \leq 4$, the rescaling step of Proposition 4.1(b) for $m \leq 5$, and Lemma 5.2 on 276 random instances.

Its second program, `r2_numeric.py`, repeats the least-squares refutation attempt for $2 \leq m \leq 5$: every computed zero satisfied $x_i y_i = 1$ and $x_0 = m$ up to 10^{-5} , and all 2^m sign patterns occurred. For $m = 1, 2$ it also approximates the set of points of $[-3, 3]^{2m}$ with $x_i y_i = 1$ for all i by grid cells and finds 2 and 4 clusters of cells. This program gives numerical evidence only.

The proofs in Sections 4 and 5 do not use computation.

Scope and priority. Theorem 1.2 and Proposition 4.1 answer [BP23, Section 4, Problem 2] and the problem of [ORSS25, pp. 445–446] in the negative for every degree bound $d \geq 4$: Theorem 1.2 for zero sets in $\mathbb{R}^n$, and Proposition 4.1 inside boxes and inside any sets with nonempty interior. For $d = 1$ a bound exists trivially. For $d \in \{2, 3\}$ the question remains open; Proposition 5.5 excludes only the particular mechanism used here. The construction is elementary and follows the pattern of [BP23, Example 2.2], and it may be known to specialists. In September 2026 we searched arXiv, with abstract and title searches for multi-affine polynomials and the recent papers of both authors of [BP23]; these searches were repeated on September 30, 2026. We also searched Crossref, zbMATH, OpenAlex, Semantic Scholar and the web. We found no resolution of the question and no publication of the construction. Semantic Scholar lists one work citing [BP23], and it is unrelated. zbMATH lists no document citing [BP23], and the Crossref and OpenAlex records of [BP23] report no citations. The OpenAlex list of citing works could not be retrieved, because anonymous access to it was rate-limited during all our searches, most recently on September 30, 2026. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the construction and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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