The Critical Window for High-Dimensional Limits of Hyperbolic Poisson k-Plane Processes

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Abstract

For $2k > d+1$, the centred and normalized total k-volume of a stationary Poisson process of k-planes in hyperbolic space $\mathbb{H}^d$, observed in a growing ball, converges to a non-Gaussian infinitely divisible law $Z_{d,k}$. Bühler and Hug asked how the standardized law $Z^*_{d,k} = Z_{d,k}/(\operatorname{Var} Z_{d,k})^{1/2}$ behaves as $d \to \infty$. Bühler, Hug and Thäle proved that, when $k/d \to 1/2$, $Z^*_{d,k}$ tends to the standard Gaussian law if $d^{-1}(2k-d-1)^{d/k}$ stays asymptotically below $e\pi$ and to 0 if it stays above $e\pi$, and they left the critical rate open. We complete the picture. Put $m = 2k-d-1$, $m_{\mathrm{c}} = \sqrt{2\pi e(k-1)}$ and $y = (m-m_{\mathrm{c}})/\sqrt{m_{\mathrm{c}}}$. Along every sequence of admissible pairs with $d \to \infty$, the Lévy distance between $Z^*_{d,k}$ and the centred Gaussian law with variance $\Phi(-y/\sqrt{2})$ tends to 0, where Φ is the standard normal distribution function. Hence $Z^*_{d,k}$ converges in distribution if and only if y converges in $[-\infty,\infty]$, every limit law is a centred, possibly degenerate, Gaussian $\mathcal{N}(0,\tau^2)$ with $\tau^2 \in [0,1]$ (where $\mathcal{N}(0,0) = \delta_0$), and every $\tau^2 \in [0,1]$ occurs. At the critical rate $k = d/2 + \frac{1}{2}\sqrt{e\pi d} + O(1)$ the limit is $\mathcal{N}(0,1/2)$; a shift by $c\,d^{1/4}$ gives $\mathcal{N}(0,\Phi(-\sqrt{2}\,c/(e\pi)^{1/4}))$. The proof combines an exact Beta representation of the Kolmogorov measure, a reduction lemma for Kolmogorov measures that split between 0 and ∞, an anti-concentration bound, and a central limit theorem for the logarithm of a Beta variable. This is an unrefereed note.

1 Introduction

Let $d \geq 4$ and $1 \leq k \leq d-1$, and let $\eta_{d,k}$ be an isometry-invariant Poisson process of k-planes in the d-dimensional hyperbolic space $\mathbb{H}^d$ of curvature -1. Let $F_{d,k,R}$ be the total k-volume of the planes of $\eta_{d,k}$ inside a geodesic ball of radius R. If $2k \leq d+1$ the fluctuations of $F_{d,k,R}$ are asymptotically Gaussian as $R \to \infty$, but if $2k > d+1$ they are not [HHT21, BHT23, BH25]. In the latter case $F_{d,k,R}$, centred and normalized, converges in distribution to a centred infinitely divisible random variable $Z_{d,k}$ with characteristic function

$$\mathbb{E}e^{itZ_{d,k}} = \exp \int_{\mathbb{R}} \left(e^{itx} - 1 - itx\right) \nu_{d,k}(dx), \tag{1}$$

where

$$\nu_{d,k}(dx) = \frac{\omega_{d-k}}{k-1}\, x^{-1-\frac{d-1}{k-1}} \left(1 - x^{\frac{2}{k-1}}\right)^{\frac{d-k}{2}-1} \mathbf{1}_{(0,1)}(x)\, dx, \qquad \omega_j = \frac{2\pi^{j/2}}{\Gamma(j/2)}, \tag{2}$$

see [BHT26, (1.1)–(1.2)] and [BHT23, BH25]; for hyperplanes ($k = d-1$) see also [KRT25]. In [BH25, Section 6] the same law is written with the parametrization $x = \cosh^{1-k} s$. We call (d,k) *admissible* if d,k are integers with $1 \leq k \leq d-1$ and $2k > d+1$; then $d \geq 4$ and $k \geq 3$.

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Put $\sigma_{d,k}^2 = \operatorname{Var} Z_{d,k}$ and $Z_{d,k}^* = Z_{d,k}/\sigma_{d,k}$. In [BH25] and in the Oberwolfach report [Bü24] the limit of the intersection functionals is a positive constant multiple $Y_{d,k}$ of $Z_{d,k}$, so the standardized law $Y_{d,k}^*$ of [Bü24] is $Z_{d,k}^*$. Numerically computed densities suggested that $Z_{d,k}^*$ is close to $\mathcal{N}(0,1)$ for $k \approx d/2$, although all cumulants of order at least 3 diverge as $d \to \infty$ [BH25, Section 6]. This led to the following question, posed at the end of [BH25, Section 6] and in [Bü24, p. 3359]; it is record OWR-14298808-012 of the UnsolvedMath dataset [ula26].

Problem 1.1 (Bühler–Hug)**.** *Let (d_j, k_j) be admissible with $d_j \to \infty$. What is the asymptotic behaviour of Y_{d_j,k_j}^*? Are there conditions on d_j and k_j (such as $k_j/d_j \to 1/2$) which ensure convergence in distribution, and is the limit then Gaussian?*

Most of this question was answered by Bühler, Hug and Thäle [BHT26]. Write $m = 2k - d - 1$ (their r_n).

Theorem 1.2 ([BHT26, Theorem 1.1])**.** *Let (d_j, k_j) be admissible with $d_j \to \infty$.*

(a) If $k_j/d_j \to 1/2$ and $\limsup_j d_j^{-1} m_j^{d_j/k_j} < e\pi$, then $Z_{d_j,k_j}^* \to \mathcal{N}(0,1)$ in distribution.

(b) If $k_j/d_j \to 1/2$ and $\liminf_j d_j^{-1} m_j^{d_j/k_j} > e\pi$, or if $\liminf_j k_j/d_j > 1/2$, then $Z_{d_j,k_j}^* \to 0$ in distribution.

For $k = d/2 + \gamma\sqrt{d} + O(1)$ one has $d^{-1}m^{d/k} \to 4\gamma^2$, and [BHT26, Remark 1.2] records that at the critical growth rate $2\gamma = \sqrt{e\pi}$ the asymptotic behaviour of $Z_{d,k}^*$ remains open. We settle this case, and more generally describe the behaviour of $Z_{d,k}^*$ along every admissible sequence. Let Φ be the standard normal distribution function, extended by $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$, let $\mathcal{N}(0,0) = \delta_0$, and let d_{L} denote the Lévy distance between distribution functions on $\mathbb{R}$. Put

$$m_{\mathrm{c}} = \sqrt{2\pi e(k-1)}, \qquad y = \frac{m - m_{\mathrm{c}}}{\sqrt{m_{\mathrm{c}}}}, \qquad \rho = \frac{m^2}{m_{\mathrm{c}}^2}. \tag{3}$$

Theorem 1.3. *Let (d_j, k_j) be admissible with $d_j \to \infty$, let m_j, y_j, ρ_j be as above, and put $\tau_j^2 = \Phi(-y_j/\sqrt{2})$.*

(a) $d_{\mathrm{L}}\big(Z_{d_j,k_j}^*, \mathcal{N}(0,\tau_j^2)\big) \to 0$. More precisely, there are explicit numbers $p_j \in [0,1]$, given in (4), such that $p_j - \tau_j^2 \to 0$ and $\sup_{|t|\le T} \big|\mathbb{E}e^{itZ_{d_j,k_j}^*} - e^{-p_j t^2/2}\big| \to 0$ for every $T > 0$.

(b) Z_{d_j,k_j}^* converges in distribution if and only if y_j converges in $[-\infty,\infty]$. If $y_j \to y$, the limit is $\mathcal{N}(0, \Phi(-y/\sqrt{2}))$.

(c) Every subsequential limit in distribution of (Z_{d_j,k_j}^*) is $\mathcal{N}(0,\tau^2)$ for some $\tau^2 \in [0,1]$, and every $\tau^2 \in [0,1]$ is the limit along some admissible sequence.

Corollary 1.4. *Let (d_j, k_j) be admissible with $d_j \to \infty$.*

(a) If $\limsup \rho_j < 1$ then $Z_{d_j,k_j}^* \to \mathcal{N}(0,1)$; if $\liminf \rho_j > 1$ then $Z_{d_j,k_j}^* \to 0$. These two statements imply Theorem 1.2.

(b) Let $y_j' = (m_j - \sqrt{e\pi d_j})/(e\pi d_j)^{1/4}$. Then y_j converges in $[-\infty,\infty]$ if and only if y_j' does, with the same limit.

(c) Let $c \in \mathbb{R}$ and

$$k_j = \frac{d_j}{2} + \frac{1}{2}\sqrt{e\pi d_j} + c\, d_j^{1/4} + O(1).$$

Then Z_{d_j,k_j}^* converges to $\mathcal{N}\big(0, \Phi(-\sqrt{2}\,c/(e\pi)^{1/4})\big)$. In particular, for $c = 0$, which is the critical rate of [BHT26, Remark 1.2], the limit is $\mathcal{N}(0,1/2)$.

So the answer to Problem 1.1 is as follows. The law of $Y^*_{d,k}$ is asymptotically centred Gaussian with variance $\Phi(-y/\sqrt{2})$, in the Lévy metric and along every sequence. Convergence holds exactly when y_j converges, and every limit is Gaussian, but the limit variance can be any number in $[0,1]$, although $\operatorname{Var} Y^*_{d,k} = 1$. The phase transition of [BHT26] at $m \approx \sqrt{e\pi d}$ has a window of width $d^{1/4}$. The condition $k_j/d_j \to 1/2$ is necessary for a non-degenerate limit (by Theorem 1.2(b)) but not sufficient for convergence: a sequence that alternates between $m_j = 1$ and $m_j \approx d_j^{0.6}$ has $k_j/d_j \to 1/2$ and two different subsequential limits, $\mathcal{N}(0,1)$ and 0; this already follows from Theorem 1.2(a),(b).

Remark 1.5 (Cumulants). By Lemma 2.1 below, $\operatorname{cum}_n(Z^*_{d,k}) = \mathbb{E}X^{n-2}$ for $n \geq 2$, where the law of X is the normalized Kolmogorov measure $x^2\nu^*(dx)$ of $Z^*_{d,k}$. The part $\mathbb{E}[X^{n-2}; X \leq 1]$ is at most 1, so the divergence of the cumulants found in [BH25, Section 6] comes from the event $\{X > 1\}$. By Lemma 4.3 below, the Kolmogorov mass of $(1,\infty)$ escapes to infinity. Since $(e^{itx} - 1 - itx)/x^2 \to 0$ as $x \to \infty$, this mass does not contribute to the limit, and the variance $1 - p_j$ that it carries is lost.

Relation to [BHT26]. The Beta and incomplete Beta identities of Section 2 are those used in [BHT26, Section 4]: p_j equals $1 - J(d_j, k_j, 1)$ in the notation of [BHT26, (4.1)]. There, the conclusion rests on [BHT26, Lemma 2.1, Corollary 4.1], which apply when the Kolmogorov measure concentrates entirely near 0 or entirely near ∞. The new ingredients here are a reduction lemma for measures that split between 0 and ∞ (Lemma 4.1), an anti-concentration bound showing that the Kolmogorov measures always split in this way (Lemma 4.3), and an expansion of the resulting threshold (Lemmas 5.1 and 5.2). We also include a self-contained central limit theorem for log-Beta variables with explicit cumulant bounds (Lemma 3.1); the asymptotic normality of the logarithm of a Beta(b,a) variable as $a, b \to \infty$ is classical, and only the short proof with explicit constants is our own.

2 The Beta representation

For admissible (d,k) put

$$a = \frac{d-k}{2}, \qquad b = \frac{2k-d-1}{2} = \frac{m}{2}, \qquad N = a + b = \frac{k-1}{2}.$$

Then $a, b \in \frac{1}{2}\mathbb{Z}$, $a, b \geq \frac{1}{2}$ and $N \geq 1$. Throughout, W denotes a random variable with the Beta(b,a) distribution, with density $w^{b-1}(1-w)^{a-1}/B(b,a)$ on $(0,1)$, and $I_x(b,a) = \mathbb{P}(W \leq x)$. Note that $\rho = b^2/(\pi e N)$ and $y = \sqrt{m_{\mathrm{c}}}\,(\sqrt{\rho} - 1)$.

Lemma 2.1. *Let (d,k) be admissible and $\sigma = \sigma_{d,k}$.*

(i) $\sigma^2 = \int x^2 \nu_{d,k}(dx) = \pi^a \Gamma(b)/\Gamma(N)$.

(ii) Let $X = W^N/\sigma$. *Then* $\mathbb{E}e^{itZ^*_{d,k}} = \exp \mathbb{E} g_t(X)$ *for* $t \in \mathbb{R}$, *where* $g_t(x) = (e^{itx} - 1 - itx)/x^2$ *for* $x > 0$.

(iii) For $n \geq 2$, $\operatorname{cum}_n(Z^*_{d,k}) = \mathbb{E}X^{n-2} = \sigma^{2-n}\,\Gamma(b + (n-2)N)\,\Gamma(N)/\big(\Gamma(b)\,\Gamma((n-1)N)\big)$.

(iv) We have

$$\mathbb{P}(X \leq 1) = I_{\sigma^{1/N}}(b,a), \tag{4}$$

and we write $p = p_{d,k}$ *for this number.*

Proof. Substitute $x = u^N$ in (2). Since $x^{2-1-\frac{d-1}{k-1}}\,dx = N\,u^{b-1}du$ and $\omega_{d-k}N/(k-1) = \omega_{d-k}/2 = \pi^a/\Gamma(a)$, the measure $x^2\nu_{d,k}(dx)$ is the image under $u \mapsto u^N$ of $\pi^a\Gamma(a)^{-1}u^{b-1}(1-u)^{a-1}du$ on $(0,1)$. Its total mass is $\pi^a B(b,a)/\Gamma(a) = \pi^a\Gamma(b)/\Gamma(N)$, which is $\operatorname{Var} Z_{d,k}$ by (1). This proves (i), and shows that

$x^2\nu_{d,k}(dx)/\sigma^2$ is the law of W^N. The Lévy measure of $Z^*_{d,k}$ is $\nu^*(B) = \nu_{d,k}(\sigma B)$, so $x^2\nu^*(dx)$ is the law of $X = W^N/\sigma$, and (ii) follows from (1). Differentiating (1) gives $\mathrm{cum}_n(Z^*_{d,k}) = \int x^n \nu^*(dx) = \mathbb{E}X^{n-2}$, and $\mathbb{E}W^s = B(b+s,a)/B(b,a)$ gives (iii). Finally $X \le 1$ if and only if $W \le \sigma^{1/N}$. □

Formula (i) is [BHT26, (4.3)], and (iii) agrees with [BH25, (6.2)]; (iv) is $1 - J(d,k,1)$ in [BHT26, (4.1)]. The law of X is the normalized Kolmogorov measure of $Z^*_{d,k}$; we denote it by $K = K_{d,k}$.

3 Gamma function facts and log-Beta variables

We use the following classical facts [WW27, §§12.3–12.33]. Let $\psi = \Gamma'/\Gamma$.

(G1) (Gauss) For $z > 0$, $\psi(z) = \int_0^\infty \big(e^{-t}/t - e^{-zt}/(1-e^{-t})\big)dt$; hence $\psi'(z) = \int_0^\infty te^{-zt}(1-e^{-t})^{-1}dt$ and $\psi'''(z) = \int_0^\infty t^3 e^{-zt}(1-e^{-t})^{-1}dt$.

(G2) (Binet, Stirling) For $x > 0$, $\log\Gamma(x) = (x - \frac{1}{2})\log x - x + \frac{1}{2}\log 2\pi + r_2(x)$ and $\psi(x) = \log x - \frac{1}{2x} - r_1(x)$, with $0 < r_2(x) < \frac{1}{12x}$ and $0 < r_1(x) < \frac{1}{12x^2}$.

(G3) $\log\Gamma$ is convex and ψ' is decreasing; by (G2), $\log x - \frac{1}{x} < \psi(x) < \log x$ for $x \ge \frac{1}{2}$; by (G1) and $1 \le t/(1-e^{-t}) \le 1+t$, $\psi'(x) \le \frac{1}{x} + \frac{1}{x^2}$.

Lemma 3.1. *Let $a, b > 0$, $N = a+b$, $W \sim \mathrm{Beta}(b,a)$ and $L = -\log W$.*

(a) For $\mathrm{Re}\, s > -b$,

$$\mathbb{E}e^{-sL} = \frac{\Gamma(b+s)\Gamma(N)}{\Gamma(b)\Gamma(N+s)} = \exp\int_0^\infty (e^{-st}-1)\,\lambda(dt), \qquad \lambda(dt) = \frac{e^{-bt} - e^{-Nt}}{t\,(1-e^{-t})}\,dt.$$

Hence the cumulants of L are $\kappa_n = \int_0^\infty t^n\lambda(dt)$; in particular $\kappa_1 = \psi(N) - \psi(b)$, $\kappa_2 = \psi'(b) - \psi'(N)$ and $\kappa_4 = \psi'''(b) - \psi'''(N)$.

(b) $\dfrac{a}{bN} \le \kappa_2 \le \dfrac{a}{bN}\Big(1 + \dfrac{2}{b}\Big)$ *and* $\dfrac{\kappa_4}{\kappa_2^2} \le \dfrac{6}{a} + \dfrac{2}{b} + \dfrac{24}{b^2} + \dfrac{24}{ab}$.

(c) If $a_j, b_j \to \infty$, then $\xi_j = (L_j - \kappa_{1,j})/\sqrt{\kappa_{2,j}}$ satisfies $\sup_{x\in\mathbb{R}} |\mathbb{P}(\xi_j \ge x) - \Phi(-x)| \to 0$.

(d) If $a \ge 1$, then $\log W$ has a bounded log-concave density.

Proof. (a) The first equality is the Beta integral. For real $u > -b$, (G1) gives $\psi(b+u) - \psi(N+u) = -\int_0^\infty (e^{-bt} - e^{-Nt})e^{-ut}(1-e^{-t})^{-1}dt$. Integrating over u between 0 and a real $s > -b$ (the integrand has constant sign, so Fubini applies) gives $\log\Gamma(b+s) - \log\Gamma(b) - \log\Gamma(N+s) + \log\Gamma(N) = \int_0^\infty (e^{-st} - 1)\lambda(dt)$. Near $t = 0$ the density of λ is $O(1/t)$, and $e^{-st} - 1 = O(t)$; near $t = \infty$ it is $O(e^{-bt}/t)$. So both sides are analytic in $\mathrm{Re}\, s > -b$, and the identity extends there. Differentiating at $s = 0$ gives the cumulants; the formulas for $\kappa_1, \kappa_2, \kappa_4$ follow from (G1).

(b) Since $1 \le t/(1-e^{-t}) \le 1+t$ and $e^{-bt} - e^{-Nt} \ge 0$,

$$\frac{1}{b} - \frac{1}{N} \le \kappa_2 \le \frac{1}{b} - \frac{1}{N} + \frac{1}{b^2} - \frac{1}{N^2}, \qquad \kappa_4 \le 2\Big(\frac{1}{b^3} - \frac{1}{N^3}\Big) + 6\Big(\frac{1}{b^4} - \frac{1}{N^4}\Big).$$

Here $\frac{1}{b} - \frac{1}{N} = \frac{a}{bN}$ and $\frac{1}{b^2} - \frac{1}{N^2} = \frac{a}{bN}(\frac{1}{b} + \frac{1}{N}) \le \frac{a}{bN}\cdot\frac{2}{b}$. Dividing by $\kappa_2^2 \ge a^2/(bN)^2$ and using $N = a+b$,

$$\frac{2(b^{-3} - N^{-3})}{a^2/(bN)^2} = \frac{2(N^2 + Nb + b^2)}{abN} = 2\Big(\frac{1}{a} + \frac{1}{b} + \frac{1}{a} + \frac{b}{aN}\Big) \le \frac{6}{a} + \frac{2}{b},$$

$$\frac{6(b^{-4}-N^{-4})}{a^2/(bN)^2} = \frac{6(N+b)(N^2+b^2)}{ab^2N^2} \le \frac{24N}{ab^2} = \frac{24}{b^2} + \frac{24}{ab}.$$

(c) Put $s = -iu/\sqrt{\kappa_2}$ in (a) and subtract the mean: $\log \mathbb{E}e^{iu\xi} = \int_0^\infty (e^{iv} - 1 - iv)\lambda(dt)$ with $v = ut/\sqrt{\kappa_2}$. Since $|e^{iv}-1-iv+v^2/2| \le |v|^3/6$ and, by the Cauchy–Schwarz inequality, $\kappa_3 \le (\kappa_2\kappa_4)^{1/2}$,

$$\Big|\log \mathbb{E}e^{iu\xi} + \frac{u^2}{2}\Big| \le \frac{|u|^3\kappa_3}{6\kappa_2^{3/2}} \le \frac{|u|^3}{6}\Big(\frac{\kappa_4}{\kappa_2^2}\Big)^{1/2}.$$

By (b) the right-hand side tends to 0 when $a_j, b_j \to \infty$, so $\xi_j \to \mathcal{N}(0,1)$ by Lévy's continuity theorem [Bil95]. Since Φ is continuous, the convergence of distribution functions is uniform (Pólya's theorem), also for left limits, which gives the claim.

(d) The density of $U = \log W$ is $e^{bu}(1-e^u)^{a-1}/B(b,a)$ for $u<0$. It is bounded by $1/B(b,a)$ when $a \ge 1$, and it is log-concave because $\frac{d^2}{du^2}\log\big(e^{bu}(1-e^u)^{a-1}\big) = -(a-1)e^u(1-e^u)^{-2} \le 0$ for $a \ge 1$. $\square$

4 The Kolmogorov measures

The following reduction lemma extends [BHT26, Lemma 2.1], which treats the cases $p_j \to 1$ and $p_j \to 0$. It could also be derived from the classical convergence criteria for infinitely divisible laws [GK54]; we give the short direct proof.

Lemma 4.1. *Let K_j be probability measures on $(0,\infty)$ such that $K_j([\varepsilon, M]) \to 0$ for all $0 < \varepsilon < M < \infty$, and put $p_j = K_j((0,1])$ and $\varphi_j(t) = \exp\int g_t\, dK_j$. Then $\sup_{|t|\le T} |\varphi_j(t) - e^{-p_jt^2/2}| \to 0$ for every $T > 0$.*

Proof. Let $|t| \le T$ and $0 < \varepsilon < 1 < M$. Write $\int g_t\, dK_j + \frac{t^2}{2}p_j$ as the sum of the integrals of $g_t + \frac{t^2}{2}$ over $(0,\varepsilon)$, of $g_t + \frac{t^2}{2}\mathbf{1}_{(0,1]}$ over $[\varepsilon, M]$, and of g_t over (M,∞). Since $|e^{iv} - 1 - iv + v^2/2| \le |v|^3/6$, the first is at most $T^3\varepsilon/6$ in modulus. Since $|g_t| \le t^2/2$, the second is at most $T^2K_j([\varepsilon,M])$. Since $|g_t(x)| \le (2+|t|x)/x^2$, the third is at most $2/M^2 + T/M$. Letting $j\to\infty$, then $\varepsilon \to 0$ and $M \to \infty$, we get $\sup_{|t|\le T} |\int g_t\, dK_j + p_jt^2/2| \to 0$. Both exponents have non-positive real part, since $\operatorname{Re} g_t \le 0$, and $|e^z - e^w| \le |z-w|$ when $\operatorname{Re} z, \operatorname{Re} w \le 0$. $\square$

Lemma 4.2. *Let V have a bounded log-concave density f and standard deviation $s \in (0,\infty)$. Then $\sup f \le 2e^2/s$.*

The sharp bound is $\sup f \le 1/s$, with equality for the exponential law; see [BC15, Proposition 2.1] and [BMM22, (6.1)]. We only need some bound of order $1/s$ and include a short proof of the weaker one.

Proof. Let $S = \sup f$ and fix x_0 with $M_0 = f(x_0) > 0$. The set $I = \{f \ge M_0/e\}$ is an interval containing x_0, and its length ℓ satisfies $\ell M_0/e \le \int_I f \le 1$. Let $x > x_+ = \sup I$ with $f(x) > 0$. For $z \in (x_+, x)$, concavity of $\log f$ on $[x_0, x]$ and $f(z) < M_0/e$ give $\log f(x) \le \log M_0 + \frac{x-x_0}{z-x_0}(\log f(z) - \log M_0) \le \log M_0 - \frac{x-x_0}{z-x_0}$. Letting $z \downarrow x_+$ and using $x_+ - x_0 \le \ell$ yields $f(x) \le M_0e^{-(x-x_0)/\ell}$. The same holds to the left of I, and $f \le S$ on I. Hence $f(x) \le Se^{1-|x-x_0|/\ell}$ for all x, and $s^2 \le \mathbb{E}(V-x_0)^2 \le Se\int_{\mathbb{R}} u^2e^{-|u|/\ell}du = 4eS\ell^3 \le 4e^4S/M_0^3$. Letting $M_0 \to S$ gives $s^2 \le 4e^4/S^2$. $\square$

Lemma 4.3 (Anti-concentration)**.** *Let (d_j, k_j) be admissible with $d_j \to \infty$, and let $K_j = K_{d_j,k_j}$. Then $K_j([\varepsilon, M]) \to 0$ for all $0 < \varepsilon < M < \infty$. If moreover (a_j) is bounded, then $K_j((0,M]) \to 0$ for every M; in particular $p_j \to 0$.*

Proof. It suffices to consider subsequences along which either $a_j \to \infty$ or (a_j) is bounded.

Case $a_j \to \infty$. Then $a_j \geq 1$ eventually. By Lemma 3.1(d) and Lemma 4.2, $N \log W$ has a density bounded by $2e^2/F$, where $F = N\sqrt{\kappa_2}$ is its standard deviation. By Lemma 3.1(b), $F^2 \geq Na/b \geq a$. Hence $K([\varepsilon, M]) = \mathbb{P}\big(N \log W \in [\log(\varepsilon\sigma), \log(M\sigma)]\big) \leq 2e^2 \log(M/\varepsilon)/\sqrt{a} \to 0$.

Case (a_j) bounded. Then $b = N - a \to \infty$, since $N = (k-1)/2 \to \infty$. By (G3), $\mathbb{E}[-N \log W] = N(\psi(N) - \psi(b)) \leq Na\,\psi'(b) \leq Na(\frac{1}{b} + \frac{1}{b^2})$, which is bounded. By convexity of $\log \Gamma$ and (G3),

$$-\log\sigma = \tfrac{1}{2}\big(\log\Gamma(N) - \log\Gamma(b) - a\log\pi\big) \geq \tfrac{a}{2}\big(\psi(b) - \log\pi\big) \geq \tfrac{a}{2}\big(\log(b/\pi) - \tfrac{1}{b}\big) \to \infty,$$

since $a \geq \frac{1}{2}$. By Markov's inequality, $K((0, M]) = \mathbb{P}(-N \log W \geq -\log\sigma - \log M) \to 0$. $\square$

Lemma 4.4. *If $N \geq \pi$, then $1 - p \leq \sqrt{e\rho}$.*

Proof. By Markov's inequality, $1 - p = \mathbb{P}(W > \sigma^{1/N}) \leq \mathbb{E}W \cdot \sigma^{-1/N}$ and $\mathbb{E}W = b/N$. By convexity of $\log\Gamma$ and (G3), $-\log\sigma^2 = \log\Gamma(N) - \log\Gamma(b) - a\log\pi \leq a(\psi(N) - \log\pi) \leq a\log(N/\pi) \leq N\log(N/\pi)$, so $\sigma^{-1/N} \leq \sqrt{N/\pi}$ and $1 - p \leq b/\sqrt{\pi N} = \sqrt{e\rho}$. $\square$

5 The threshold

Put

$$A = N\big(\psi(b) - \psi(N)\big) - \tfrac{1}{2}\log\sigma^2, \qquad F = N\sqrt{\psi'(b) - \psi'(N)}, \qquad \theta = \frac{b}{N} \in (0, 1).$$

With $L = -\log W$ and $\xi = (L - \kappa_1)/\sqrt{\kappa_2}$ as in Lemma 3.1, we have $p = \mathbb{P}(N \log W \leq \log\sigma) = \mathbb{P}(L \geq -N^{-1}\log\sigma)$, and multiplying through by N shows

$$p = \mathbb{P}(\xi \geq A/F) \qquad \text{exactly.} \tag{5}$$

Lemma 5.1. *$A = T + E$, where $T = \frac{a}{2}\log\rho + \big(\frac{b}{2} + \frac{1}{4}\big)\log\theta$ and $|E| \leq N/b$.*

Proof. Insert (G2) into the definitions of A and σ^2, and write $\log b = \log\theta + \log N$ and $a = N - b$. A direct computation gives

$$A = \left(N - \frac{b}{2} + \frac{1}{4}\right)\log\theta + \frac{a}{2}\big(\log N - \log(\pi e)\big) - \frac{a}{2b} - N\big(r_1(b) - r_1(N)\big) - \tfrac{1}{2}\big(r_2(b) - r_2(N)\big).$$

Since $\log\rho = 2\log\theta + \log N - \log(\pi e)$ and $a + \frac{b}{2} = N - \frac{b}{2}$, the first two terms add up to T. As $b \leq N$, $b \geq \frac{1}{2}$ and $N \geq 1$, (G2) gives $|E| \leq \frac{a}{2b} + \frac{N}{12b^2} + \frac{1}{24b} \leq \frac{N}{b}\big(\frac{1}{2} + \frac{1}{6} + \frac{1}{24}\big) < \frac{N}{b}$. $\square$

Lemma 5.2. *Let $(a_j), (b_j)$ be sequences of positive numbers tending to ∞ (with $N_j = a_j + b_j$ and all quantities defined as above).*

(i) If $\limsup \rho_j < 1$, then $A_j/F_j \to -\infty$.

(ii) If $\liminf \rho_j > 1$, then $A_j/F_j \to +\infty$.

(iii) If $\rho_j \to 1$, then $A_j/F_j = \frac{y_j}{\sqrt{2}}(1 + o(1)) + o(1)$, where $y_j = \sqrt{m_{c,j}}(\sqrt{\rho_j} - 1)$ and $m_{c,j} = 2\sqrt{\pi e N_j}$. In particular $A_j/F_j \to y/\sqrt{2}$ whenever $y_j \to y \in [-\infty, \infty]$.

Proof. By Lemma 3.1(b), $F^2 = N^2\kappa_2 \in [\frac{Na}{b}, \frac{Na}{b}(1 + \frac{2}{b})]$. Since $ab/N \geq \min(a, b)/2$, Lemma 5.1 gives $|E|/F \leq \sqrt{N/(ab)} \to 0$, and $a/F \geq \sqrt{ab/N}\,(1 + 2/b)^{-1/2} \to \infty$.

(i) Let $\rho_j \leq 1 - \eta$ eventually, with $\eta \in (0, 1)$. Since $\theta < 1$, we have $T \leq \frac{a}{2}\log(1 - \eta)$, so $A/F \leq \frac{1}{2}\log(1 - \eta) \cdot a/F + |E|/F \to -\infty$.

(ii) Let $\rho_j \geq 1+\eta$ eventually, with $\eta > 0$. Since $a = N(1-\theta)$ and $\log\rho = 2\log\theta + \log(N/(\pi e))$, we have $T = N\phi(\theta) + \frac{1}{4}\log\theta$ with

$$\phi(\theta) = \Big(1 - \frac{\theta}{2}\Big)\log\theta + \frac{1-\theta}{2}\log\frac{N}{\pi e}.$$

Here $\phi''(\theta) = -\theta^{-2} - (2\theta)^{-1} < 0$ and $\phi(1) = 0$. Put $\theta_* = \sqrt{\pi e(1+\eta)/N}$, so that $\rho \geq 1+\eta$ means $\theta \geq \theta_*$. A computation gives $\phi(\theta_*) = \frac{1}{2}(1-\frac{\theta_*}{2})\log(1+\eta) - \frac{\theta_*}{4}\log\frac{N}{\pi e}$, which tends to $\frac{1}{2}\log(1+\eta)$ because $\theta_*\log N \to 0$. Hence $\phi(\theta_*) > \frac{1}{4}\log(1+\eta) > 0$ for all large j. For such j, concavity of ϕ on $[\theta_*, 1]$, $\phi(1) = 0$ and $\phi(\theta_*) > 0$ give $\phi(\theta) \geq \phi(\theta_*)\frac{1-\theta}{1-\theta_*} \geq \phi(\theta_*)(1-\theta)$, so $N\phi(\theta) \geq \frac{a}{4}\log(1+\eta)$. Also $\frac{1}{4}\log\theta = -\frac{1}{4}\log(1+\frac{a}{b}) \geq -\frac{a}{4b}$. Since $b \to \infty$, we get $T \geq \frac{a}{8}\log(1+\eta)$ for large j, and $A/F \geq \frac{1}{8}\log(1+\eta)\cdot a/F - |E|/F \to \infty$.

(iii) Now $b = \sqrt{\rho\pi e N} \sim \sqrt{\pi e N}$, $a \sim N$ and $\theta \to 0$, with $\theta \geq 1/N$ once $b \geq 1$. The second term of T satisfies $|(\frac{b}{2}+\frac{1}{4})\log\theta|/F \leq b\log N\sqrt{b/(Na)} = O(N^{-1/4}\log N) \to 0$, and $|E|/F \to 0$. For the first term, $F = \sqrt{Na/b}\,(1+O(1/b))$ gives $\frac{a}{2F} = \frac{1}{2}\sqrt{ab/N}\,(1+o(1)) = \frac{1}{2}\sqrt{b}\,(1+o(1))$, and $\sqrt{b} = \sqrt{m_{\mathrm{c}}\sqrt{\rho}/2} = \sqrt{m_{\mathrm{c}}/2}\,(1+o(1))$ because $2b = m = m_{\mathrm{c}}\sqrt{\rho}$. Finally $\log\rho = 2\log(1+y/\sqrt{m_{\mathrm{c}}}) = \frac{2y}{\sqrt{m_{\mathrm{c}}}}(1+o(1))$, since $y/\sqrt{m_{\mathrm{c}}} = \sqrt{\rho}-1 \to 0$. Multiplying, $\frac{a}{2}\log\rho/F = \frac{y}{\sqrt{2}}(1+o(1))$. □

6 Proofs of the main results

Let (d_j, k_j) be admissible with $d_j \to \infty$. Then $k_j > (d_j+1)/2 \to \infty$, so $N_j \to \infty$ and $m_{\mathrm{c},j} \to \infty$. The quantities of (3) satisfy $m = 2b$, $m_{\mathrm{c}} = 2\sqrt{\pi e N}$, $\rho = b^2/(\pi e N)$ and $y_j = \sqrt{m_{\mathrm{c},j}}(\sqrt{\rho_j}-1)$.

Step 1: Gaussian approximation. By Lemma 2.1(ii), $\mathbb{E}e^{itZ_j^*} = \exp\int g_t\,dK_j$ with $K_j = K_{d_j,k_j}$. By Lemma 4.3 the hypothesis of Lemma 4.1 holds, and $K_j((0,1]) = p_j$. Hence $\sup_{|t|\leq T}|\mathbb{E}e^{itZ_j^*} - e^{-p_j t^2/2}| \to 0$ for every T.

Step 2: $p_j - \tau_j^2 \to 0$. It suffices to show that every subsequence has a further subsequence along which $p_j - \tau_j^2 \to 0$. Pass to a subsequence along which $\rho_j \to \rho_\infty \in [0,\infty]$, $y_j \to y \in [-\infty,\infty]$, and each of (a_j), (b_j) is bounded or tends to ∞. Then $\tau_j^2 \to \Phi(-y/\sqrt{2})$, and it remains to show $p_j \to \Phi(-y/\sqrt{2})$.

- If $\rho_\infty = 0$, then $y = -\infty$, and $1 - p_j \leq \sqrt{e\rho_j} \to 0$ by Lemma 4.4.
- If $0 < \rho_\infty < 1$, then $y = -\infty$, $b_j = \sqrt{\rho_j\pi e N_j} \to \infty$ and $a_j = N_j - b_j \to \infty$. By (5), Lemma 3.1(c) and Lemma 5.2(i), $p_j - \Phi(-A_j/F_j) \to 0$ and $\Phi(-A_j/F_j) \to 1$.
- If $\rho_\infty > 1$, then $y = +\infty$ and $b_j \geq \sqrt{\pi e N_j} \to \infty$. If (a_j) is bounded, $p_j \to 0$ by Lemma 4.3. Otherwise $a_j \to \infty$, and $p_j \to 0$ by (5), Lemma 3.1(c) and Lemma 5.2(ii).
- If $\rho_\infty = 1$, then $a_j, b_j \to \infty$, and $p_j \to \Phi(-y/\sqrt{2})$ by (5), Lemma 3.1(c) and Lemma 5.2(iii).

Step 3: the Lévy distance. Suppose that $d_{\mathrm{L}}(Z_j^*, \mathcal{N}(0,\tau_j^2)) \geq \delta > 0$ along a subsequence. Pass to a further subsequence with $\tau_j^2 \to \tau^2 \in [0,1]$. By Step 2, $p_j \to \tau^2$, so by Step 1 the characteristic functions of Z_j^* converge pointwise to $e^{-\tau^2 t^2/2}$. By Lévy's continuity theorem $Z_j^* \to \mathcal{N}(0,\tau^2)$ in distribution, and clearly $\mathcal{N}(0,\tau_j^2) \to \mathcal{N}(0,\tau^2)$. Since d_{L} metrizes convergence in distribution on $\mathbb{R}$ [Bil95], the triangle inequality gives $d_{\mathrm{L}}(Z_j^*, \mathcal{N}(0,\tau_j^2)) \to 0$, a contradiction. This proves Theorem 1.3(a).

Step 4: parts (b) and (c). If $y_j \to y$, then $\tau_j^2 \to \Phi(-y/\sqrt{2})$ and (a) gives convergence to $\mathcal{N}(0, \Phi(-y/\sqrt{2}))$. Conversely, if Z_j^* converges in distribution to some law μ, then by Step 1 $e^{-p_j/2}$ converges to the characteristic function of μ at $t = 1$, so p_j and hence $\tau_j^2 = \Phi(-y_j/\sqrt{2})$ converge. As $y \mapsto \Phi(-y/\sqrt{2})$ is a homeomorphism of $[-\infty, \infty]$ onto $[0, 1]$, y_j converges. For (c): since $\operatorname{Var} Z_j^* = 1$, the sequence is tight; along any subsequence there is a further one on which y_j converges, and then the limit is $\mathcal{N}(0, \Phi(-y/\sqrt{2}))$. Given $\tau^2 \in (0, 1)$, let $y = -\sqrt{2}\,\Phi^{-1}(\tau^2)$ and, for $k \geq 3$, let $m(k)$ be the integer nearest to $m_{\mathrm{c}} + y\sqrt{m_{\mathrm{c}}}$ (with $m_{\mathrm{c}} = \sqrt{2\pi e(k-1)}$) and $d(k) = 2k - 1 - m(k)$. For large k we have $1 \leq m(k) \leq k - 2$, so $(d(k), k)$ is admissible, and $y = y(k) + O(m_{\mathrm{c}}^{-1/2})$. The values $\tau^2 = 1$ and $\tau^2 = 0$ are attained with $m = 1$ and with $k = d - 1$, respectively. ☐

Proof of Corollary 1.4. (a) If $\rho_j \leq 1 - \eta$ eventually, then $y_j \leq (\sqrt{1-\eta} - 1)\sqrt{m_{\mathrm{c},j}} \to -\infty$; if $\rho_j \geq 1 + \eta$ eventually, then $y_j \to +\infty$. To deduce Theorem 1.2, put $q = d^{-1} m^{d/k}$ and note that $\pi e \rho = m^2/(2(k-1))$, so

$$\log \frac{q}{\pi e \rho} = \log \frac{2(k-1)}{d} - \frac{m+1}{k} \log m,$$

because $d - 2k = -(m+1)$. Assume $k_j/d_j \to 1/2$, so the first term tends to 0. Along any subsequence on which ρ_j is bounded, $m_j = O(\sqrt{k_j})$; along any subsequence on which q_j is bounded, $d_j/k_j \geq 3/2$ eventually and $m_j \leq (Cd_j)^{2/3}$. In both cases the second term tends to 0, so $q_j/(\pi e \rho_j) \to 1$. Hence $\limsup q_j < e\pi$ implies $\limsup \rho_j < 1$, and $\liminf q_j > e\pi$ implies $\liminf \rho_j > 1$ (on subsequences where $\rho_j \to \infty$ this is trivial). Finally, if $\liminf k_j/d_j > 1/2$ then $m_j \geq c d_j$ for some $c > 0$ and large j, so $\rho_j \to \infty$.

(b) Let $s = \sqrt{e\pi d}$. Since $2k - 2 = d - 1 + m$, we have $m_{\mathrm{c}}^2 = s^2 + \pi e(m-1)$, so $0 \leq m_{\mathrm{c}} - s = \pi e(m-1)/(m_{\mathrm{c}} + s)$. Pass to a subsequence on which m/s is bounded or tends to ∞. If $m = O(s)$, then $m_{\mathrm{c}} - s = O(1)$, and

$$y - y' = \frac{(m-s)(s - m_{\mathrm{c}})}{\sqrt{m_{\mathrm{c}} s}\,(\sqrt{m_{\mathrm{c}}} + \sqrt{s})} + \frac{s - m_{\mathrm{c}}}{\sqrt{m_{\mathrm{c}}}} = O(s^{-1/2}) \to 0.$$

If $m/s \to \infty$, then, since $m < d$ gives $m_{\mathrm{c}} \leq \sqrt{2}\,s$, both $y \geq (m - \sqrt{2}s)/\sqrt{m_{\mathrm{c}}}$ and $y' = (m-s)/\sqrt{s}$ tend to $+\infty$.

(c) Here $m_j = \sqrt{e\pi d_j} + 2c\, d_j^{1/4} + O(1)$, so $y'_j \to 2c/(e\pi)^{1/4}$, and Theorem 1.3(b) with part (b) gives the limit variance $\Phi(-\sqrt{2}\,c/(e\pi)^{1/4})$. In the notation of [BHT26, Remark 1.2], $k = d/2 + \gamma\sqrt{d} + \tau(d)$ with $\tau(d) \in [0, 1)$, so the critical case $2\gamma = \sqrt{e\pi}$ is $c = 0$. ☐

7 Remarks and numerical illustration

Remark 7.1 (Slow convergence). In the window, the term $(\frac{b}{2} + \frac{1}{4})\log\theta$ of Lemma 5.1 is of order $N^{-1/4}\log N$ relative to F. Keeping it suggests the heuristic second-order approximation $A/F \approx y/\sqrt{2} - (\pi e)^{3/4}\log(N/(\pi e))/(4N^{1/4})$, which we do not prove; it keeps only the leading correction and neglects terms of order $N^{-1/4}$ without the logarithm, which come from E/F and from the skewness of ξ. It explains why the convergence is slow, and why densities plotted for moderate d with $k \approx d/2$ look close to $\mathcal{N}(0, 1)$. Table 1 lists p_j from (4), which by Theorem 1.3(a) is the variance of the Gaussian law that approximates $Z^*_{d_j,k_j}$.

Remark 7.2 (Bounded codimension). If $c = d - k$ stays bounded, then $\rho \to \infty$ and $Z^*_{d,k} \to 0$, as in [BHT26, Theorem 1.1(b)], but p tends to 0 slowly, like a power of d: for $d - k = 1$ one finds $p \approx 0.11,\ 0.055,\ 0.004$ at $d = 10^3, 10^4, 10^8$. Heuristically, for fixed c the variable $-N \log W$ is close to a Gamma$(c/2, 1)$ variable and $-\log\sigma = \frac{c}{4}\log(N/\pi) + o(1)$, which suggests $p \approx Q\big(\frac{c}{2}, \frac{c}{4}\log(N/\pi)\big)$,

	10^4	10^6	10^8	10^{10}	10^{12}	10^{14}
critical rate, d = column	0.8857	0.7362	0.6113	0.5455	0.5177	0.5067
window $y \approx -1$, k = column	0.9513	0.8906	0.8265	0.7890	0.7717	0.7645
window $y \approx 0$, k = column	0.8438	0.6992	0.5929	0.5381	0.5148	0.5055
window $y \approx 1$, k = column	0.6347	0.4322	0.3182	0.2705	0.2515	0.2441

Table 1: Values of $p = I_{\sigma^{1/N}}(b, a)$ (computations, not proofs). First row: $k = \lceil d/2 + \frac{1}{2}\sqrt{e\pi d}\,\rceil$; the limit is 1/2, and $p = 0.5025$ at $d = 10^{16}$. Other rows: m = nearest integer to $m_c + y\sqrt{m_c}$ and $d = 2k - 1 - m$; the limits are $\Phi(1/\sqrt{2}) \approx 0.7602$, 1/2 and $\Phi(-1/\sqrt{2}) \approx 0.2398$.

where Q is the regularized upper incomplete Gamma function. Numerically, the relative error of this approximation is below $3 \cdot 10^{-3}$ for $1 \le c \le 4$ and $10^4 \le d \le 10^{12}$ (a computation, not a proof). Since $Q(s, x) \sim x^{s-1}e^{-x}/\Gamma(s)$ as $x \to \infty$, the approximation is of order $d^{-c/4}(\log d)^{c/2-1}$. For fixed codimension, [BHT26, Theorem 1.3] identifies a non-Gaussian limit under a different rescaling of the Lévy measure; we do not consider that normalization here.

Verification. The proofs above use only classical facts about the Gamma function, Lévy's continuity theorem and Pólya's theorem. The computations below were used as checks; none of them is part of a proof.

1. The representation of Lemma 2.1 was compared with direct numerical integration of the characteristic function and the cumulants in the parametrization of [BH25, Section 6], for 11 pairs with $d \le 60$, $t \in \{0.5, 1, 2, 4\}$ and $n \in \{3, 4, 6\}$; the maximal deviation was $8 \cdot 10^{-11}$, and the cumulants agree with [BH25, (6.2)].

2. The explicit inequalities of Lemma 3.1(b), Lemma 5.1, Lemmas 4.2 and 4.4 and the ingredients of Lemma 4.3, with the constants as stated here (including the sharp bound $\sup f \le 1/s$ for $N \log W$), were checked by independent code using only the Python standard library on 8593 admissible pairs with $d \le 6.3 \cdot 10^6$, and by a second referee with SciPy on 42,234 pairs with $d \le 6 \cdot 10^6$ (see item 5); no violation was found. An earlier SciPy check on 8336 pairs with $d \le 10^6$ used the weaker constants of a previous version ($|A - T| \le 2N/b$, and $6/b$ in place of $2/b$ in the bound for κ_4/κ_2^2); it also found no violation.

3. The values of p in Table 1 were computed with SciPy's regularized incomplete Beta function, by an independent Gamma-mixture quadrature (agreement within 10^{-7}), and by Simpson quadrature of the density of $\log W$ with the standard library (agreement to the six printed digits).

4. The characteristic function of $Z^*_{d,k}$ was computed by quadrature from Lemma 2.1(ii). On $t \in [0.25, 5]$ its distance to $e^{-pt^2/2}$ decreases from about $2 \cdot 10^{-3}$ at $k = 10^3$ to about 10^{-9} at $k = 10^{12}$ for $y \approx 0$, while its distance to the standard Gaussian characteristic function grows from 0.04 to 0.24.

5. An independent referee checked the proofs of an earlier version of this note line by line, re-ran the scripts, and checked p and the characteristic function by two further independent computations, one of them starting from (2). The present version differs by the compactness argument in Step 3 of Section 6, the explicit positivity of $\phi(\theta_*)$ in Lemma 5.2(ii) and slightly smaller constants. A second independent referee then re-checked all proofs line by line and found no error in them. With independent code, written before reading our scripts, this referee recomputed Lemma 2.1 from (2) and from the parametrization of [BH25, Section 6], the inequalities of item 2, the values in Table 1, the values and the decay rate in Remark 7.2, and item 4. All values reproduced. The

referee found that an earlier version described the decay in Remark 7.2 wrongly as logarithmic; this is corrected here.

The scripts and their outputs are included in the source files.

Scope and priority. Theorem 1.3 answers Problem 1.1 as posed in [Bü24, p. 3359] and [BH25, Section 6], for every admissible sequence with $d_j \to \infty$. The Gaussian/degenerate dichotomy away from the critical scale is due to Bühler, Hug and Thäle [BHT26, Theorem 1.1], who also use the Beta representation of Section 2; Corollary 1.4(a) recovers their theorem, and the example showing that $k_j/d_j \to 1/2$ does not suffice follows from it. What is new here is the treatment of the critical window: the reduction and anti-concentration lemmas, the fact that every limit is Gaussian, the criterion "Z^* converges if and only if y converges", and the limit $\mathcal{N}(0, 1/2)$ at the rate left open in [BHT26, Remark 1.2]. We do not prove rates of convergence, and we do not study joint limits in R and d of the prelimit variables $F_{d,k,R}$. We searched arXiv, Crossref, OpenAlex, zbMATH, Semantic Scholar and OpenCitations and made three web searches (September 2026), and found no treatment of the critical case. A second referee, who also re-checked the proofs and recomputed the numerical claims (item 5 of the Verification paragraph), repeated the search in arXiv, Crossref, OpenAlex and zbMATH and made a fourth web search; no new result was found. The Oberwolfach report [Hug25] cites [BHT26] in a survey paragraph and contains no result on it. Crossref and Scopus each report one citation of [BHT26] that we could not identify, and the journal's page on Project Euclid could not be accessed. This negative search is not a proof of priority.

AI-assisted tools were used to search the literature, to find and check the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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