

A Negative Answer to the Strong Geography Question of Alfieri and Binns

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Abstract

Alfieri and Binns say that an $\mathbb{F}[U]$ -module M satisfies the strong geography restriction if it has a direct summand $\mathbb{F}[U]/U^\ell \oplus \mathbb{F}[U]/U^{\ell-1} \oplus \cdots \oplus \mathbb{F}[U]/U$, where ℓ is the least integer with $U^\ell M_{\text{red}} = 0$. They showed that $HF^-(Y)$ satisfies it when Y is surgery on a knot in S^3 or large surgery on a link, and asked whether it holds for every rational homology sphere Y . We show that the answer is no. For the Brieskorn sphere $Y = \Sigma(30, 47, 83)$, with either orientation, the reduced Heegaard Floer homology is $\mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16} \oplus \cdots \oplus \mathcal{T}_1^{92}$, where $\mathcal{T}_k = \mathbb{F}[U]/U^k$. So $\ell = 16$, but $\mathbb{F}[U]/U^{15}$ is not a direct summand of $HF^-(Y)$. The proof combines the Ozsváth–Szabó description of HF^+ for plumbed manifolds, Némethi’s reduction to the graded root of an explicit function τ , and an exact computer calculation. We give a certificate, the 1707 turning points of τ , from which the summand lengths can be rechecked by a short program. The manifold Y is an irreducible integral homology sphere, is not an L -space, and satisfies Lin’s weaker restriction. It also shows that the word “large” cannot be removed from the link-surgery theorem of Alfieri and Binns. Computer searches with two independent programs show that among Seifert fibred integral homology spheres $\Sigma(a_1, \dots, a_n)$, the counterexamples with the smallest product $a_1 \cdots a_n$ are $\Sigma(30, 47, 83)$ and three spheres with four singular fibres, all of product 117030. This is an unrefereed note.

1 Introduction

Throughout, $\mathbb{F} = \mathbb{Z}/2$. Heegaard Floer homology [OS04a, OS04b] assigns to a rational homology sphere Y a finitely generated graded $\mathbb{F}[U]$ -module $HF^-(Y)$. Since $\mathbb{F}[U]$ is a principal ideal domain, $HF^-(Y)$ is the direct sum of a free module and of its U -torsion submodule, the reduced part $HF_{\text{red}}^-(Y)$. Lin [Lin24] proved that if Y admits a taut foliation, then $HF^-(Y)$ has a direct summand $\mathbb{F}[U]/U$. Alfieri and Binns [AB24] call this conclusion (or $HF_{\text{red}}^-(Y) = 0$) the *Lin geography restriction* and introduce a stronger one.

Definition 1.1 ([AB24, Definition 1.6], [Alf24, p. 1952]). An $\mathbb{F}[U]$ -module M satisfies the *strong geography restriction* if it contains a direct summand

$$\mathbb{F}[U]/U^\ell \oplus \mathbb{F}[U]/U^{\ell-1} \oplus \cdots \oplus \mathbb{F}[U]/U^2 \oplus \mathbb{F}[U]/U, \quad \text{where } \ell = \min\{\ell \geq 0 : U^\ell \cdot M_{\text{red}} = 0\}.$$

For $M = HF^-(Y)$ the module M_{red} is $HF_{\text{red}}^-(Y)$, the torsion submodule; it is isomorphic to $HF_{\text{red}}^+(Y)$ [AB24, Remark 2.2]. If $M_{\text{red}} = 0$, then $\ell = 0$ and the condition is empty. Alfieri and Binns proved that $HF^-(Y)$ satisfies the strong geography restriction when Y is surgery on a knot in S^3 [AB24, Theorem 1.7] and when Y is large surgery on a link in S^3 [AB24, Theorem 1.8], and

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more generally for null-homologous knots and links in L -spaces [AB24, Remark 1.9]. They asked the following question, which also appears in the Oberwolfach report [Alf24] and as record OWR-14298580-008 of the UnsolvedMath dataset [ula26].

Question 1.2 ([AB24, Question 1.10], [Alf24, p. 1953]). *Does $HF^-(Y)$ satisfy the strong geography restriction for every rational homology sphere Y ?*

Both sources remark that every rational homology sphere is integral surgery on some link in S^3 , so removing the word “large” from [AB24, Theorem 1.8] would give a positive answer. We show that the answer is negative. Write $\mathcal{T}_k = \mathbb{F}[U]/U^k$.

Theorem 1.3. *Let $Y = \Sigma(30, 47, 83)$. For both orientations of Y there are isomorphisms of $\mathbb{F}[U]$ -modules $HF^-(\pm Y) \cong \mathbb{F}[U] \oplus HF_{\text{red}}^-(\pm Y)$ and*

$$\begin{aligned} HF_{\text{red}}^-(\pm Y) \cong HF_{\text{red}}^+(\pm Y) \cong & \mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16} \oplus \mathcal{T}_{12}^{24} \oplus \mathcal{T}_{11}^{36} \oplus \mathcal{T}_{10}^{46} \oplus \mathcal{T}_9^{56} \oplus \mathcal{T}_8^{66} \\ & \oplus \mathcal{T}_7^{76} \oplus \mathcal{T}_6^{82} \oplus \mathcal{T}_5^{84} \oplus \mathcal{T}_4^{86} \oplus \mathcal{T}_3^{88} \oplus \mathcal{T}_2^{90} \oplus \mathcal{T}_1^{92}. \end{aligned}$$

Hence $\ell = 16$, and $HF^-(\pm Y)$ has no direct summand isomorphic to $\mathbb{F}[U]/U^{15}$, graded or not. In particular $HF^-(\pm Y)$ does not satisfy the strong geography restriction, and the answer to Question 1.2 is negative.

The proof is computer-assisted. It uses the theorems of Ozsváth and Szabó [OS03b] and Némethi [Né05], which reduce $HF^+(-Y)$ to the graded root of an explicit integer-valued function τ , and an exact finite computation with τ . Appendix A gives a certificate for this computation: the 1707 turning points of τ . From them the summand lengths can be recomputed without the programs used to find the example.

Remark 1.4. (a) Y is an integral homology sphere. It is irreducible, as is every Seifert fibred integral homology sphere [Hat, Proposition 1.13]; a direct argument is given in the proof in Section 4. It is not an L -space. So the answer stays negative if Question 1.2 is restricted to integral homology spheres, to irreducible manifolds, or to manifolds that are not L -spaces. $HF_{\text{red}}^-(Y)$ has 853 cyclic summands and total rank 4864, and $d(Y) = 0$.

(b) $HF^-(Y)$ has 92 summands $\mathbb{F}[U]/U$. So Y satisfies the Lin geography restriction. The example is consistent with Lin’s theorem [Lin24] and says nothing about the L -space conjecture [BGW13].

The plumbing graph of Y (Section 3) gives an explicit description of Y as integral surgery on a link of unknots in S^3 . Hence the following statement does not depend on the results of [AB24].

Corollary 1.5. *There are integral surgeries on links in S^3 , for instance $\pm Y$, whose HF^- does not satisfy the strong geography restriction. So the hypothesis “large” cannot be removed from [AB24, Theorem 1.8].*

The next statement is conditional: it depends on [AB24, Theorems 1.7 and 1.8, Remark 1.9], which are stated in an unrefereed preprint (arXiv version 1).

Corollary 1.6. *Assume [AB24, Theorems 1.7 and 1.8 and Remark 1.9]. Then neither Y nor $-Y$ is surgery on a knot in S^3 , surgery on a null-homologous knot in an L -space, or large surgery (in the sense of [AB24, Definition 3.6]) on a link in S^3 .*

We claim no novelty for Corollary 1.6: other obstructions, for example those of Hom, Karakurt and Lidman [HKL16], may already rule out some of these descriptions.

Up to orientation, the Seifert fibred integral homology spheres other than S^3 are the manifolds $\Sigma(a_1, \dots, a_n)$ with $n \geq 3$ and pairwise coprime $a_i \geq 2$. We searched them by computer.

Proposition 1.7 (computer search). *Let $2 \leq a_1 < \dots < a_n$ be pairwise coprime, $n \geq 3$, and $A = a_1 \dots a_n$. Exact computations with two independent programs give the following.*

- (a) *If $A < 117030$, then $HF^-(\Sigma(a_1, \dots, a_n))$ satisfies the strong geography restriction. For $A = 117030$ it fails exactly for $\Sigma(30, 47, 83)$, $\Sigma(2, 15, 47, 83)$, $\Sigma(3, 10, 47, 83)$ and $\Sigma(5, 6, 47, 83)$.*
- (b) *For $n = 3$ and $A \leq 200000$ (423020 spheres), $\Sigma(30, 47, 83)$ is the only failure. For $n = 4$ and $A \leq 600000$ (401831 spheres) there are exactly 12 failures, listed in Table 2. For $n = 5$ and $A \leq 250000$ (9556 spheres), $n = 6$ and $A \leq 130000$ (46 spheres), and $n = 7$ and $A \leq 600000$ (2 spheres) there are none.*

Every sphere with $n \geq 7$ has $A \geq 510510$, so (a) follows from (b). Proposition 1.7 rests on the same reduction as Theorem 1.3 and on exact integer arithmetic, but we give no certificates for the individual spheres.

Related work. Hanselman, Kutluhan and Lidman [HKL19] obtained restrictions of a different kind on the geography of Heegaard Floer homology. Alfieri and Binns credit Bodnár and Plamenevskaya [BP21] with the first verification of Lin’s restriction for a large class of graph manifolds, and recall that Rustamov [Rus03] found summands $\mathbb{F}[U]/U^2$ in $HF^-(\Sigma(2, 2n+1, 4n+3))$ for $n \geq 3$. Graded roots have also been used to study the Heegaard Floer homology of Seifert fibred homology spheres and of almost rational plumbings, for instance by Karakurt and Lidman [KL15] and by Dai and Manolescu [DM19].

2 Algebraic preliminaries

Let $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$. For an $\mathbb{F}[U]$ -module M and $j \geq 0$ put $f_M(j) = \dim_{\mathbb{F}}(\ker U \cap U^j M)$. When $f_M(0) < \infty$, put $m_k(M) = f_M(k-1) - f_M(k)$ for $k \geq 1$.

Lemma 2.1. (a) *f and m_k are additive under finite direct sums.*

- (b) *$f_{\mathbb{F}[U]} \equiv 0$ and $f_{\mathcal{T}^+} \equiv 1$; $f_{\mathcal{T}_n}(j) = 1$ for $j < n$ and $f_{\mathcal{T}_n}(j) = 0$ for $j \geq n$. So $m_k(\mathbb{F}[U]) = m_k(\mathcal{T}^+) = 0$ and $m_k(\mathcal{T}_n) = 1$ if $k = n$, 0 otherwise.*

- (c) *f_M is non-increasing, so $m_k(M) \geq 0$.*

- (d) *If $f_M(0) < \infty$ and $\mathcal{T}_k$ is isomorphic to a direct summand of M , then $m_k(M) \geq 1$.*

Proof. (a) holds because $\ker U$ and U^j respect direct sums, and (b) is a direct computation. (c) holds because $U^{j+1}M \subseteq U^jM$. For (d) write $M = N \oplus P$ with $N \cong \mathcal{T}_k$; by (a)–(c), $m_k(M) = m_k(N) + m_k(P) \geq 1$. $\square$

Lemma 2.2. *Let $M \cong \mathbb{F}[U]^r \oplus D \oplus \bigoplus_{k \geq 1} \mathcal{T}_k^{\mu_k}$, where D is a direct sum of finitely many copies of $\mathcal{T}^+$ and only finitely many μ_k are non-zero. Then $m_k(M) = \mu_k$ for all $k \geq 1$. Let ℓ be the largest k with $\mu_k > 0$ ($\ell = 0$ if there is none). If $\mu_k = 0$ for some $1 \leq k \leq \ell$, then M has no direct summand isomorphic to $\mathcal{T}_k$, and in particular no direct summand isomorphic to $\mathcal{T}_\ell \oplus \dots \oplus \mathcal{T}_1$. If $\mu_k \geq 1$ for all $k \leq \ell$, then it has such a summand.*

Proof. The first claim follows from Lemma 2.1(a),(b), the second from Lemma 2.1(d) and the fact that a summand of a summand is a summand. The last claim is clear: take one copy of each $\mathcal{T}_k$. $\square$

For $M = HF^-(Y) \cong \mathbb{F}[U]^r \oplus \bigoplus_k \mathcal{T}_k^{\mu_k}$ we have $M_{\text{red}} \cong \bigoplus_k \mathcal{T}_k^{\mu_k}$, so the ℓ of Definition 1.1 is the ℓ of Lemma 2.2. Hence $HF^-(Y)$ satisfies the strong geography restriction if and only if $m_k(HF^-(Y)) \geq 1$ for $1 \leq k \leq \ell$. The criterion does not involve the grading: if $\mu_k = 0$, there is not even an ungraded summand $\mathcal{T}_k$.

We recall the relevant parts of Heegaard Floer theory [OS04a]. Let Y be a rational homology sphere with a Spin^c structure $\mathfrak{s}$. The chain complex $CF^-(Y, \mathfrak{s})$ is a finitely generated free $\mathbb{F}[U]$ -module, $CF^\infty = CF^- \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ and $CF^+ = CF^\infty / CF^-$. (In [OS04a] CF^- is the subcomplex of CF^∞ spanned by the generators $[\mathbf{x}, i]$ with $i < 0$; this changes only the grading.) There is a U -equivariant exact triangle

$$\cdots \rightarrow HF^-(Y, \mathfrak{s}) \xrightarrow{\iota} HF^\infty(Y, \mathfrak{s}) \xrightarrow{\pi} HF^+(Y, \mathfrak{s}) \xrightarrow{\delta} HF^-(Y, \mathfrak{s}) \rightarrow \cdots,$$

where $HF^\infty(Y, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}]$ [OS04b, Theorem 10.1] (stated there over $\mathbb{Z}$; the version over $\mathbb{F}$ follows by the universal coefficient theorem). Every element of $HF^+(Y, \mathfrak{s})$ is annihilated by a power of U [OS04b, Section 2.1].

Lemma 2.3. *Let $D = \bigcap_{k \geq 0} U^k HF^+(Y, \mathfrak{s})$. Then $D \cong \mathcal{T}^+$, $HF^+(Y, \mathfrak{s}) \cong D \oplus HF^+(Y, \mathfrak{s})/D$, and δ induces an isomorphism of $\mathbb{F}[U]$ -modules $HF^+(Y, \mathfrak{s})/D \cong \text{Tor } HF^-(Y, \mathfrak{s})$. Moreover $HF^-(Y, \mathfrak{s}) \cong \mathbb{F}[U] \oplus \text{Tor } HF^-(Y, \mathfrak{s})$. Consequently $m_k(HF^+(Y, \mathfrak{s})) = m_k(HF^-(Y, \mathfrak{s})) = m_k(\text{Tor } HF^-(Y, \mathfrak{s}))$ for all $k \geq 1$.*

Proof. Since HF^∞ has no U -torsion, $\text{Tor } HF^- \subseteq \ker \iota$. Since every element of HF^+ is U -torsion and δ is U -equivariant, $\text{im } \delta \subseteq \text{Tor } HF^-$. Exactness gives $\ker \iota = \text{im } \delta$, so $\text{Tor } HF^- = \text{im } \delta \cong HF^+ / \ker \delta = HF^+ / \text{im } \pi$. The image of ι is a finitely generated graded submodule of $\mathbb{F}[U, U^{-1}]$. It is non-zero, since otherwise π would embed $\mathbb{F}[U, U^{-1}]$ into the U -torsion module HF^+ . Hence it equals $U^m \mathbb{F}[U]$ for some m , and $\text{im } \pi \cong \mathbb{F}[U, U^{-1}] / U^m \mathbb{F}[U] \cong \mathcal{T}^+$. Clearly $\text{im } \pi \subseteq D$. Conversely $HF^+ / \text{im } \pi \cong \text{Tor } HF^-$ has finite length, so $U^N HF^+ \subseteq \text{im } \pi$ for some N , and $D \subseteq \text{im } \pi$. Thus $D = \text{im } \pi \cong \mathcal{T}^+$. As $\mathcal{T}^+$ is divisible, it is an injective module over the principal ideal domain $\mathbb{F}[U]$, so D is a direct summand. Finally, HF^- is finitely generated over $\mathbb{F}[U]$, so $HF^- \cong \mathbb{F}[U]^r \oplus \text{Tor } HF^-$ for some $r \geq 0$. The localization $\mathbb{F}[U, U^{-1}]$ is a flat $\mathbb{F}[U]$ -module, so taking homology commutes with $- \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$, and

$$HF^\infty = H_*(CF^- \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]) \cong HF^- \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}] \cong \mathbb{F}[U, U^{-1}]^r,$$

because every element of $\text{Tor } HF^-$ is annihilated by a power of U , which becomes invertible. As $HF^\infty \cong \mathbb{F}[U, U^{-1}]$, tensoring with the field $\mathbb{F}(U)$ of rational functions gives $r = 1$. Hence $HF^- \cong \mathbb{F}[U] \oplus \text{Tor } HF^-$, and the statement about m_k follows from Lemma 2.1(a),(b). $\square$

So $HF_{\text{red}}^+(Y) := HF^+(Y) / \text{im } \pi$ is isomorphic to $HF_{\text{red}}^-(Y)$, as recalled in [AB24, Remark 2.2], and the answer to Question 1.2 does not depend on the flavour used.

Lemma 2.4. *$m_k(HF^-(-Y, \mathfrak{s})) = m_k(HF^-(Y, \mathfrak{s}))$ for all $k \geq 1$.*

Proof. Let $CF^-(Y, \mathfrak{s})^\vee = \bigoplus_d \text{Hom}_{\mathbb{F}}(CF_d^-(Y, \mathfrak{s}), \mathbb{F})$ be the graded dual of $CF^-(Y, \mathfrak{s})$, with the dual differential and with U acting by the dual of U . By [OS04b, Proposition 2.5] (all Spin^c structures on a rational homology sphere are torsion, and the proof there works over $\mathbb{F}$), $HF^+(-Y, \mathfrak{s})$ is isomorphic to the homology of this complex. The isomorphism comes from an identification of $CF^+(-Y, \mathfrak{s})$ with $CF^-(Y, \mathfrak{s})^\vee$ that is made degree by degree: each homogeneous piece of $CF^+(-Y, \mathfrak{s})$ is identified with $\text{Hom}_{\mathbb{F}}(CF_d^-(Y, \mathfrak{s}), \mathbb{F})$ for one degree d , and U corresponds to the dual of U (see the proof there). So the dual has to be taken in this graded sense; the full $\text{Hom}_{\mathbb{F}}(CF^-(Y, \mathfrak{s}), \mathbb{F})$ is the product of these spaces over all d . Each graded piece of $CF^-(Y, \mathfrak{s})$ is finite dimensional and $\mathbb{F}$ is a field, so $HF^+(-Y, \mathfrak{s})$ is the graded dual of $HF^-(Y, \mathfrak{s})$, with U acting by the dual map. Decompose $HF^-(Y, \mathfrak{s}) \cong \mathbb{F}[U] \oplus \bigoplus_k \mathcal{T}_k^{\mu_k}$ (Lemma 2.3) into homogeneous cyclic summands. The graded dual of $\mathbb{F}[U]$ is isomorphic to $\mathcal{T}^+$ and that of $\mathcal{T}_k$ to $\mathcal{T}_k$. Hence $m_k(HF^+(-Y, \mathfrak{s})) = \mu_k = m_k(HF^-(Y, \mathfrak{s}))$, and Lemma 2.3 for $-Y$ gives the claim. $\square$

3 From Heegaard Floer homology to the function τ

3.1 The plumbing

Let $n \geq 3$ and let $a_1, \dots, a_n \geq 2$ be pairwise coprime, $A = a_1 \cdots a_n$. The normalized Seifert invariants $(e_0; (a_l, \omega_l))$ of $\Sigma(a_1, \dots, a_n)$ are the integers with $0 < \omega_l < a_l$ and

$$e_0 A + \sum_{l=1}^n \omega_l \frac{A}{a_l} = -1, \quad \text{equivalently} \quad e = e_0 + \sum_l \frac{\omega_l}{a_l} = -\frac{1}{A},$$

as in [CK14, Section 1] and [Né05, Section 11]. The manifold is the boundary $Y(G)$ of the plumbing along the star-shaped graph G with a central vertex of weight e_0 and, for each l , an arm of vertices with weights $-k_1, \dots, -k_s$, where $a_l/\omega_l = k_1 - 1/(k_2 - 1/(\cdots - 1/k_s))$ with all $k_j \geq 2$. For $n = 3$ it is the link of the Brieskorn singularity $z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0$.

For $(a_1, a_2, a_3) = (30, 47, 83)$ we have $A = 117030$ and

$$e_0 = -1, \quad (\omega_1, \omega_2, \omega_3) = (29, 1, 1),$$

since $-117030 + 29 \cdot 3901 + 2490 + 1410 = -1$. As $30/29 = [2, 2, \dots, 2]$ (29 twos), G has 32 vertices: a centre of weight -1 , an arm of 29 vertices of weight -2 , and two single vertices of weights -47 and -83 . Its intersection form is negative definite (as $e < 0$; an exact LDL^T decomposition confirms this) and has determinant 1. The centre is the only bad vertex in the sense of [OS03b, Definition 1.1]: its weight -1 is larger than minus its degree 3, while every other vertex v has $-e_v \geq 2 \geq \deg v$. Surgery on the link of unknots associated with G , with framings the weights, gives $Y(G)$; this is the explicit surgery description used in Corollary 1.5.

3.2 The graded root

We use Némethi's graded roots [Né05, Section 3]. Let $t: \{0, 1, \dots, L\} \rightarrow \mathbb{Z}$ be a function. For $q \in \mathbb{Z}$ let $\mathcal{C}_q$ be the set of maximal intervals $I \subseteq \{0, \dots, L\}$ with $t \leq q$ on I (the connected components of the sublevel set in the path $0 - 1 - \dots - L$). The graded root (R_t, χ_t) of [Né05, Example 3.4(3)] has the elements of $\mathcal{C}_q$ as its vertices of grading q , and an edge from $I \in \mathcal{C}_q$ to the interval of $\mathcal{C}_{q+1}$ containing I . Némethi's module $\mathbb{H}(R, \chi)$ [Né05, Definition 3.5] consists of the functions ϕ from the vertices to $\mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$ with $U\phi(v) = \phi(w)$ for every edge $[v, w]$ with $\chi(v) < \chi(w)$. Némethi also allows functions $t: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}$ that are eventually non-decreasing; the values after the last decrease do not change the root [Né05, Example 3.4(3)].

Theorem 3.1 (Ozsváth–Szabó, Némethi). *Let $Y = \Sigma(a_1, \dots, a_n)$ and define $\tau: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}$ by $\tau(0) = 0$ and $\tau(i+1) = \tau(i) + \Delta(i)$, where*

$$\Delta(i) = 1 - e_0 i - \sum_{l=1}^n \left\lceil \frac{i \omega_l}{a_l} \right\rceil.$$

Then $HF_{\text{odd}}^+(-Y; \mathbb{Z}) = 0$, and $HF^+(-Y; \mathbb{Z})$ is isomorphic to $\mathbb{H}(R_\tau, \chi_\tau)$ as a $\mathbb{Z}[U]$ -module, up to a shift of the grading.

References. The graph G is negative definite with at most one bad vertex, so $HF^+(-Y(G))$ is isomorphic to the Ozsváth–Szabó module $\mathbb{H}^+(G)$ by [OS03b, Theorem 1.2] (stated over $\mathbb{Z}$; the version over $\mathbb{F}$ is [OS03b, Theorem 2.1]). A star-shaped graph is almost rational [Né05, Example 8.2(3)], and for almost rational graphs [Né05, Theorem 8.3] gives $HF_{\text{odd}}^+(-Y) = 0$ and $HF_{\text{even}}^+(-Y) \cong \mathbb{H}^+(G)$. By [Né05, Proposition 4.7] and [Né05, Theorem 9.3], $\mathbb{H}^+(G)$ is, up to a shift, the module $\mathbb{H}(R_\tau, \chi_\tau)$

of the graded root of $\tau(i) = \chi_K(x(i))$, where $x(i)$ is the generalized Laufer sequence with central coefficient i and K is the canonical class (for an integral homology sphere the only Spin^c structure is the canonical one). For Seifert manifolds and the canonical Spin^c structure, [Né05, Sections 11.12 and 11.13] give $\tau(i+1) - \tau(i) = 1 - e_0 i + \sum_l [-i\omega_l/a_l] = \Delta(i)$ and state $HF_{\text{even}}^+(-Y) \cong \mathbb{H}^+(G) \cong \mathbb{H}(R_\tau, \chi_\tau)[- (K^2 + s)/4]$, where s is the number of vertices. The same statement for Seifert homology spheres is [CK14, Theorem 2.3]. $\square$

For $Y = \Sigma(30, 47, 83)$, since $i - \lceil 29i/30 \rceil = \lfloor i/30 \rfloor$,

$$\Delta(i) = 1 + \left\lfloor \frac{i}{30} \right\rfloor - \left\lfloor \frac{i}{47} \right\rfloor - \left\lfloor \frac{i}{83} \right\rfloor. \quad (1)$$

Némethi also decomposes $\mathbb{H}(R, \chi)$ [Né05, Proposition 3.5.2]. Order the leaves (vertices of degree one) of R as $v_1, v_2, \dots$ with $\chi(v_1) \leq \chi(v_2) \leq \dots$, and let w_k ($k \geq 2$) be the lowest vertex that lies above both v_k and some v_j with $j < k$. Then

$$\mathbb{H}(R, \chi) \cong \mathcal{T}_{\mathbb{Z}}^+ \oplus \bigoplus_{k \geq 2} \mathbb{Z}[U]/U^{\chi(w_k) - \chi(v_k)} \quad (2)$$

as $\mathbb{Z}[U]$ -modules, where $\mathcal{T}_{\mathbb{Z}}^+ = \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$. In particular $HF^+(-Y; \mathbb{Z})$ is a free abelian group in each degree. The complex $CF^+(-Y; \mathbb{Z})$ is free abelian and finitely generated in each degree, so by the universal coefficient theorem the natural U -equivariant map $HF^+(-Y; \mathbb{Z}) \otimes \mathbb{F} \rightarrow HF^+(-Y; \mathbb{F})$ is an isomorphism. Therefore

$$HF^+(-Y; \mathbb{F}) \cong \mathcal{T}^+ \oplus \bigoplus_{k \geq 2} \mathcal{T}_{\chi(w_k) - \chi(v_k)}. \quad (3)$$

3.3 Reading the lengths from τ

For $t: \{0, \dots, L\} \rightarrow \mathbb{Z}$, $q \in \mathbb{Z}$ and $b \leq q$ let $N_q(b)$ be the number of intervals $I \in \mathcal{C}_q$ with $\min_I t = b$, and for $b < d$ put

$$n_t(b, d) = N_{d-1}(b) - N_d(b).$$

Lemma 3.2. *For the graded root R_t , the number of $k \geq 2$ with $\chi(v_k) = b$ and $\chi(w_k) = d$ equals $n_t(b, d)$. In particular $n_t(b, d) \geq 0$, and the multiset $\{\chi(w_k) - \chi(v_k) : k \geq 2\}$ contains $\sum_b n_t(b, b + L')$ copies of each $L' \geq 1$.*

Proof. A vertex x of grading q is an interval $I \in \mathcal{C}_q$; it lies above a leaf of grading $\leq p$ if and only if $\min_I t \leq p$. Every vertex lies above at least one leaf. For a vertex x let $k(x)$ be the least k such that x lies above v_k . Then x lies above no v_j with $j < k(x)$, so $\chi(x) < \chi(w_{k(x)})$ (with $\chi(w_1) = +\infty$). Conversely, if $\chi(v_k) \leq q < \chi(w_k)$, the vertex of grading q above v_k lies above no earlier leaf, so $k(x) = k$ for it. Vertices on the path above v_k are unique in each grading, so $x \mapsto k(x)$ is a bijection from the vertices of grading q onto $\{k : \chi(v_k) \leq q < \chi(w_k)\}$. Since the $\chi(v_k)$ are non-decreasing, x lies above a leaf of grading $\leq p$ if and only if $\chi(v_{k(x)}) \leq p$. Hence, for $p \leq q$,

$$r(p, q) := \#\{I \in \mathcal{C}_q : \min_I t \leq p\} = \#\{k : \chi(v_k) \leq p \text{ and } q < \chi(w_k)\}.$$

By inclusion and exclusion, the number of k with $\chi(v_k) = b$ and $\chi(w_k) = d$ is

$$r(b, d-1) - r(b-1, d-1) - r(b, d) + r(b-1, d).$$

Since $r(b, q) - r(b-1, q) = N_q(b)$, this equals $N_{d-1}(b) - N_d(b)$. $\square$

In the language of persistent homology, the numbers $\chi(w_k) - \chi(v_k)$ are the lengths of the finite bars of the sublevel-set filtration of t on the path, and the lemma is the elder rule. The formula $n_t(b, d) = N_{d-1}(b) - N_d(b)$ is the one used by the checker of Appendix A.

Lemma 3.3. *Let $0 < j < L$ and suppose that $t(j)$ lies between $t(j-1)$ and $t(j+1)$ (in the weak sense). Delete j and make $j-1$ and $j+1$ adjacent. Then all numbers $N_q(b)$ are unchanged.*

Proof. Fix q . If $t(j) > q$, then one of $j \pm 1$ also has $t > q$, so no two intervals of $\mathcal{C}_q$ become adjacent. If $t(j) \leq q$, then the neighbour with the smaller value lies in the same interval I as j , so $I \setminus \{j\}$ is still an interval with the same minimum. In both cases $I \mapsto I \setminus \{j\}$ is a bijection of intervals that preserves minima. $\square$

We call $i \in \{1, \dots, L-1\}$ a *turning point* of t if t changes there from strictly increasing to strictly decreasing or vice versa; a plateau at such a change is represented by its first index. The end points 0 and L are also counted as turning points. Between two consecutive turning points t is weakly monotone, so repeated use of Lemma 3.3 shows that $N_q(b)$, and hence $n_t(b, d)$, can be computed from the sequence of values of t at the turning points alone.

3.4 A finite range

Lemma 3.4. *Let $a_1, \dots, a_n$, e_0 , ω_l and Δ be as in Theorem 3.1, with $\Delta(i)$ defined by the same formula for all $i \in \mathbb{Z}$, and let $N_0 = (n-2)A - \sum_l A/a_l$. Then:*

- (a) $\Delta(i) + \Delta(N_0 - i) = 0$ for all $i \in \mathbb{Z}$;
- (b) $\Delta(i) \geq 0$ for all $i > N_0$;
- (c) if $N_0 \geq 0$, then $\tau(N_0 + 1 - i) = \tau(i)$ for $0 \leq i \leq N_0 + 1$.

For $n = 3$, part (b) and the case $0 \leq i \leq N_0$ of part (a) are contained in [CK14, Theorem 1.3].

Proof. (a) Reducing $e_0 A + \sum_l \omega_l A/a_l = -1$ modulo a_l gives $\omega_l(A/a_l) \equiv -1$, and $N_0 \equiv -A/a_l \pmod{a_l}$, so $N_0 \omega_l \equiv 1 \pmod{a_l}$. For integers y, c and $a \geq 1$ with $c \equiv 1 \pmod{a}$ one checks, writing $y = ka + \rho$ with $0 \leq \rho < a$, that $\lceil y/a \rceil + \lceil (c-y)/a \rceil = (c-1)/a + 1$. With $y = i\omega_l$ and $c = N_0\omega_l$ this gives

$$\Delta(i) + \Delta(N_0 - i) = 2 - e_0 N_0 - \sum_l \left(\frac{N_0 \omega_l - 1}{a_l} + 1 \right) = 2 - n + \frac{N_0}{A} + \sum_l \frac{1}{a_l} = 0,$$

using $e_0 + \sum_l \omega_l/a_l = -1/A$ and $N_0/A = n - 2 - \sum_l 1/a_l$.

(b) For $j < 0$, $\Delta(j) \leq 1 - e_0 j - \sum_l j \omega_l/a_l = 1 + j/A < 1$, so $\Delta(j) \leq 0$. For $i > N_0$, (a) gives $\Delta(i) = -\Delta(N_0 - i) \geq 0$.

(c) Pairing j with $N_0 - j$ in (a) gives $\tau(N_0 + 1) = \sum_{j=0}^{N_0} \Delta(j) = 0$, and then, with $j' = N_0 - j$,

$$\tau(N_0 + 1 - i) = \tau(N_0 + 1) - \sum_{j=N_0+1-i}^{N_0} \Delta(j) = \sum_{j'=0}^{i-1} \Delta(j') = \tau(i). \quad \square$$

By (b), τ is non-decreasing on $[N_0 + 1, \infty)$. So the graded root of τ is that of its restriction to $\{0, \dots, N_0 + 1\}$: for $q \geq \tau(N_0 + 1)$ the points $i > N_0 + 1$ with $\tau(i) \leq q$ join the interval containing $N_0 + 1$ without changing its minimum, and for $q < \tau(N_0 + 1)$ there are none.

4 The computation for $\Sigma(30, 47, 83)$

For $Y = \Sigma(30, 47, 83)$ we have $N_0 = 117030 - 1410 - 2490 - 3901 = 109229$, and we work with τ on $\{0, \dots, N_0 + 1\} = \{0, \dots, 109230\}$, defined by (1).

Proposition 4.1 (exact computation). *(a) On $\{0, \dots, 109230\}$, Δ takes only the values $-1, 0, 1$, and τ takes values between -12755 and 1 . The minimum -12755 is attained exactly on the four intervals $[54390, 54420]$, $[54480, 54510]$, $[54720, 54750]$ and $[54810, 54840]$.*

(b) τ has exactly 1707 turning points, listed in Appendix A.

(c) The numbers $n_\tau(b, d)$ of Lemma 3.2 have sum 853, and the multiset in which each difference $d - b$ is counted $n_\tau(b, d)$ times is (exponents denote multiplicities)

$$\{16^1, 14^{10}, 13^{16}, 12^{24}, 11^{36}, 10^{46}, 9^{56}, 8^{66}, 7^{76}, 6^{82}, 5^{84}, 4^{86}, 3^{88}, 2^{90}, 1^{92}\},$$

with sum 4864. In particular no difference equals 15, and none exceeds 16.

Proof. This is a finite computation in exact integer arithmetic. Appendix A describes how (b) and (c) can be checked from the certificate with a short program, and the paragraph “Verification” at the end lists the independent computations that agree with it. $\square$

Proof of Theorem 1.3. By Theorem 3.1 and (3), $HF^+(-Y) \cong \mathcal{T}^+ \oplus M_0$ with $M_0 = \bigoplus_{k \geq 2} \mathcal{T}_{\chi(w_k) - \chi(v_k)}$ for the graded root of τ . By Lemma 3.4 this root is the root of τ restricted to $\{0, \dots, N_0 + 1\}$, and by Lemma 3.2 and Proposition 4.1(c),

$$M_0 \cong \mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16} \oplus \dots \oplus \mathcal{T}_1^{92}.$$

Then $\bigcap_k U^k HF^+(-Y) = \mathcal{T}^+$, so $HF_{\text{red}}^+(-Y) \cong M_0$, and by Lemma 2.3 $HF_{\text{red}}^-(-Y) \cong M_0$ and $HF^-(Y) \cong \mathbb{F}[U] \oplus M_0$. By Lemma 2.4 and Lemma 2.3, $m_k(HF^-(Y)) = m_k(M_0)$ for all k . A finitely generated torsion $\mathbb{F}[U]$ -module is determined up to isomorphism by these numbers, so $HF_{\text{red}}^-(Y) \cong HF_{\text{red}}^+(Y) \cong M_0$, and $HF^-(Y) \cong \mathbb{F}[U] \oplus M_0$ by Lemma 2.3. Now $\ell = 16$ and $m_{15}(M_0) = 0$, so by Lemma 2.2 neither $HF^-(Y)$ nor $HF^-(-Y)$ satisfies the strong geography restriction. $\square$

Proof of Remark 1.4 and Corollaries 1.5 and 1.6. Every closed orientable Seifert fibred space other than $S^1 \times S^2$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$ is irreducible [Hat, Proposition 1.13]. Neither of these two is an integral homology sphere, so Y is irreducible. For Y there is also a direct argument. Since $1/30 + 1/47 + 1/83 < 1$, the base orbifold $S^2(30, 47, 83)$ of the Seifert fibration of Y is hyperbolic, and the Euler number $e = -1/117030$ is non-zero. So Y carries the geometry $\widetilde{\text{SL}}_2(\mathbb{R})$ [Sco83], and its universal cover is $\mathbb{R}^3$. By Alexander’s theorem $\mathbb{R}^3$ is irreducible, and a manifold with an irreducible covering space is irreducible [Hat, Proposition 1.6]. Y is not an L -space because $HF_{\text{red}}^-(Y) \neq 0$. By [Né05, Theorem 8.3], $d(Y) = (K^2 + s)/4 - 2 \min \tau$; here $K^2 + s = -102040$ and $\min \tau = -12755$, so $d(Y) = 0$. Corollary 1.5 follows from Theorem 1.3 and the surgery description of $Y(G)$ in Section 3. Corollary 1.6 is the contrapositive of [AB24, Theorems 1.7, 1.8 and Remark 1.9] applied to $\pm Y$. $\square$

Remark 4.2. The bottom of the graded root explains the long summands (Table 1). The four intervals where $\tau = -12755$ are separated by barriers of heights 7, 16 and 7 above the minimum. The two outer barriers give two summands $\mathcal{T}_7$, and the middle one gives the unique $\mathcal{T}_{16}$. With the tie-breaking rule of Table 1, the component born on $[54720, 54750]$ merges at level 16 with the one chosen as elder by this rule, the component born on $[54390, 54420]$. The ten summands $\mathcal{T}_{14}$ come in five symmetric pairs (Lemma 3.4(c)), born 1, 6, 10, 28 and 67 levels above the minimum, and each dies exactly 14 levels higher. No component dies exactly 15 levels after its birth.

length	index	birth	death	length	index	birth	death
16	54720	0	16	13	46770	263	276
14	50670	67	81	13	48180	177	190
14	52080	28	42	13	49260	123	136
14	53070	10	24	13	49590	108	121
14	53400	6	20	13	51000	56	69
14	54060	1	15	13	51750	36	49
14	55050	1	15	13	52410	21	34
14	55710	6	20	13	53730	3	16
14	56040	10	24	13	55380	3	16
14	57120	28	42	13	56790	21	34
14	58530	67	81	13	57450	36	49
				13	58200	56	69
				13	59610	108	121
				13	59940	123	136
				13	61020	177	190
				13	62430	263	276

Table 1: The 27 summands of length at least 13 of $HF_{\text{red}}^-(Y)$ for $Y = \Sigma(30, 47, 83)$, computed with the elder rule. When two components with the same minimum value merge, we break the tie in favour of the smaller index: the elder is the component whose minimum is attained at the smaller index. “Index” is the smallest index at which the younger component, whose bar ends at the merge, attains its minimum. “Birth” and “death” are the levels $b - \min \tau$ and $d - \min \tau$ with $\min \tau = -12755$. The tie-breaking rule only fixes the index column; the lengths do not depend on it (Lemma 3.2).

5 Further examples

Table 2 lists the failures found by the searches of Proposition 1.7 and some further Brieskorn spheres. For each of them the two independent programs give the same decomposition of HF_{red}^- ; the table shows the top terms, the missing lengths and the total rank.

The three spheres with four fibres and $A = 117030$ arise from $\Sigma(30, 47, 83)$ by splitting the fibre 30 into coprime factors $2 \cdot 15$, $3 \cdot 10$, $5 \cdot 6$. They have $e_0 = -1$ and the same invariants $(47, 1)$ and $(83, 1)$, and the same top terms $\mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16}$.

Remark 5.1 (observed, not proved). All Brieskorn spheres $\Sigma(p, q, r)$ in Table 2 satisfy $qr - pq - pr = 1$. For such spheres $e_0 = -1$ and $\omega = (p - 1, 1, 1)$, so $\Delta(i) = 1 + \lfloor i/p \rfloor - \lfloor i/q \rfloor - \lfloor i/r \rfloor$. Writing $q = p + d$ and $r = p + d'$, the condition is $dd' = p^2 + 1$; we take $1 < d < d'$ (for $d = 1$, $\Sigma(p, p + 1, p^2 + p + 1)$ is surgery on a torus knot). We computed, with both programs, the 346 members with $p \leq 230$ and $pqr \leq 4 \cdot 10^7$ (for $p \leq 130$) or $pqr \leq 1.2 \cdot 10^8$ (for $p > 130$). Of these, 47 fail the strong geography restriction: 45 of the 106 with p even, and 2 of the 240 with p odd, namely $\Sigma(187, 317, 456)$ and $\Sigma(207, 257, 1064)$. We have no proof that infinitely many members fail.

6 Concluding remarks

Theorem 1.3 answers Question 1.2 in the negative with an explicit, irreducible integral homology sphere that satisfies Lin’s restriction. The Lin geography restriction itself [AB24, Conjecture 1.5] is untouched by our examples. Two questions remain.

Question 6.1. *Are there infinitely many Brieskorn spheres (or Seifert fibred homology spheres) whose HF^- fails the strong geography restriction?*

Y	ℓ	missing	top terms of $HF_{\text{red}}^-(Y)$	rank
$\Sigma(30, 47, 83)$	16	15	$\mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16} \oplus \dots$	4864
$\Sigma(2, 15, 47, 83)$	16	15	$\mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16} \oplus \dots$	8504
$\Sigma(3, 10, 47, 83)$	16	15	$\mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16} \oplus \dots$	9154
$\Sigma(5, 6, 47, 83)$	16	15	$\mathcal{T}_{16} \oplus \mathcal{T}_{14}^{10} \oplus \mathcal{T}_{13}^{16} \oplus \dots$	9414
$\Sigma(4, 19, 31, 55)$	6	5	$\mathcal{T}_6 \oplus \mathcal{T}_4^{116} \oplus \mathcal{T}_3^{890} \oplus \dots$	10436
$\Sigma(2, 19, 55, 123)$	22	21	$\mathcal{T}_{22} \oplus \mathcal{T}_{20}^4 \oplus \mathcal{T}_{19}^{10} \oplus \dots$	18704
$\Sigma(2, 29, 65, 93)$	11	10	$\mathcal{T}_{11} \oplus \mathcal{T}_9^{70} \oplus \mathcal{T}_8^{258} \oplus \dots$	25539
$\Sigma(7, 23, 25, 97)$	11	10	$\mathcal{T}_{11} \oplus \mathcal{T}_9^{41} \oplus \mathcal{T}_8^{240} \oplus \dots$	32140
$\Sigma(2, 23, 75, 119)$	24	23	$\mathcal{T}_{24} \oplus \mathcal{T}_{22}^{10} \oplus \mathcal{T}_{21}^{16} \oplus \dots$	29892
$\Sigma(16, 21, 23, 59)$	12	11	$\mathcal{T}_{12}^2 \oplus \mathcal{T}_{10}^{25} \oplus \mathcal{T}_9^{228} \oplus \dots$	37833
$\Sigma(4, 13, 43, 225)$	7	6	$\mathcal{T}_7^2 \oplus \mathcal{T}_5^{88} \oplus \mathcal{T}_4^{607} \oplus \dots$	40476
$\Sigma(3, 37, 50, 91)$	19	18	$\mathcal{T}_{19} \oplus \mathcal{T}_{17}^{18} \oplus \mathcal{T}_{16}^{42} \oplus \dots$	39715
$\Sigma(3, 5, 77, 452)$	10	9	$\mathcal{T}_{10} \oplus \mathcal{T}_8^{182} \oplus \mathcal{T}_7^{206} \oplus \dots$	40210
$\Sigma(38, 55, 123)$	22	21	$\mathcal{T}_{22} \oplus \mathcal{T}_{20}^4 \oplus \mathcal{T}_{19}^{10} \oplus \dots$	10694
$\Sigma(46, 75, 119)$	24	23	$\mathcal{T}_{24} \oplus \mathcal{T}_{22}^{10} \oplus \mathcal{T}_{21}^{16} \oplus \dots$	17088
$\Sigma(64, 81, 305)$	48	47	$\mathcal{T}_{48} \oplus \mathcal{T}_{46}^2 \oplus \mathcal{T}_{45}^{14} \oplus \dots$	65840
$\Sigma(72, 89, 377)$	58	57	$\mathcal{T}_{58} \oplus \mathcal{T}_{56}^2 \oplus \mathcal{T}_{55}^4 \oplus \dots$	100610
$\Sigma(98, 115, 663)$	97	96	$\mathcal{T}_{97} \oplus \mathcal{T}_{95}^2 \oplus \mathcal{T}_{94}^2 \oplus \dots$	311249
$\Sigma(122, 187, 351)$	67	64, 66	$\mathcal{T}_{67} \oplus \mathcal{T}_{65}^2 \oplus \mathcal{T}_{63}^6 \oplus \dots$	333603
$\Sigma(144, 233, 377)$	76	75	$\mathcal{T}_{76} \oplus \mathcal{T}_{74}^6 \oplus \mathcal{T}_{73}^4 \oplus \dots$	526988
$\Sigma(187, 317, 456)$	95	94	$\mathcal{T}_{95}^2 \oplus \mathcal{T}_{93}^{10} \oplus \mathcal{T}_{92}^8 \oplus \dots$	1126228
$\Sigma(207, 257, 1064)$	162	161	$\mathcal{T}_{162}^4 \oplus \mathcal{T}_{160}^4 \oplus \mathcal{T}_{159}^6 \oplus \dots$	2358350

Table 2: Seifert fibred homology spheres whose HF^- fails the strong geography restriction. Above the line: all failures with $n = 3$, $A \leq 200000$ and $n = 4$, $A \leq 600000$ (Proposition 1.7). Below the line: Brieskorn spheres from the family of Remark 5.1.

Question 6.2. *For which surgeries on links, beyond the large ones of [AB24, Theorem 1.8], does the strong geography restriction hold?*

Verification. The computations were checked as follows.

1. The programs with which the example was found (Python and C) compute τ for $\Sigma(30, 47, 83)$ in three ways: from (1), from the Hilbert series of the Brieskorn complete intersection, and by Laufer's algorithm on the plumbing lattice. They compute the lengths with two barcode algorithms. A brute-force computation on the full lattice, which does not use Némethi's reduction, agrees with the τ -method on 15 small Seifert fibred spheres. A C search program gave Proposition 1.7.
2. A standalone checker (standard library only, a few seconds) recomputes τ from (1), validates the certificate of Appendix A, and computes the lengths from the 1707 certified values with the formula $n_t(b, d) = N_{d-1}(b) - N_d(b)$ of Lemma 3.2, without the elder rule.
3. A second implementation, written separately for this note, recomputes τ on $\{0, \dots, N_0 + 1\}$ from the generalized Laufer sequence on the 32-vertex plumbing lattice and compares it with (1), with Némethi's explicit cycles [Né05, Proposition 11.11] and with the semigroup description of [CK14, Theorem 1.3]. It computes the lengths by union-find and by a stack algorithm. It checks the total rank against the Casson invariant λ : Brieskorn's formula for the signature σ of the Milnor fibre [Bri66] gives $\lambda(Y) = \sigma/8 = -4864$ [NW90], and the formula $\lambda(Y) = \chi(HF_{\text{red}}^+(Y)) - d(Y)/2$ of [OS03a, Theorem 1.3] gives $-4864 - 0$, since $d(Y) = 0$ and $HF_{\text{red}}^+(Y)$ lies in odd degrees (by

[OS03b, Corollary 1.4] and [OS04b, Proposition 2.5]). Its C search program, which uses union-find on the turning points, reproduces all counts of Proposition 1.7 and all decompositions in Table 2. It also compares the lengths for the 102308 spheres $\Sigma(p, q, pqn - 1)$ with $pq(pqn - 1) \leq 10^6$, which are $1/n$ -surgeries on torus knots [Mos71], with the prediction $\mathcal{T}_{V_0}^{n-1} \oplus \bigoplus_{k \geq 1} \mathcal{T}_{V_k}^{2n}$ of the rational surgery formula [OS11] for L -space knots, with V_k as in [NW15] computed from the Alexander polynomial. All 102308 agree; the longest summand has length 125.

4. Two independent verification runs, which were AI-assisted and used their own code, checked the source statements, the proofs and the computations. The first reproduced the decomposition (853 summands, total rank 4864, exactly one summand of length at least 15), the Casson invariant, the searches of Proposition 1.7 (for $n = 4$ up to $A = 200000$), and the agreement with the $1/n$ -surgery formula for 265 spheres. The second computed τ in three ways and the lengths by three methods, one of them through the module $\mathbb{H}(R_\tau, \chi_\tau)$ of the explicit graded root over $\mathbb{F}$. It matched the published groups of small Brieskorn spheres, including the gradings, and reproduced Table 1, all counts of Proposition 1.7, all rows of Table 2, the counts of Remark 5.1 and the agreement with the $1/n$ -surgery formula for all 102308 spheres. It also reran the programs of the source archive. Its code and outputs are in the folder `reproducibility/independent_run_2` of the source archive.

Scope and priority. Theorem 1.3 answers [AB24, Question 1.10] and the question of [Alf24] as stated: the counterexample is a rational, indeed integral, homology sphere. The proof is computer-assisted. It uses [OS03b, Né05] as published theorems and an exact finite computation, for which Appendix A gives a certificate. Corollary 1.5 is unconditional. Corollary 1.6 depends on results of the unrefereed preprint [AB24]. Proposition 1.7 is a computer search result without certificates for the individual spheres, and Remark 5.1 is an observation. We read [AB24] (version 1, the only version on arXiv as of September 30, 2026) and the Oberwolfach report [Alf24]. We searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar and ran web searches in September 2026. We found no answer to the question, and the works citing [AB24] that we found do not address it. We did not see the contents of the K3 problem list (AMS Surveys 295). This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexample and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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A The certificate

The certificate is the following file in the folder `reproducibility/verifier` of the source archive; the second line is its SHA-256 checksum:

```
certificate_Sigma_30_47_83.txt
6d7130dbc0ecf948286ed5d2b7e51a2cdad322da0588a97f3c9ddf7a43f72943
```

It has three comment lines, followed by 1707 lines “ $i \ \tau(i)$ ”. The indices i are the turning points of τ on $\{0, \dots, 109230\}$ in the sense of Section 3, in increasing order, and $\tau(i)$ is defined by $\tau(0) = 0$ and (1). (The header writes the Seifert invariants as $b = -e_0 = 1$ and $\omega = (29, 1, 1)$.) The list is printed below, row by row.

A reader can check the certificate as follows. First, compute τ on $\{0, \dots, 109230\}$ from (1). Check that the listed values are correct, that the first and last indices are 0 and 109230, and that τ is weakly monotone between consecutive listed indices (in fact every listed interior index is a turning point). Second, working only with the 1707 listed values $t_0, \dots, t_{1706}$, compute for each level q from -12755 to 1 the maximal runs of consecutive $t_j \leq q$ and their minima, and hence $N_q(b)$. By Lemma 3.3, these are the numbers $N_q(b)$ of τ . The multiset of Proposition 4.1(c) is then $\{d - b \text{ with multiplicity } N_{d-1}(b) - N_d(b)\}$. The program `check_certificate.py` does exactly this; it uses only the Python standard library and runs in a few seconds. Parts (a) and (c) of Lemma 3.4 can also be spot-checked on the list: the values are symmetric under $i \mapsto 109230 - i$ up to the position of plateaus.

0	0	1	1	1380	-650	1411	-649	2460	-1136	2491	-1135	2790	-1283	2821	-1282	3870	-1756
3902	-1754	4200	-1899	4231	-1898	4950	-2219	4981	-2218	5280	-2359	5312	-2357	5610	-2498	5641	-2497
6360	-2809	6392	-2807	6690	-2945	6722	-2943	7020	-3080	7051	-3079	7440	-3249	7471	-3248	7770	-3382
7803	-3379	8100	-3514	8132	-3512	8430	-3645	8461	-3644	8850	-3809	8882	-3807	9180	-3938	9213	-3935
9510	-4066	9542	-4064	9840	-4193	9871	-4192	9930	-4226	9961	-4225	10260	-4352	10293	-4349	10590	-4477
10623	-4474	10920	-4601	10952	-4599	11250	-4724	11281	-4723	11340	-4756	11372	-4754	11670	-4878	11704	-4874
12000	-4999	12033	-4996	12330	-5119	12362	-5117	12420	-5150	12451	-5149	12660	-5238	12691	-5237	12750	-5269
12783	-5266	13080	-5387	13114	-5383	13410	-5504	13443	-5501	13740	-5620	13772	-5618	13830	-5650	13862	-5648
14070	-5735	14101	-5734	14160	-5765	14194	-5761	14490	-5879	14524	-5875	14820	-5992	14853	-5989	14910	-6021
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19140	-7380	19174	-7376	19380	-7454	19412	-7452	19470	-7480	19506	-7474	19710	-7553	19741	-7552	19800	-7579
19835	-7574	19890	-7604	19921	-7603	20130	-7677	20164	-7673	20220	-7702	20253	-7699	20460	-7774	20493	-7771
20550	-7799	20585	-7794	20790	-7870	20822	-7868	20880	-7895	20916	-7889	21120	-7965	21151	-7964	21210	-7990
21245	-7985	21300	-8014	21332	-8012	21540	-8084	21574	-8080	21630	-8108	21664	-8104	21870	-8177	21903	-8174
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24450	-8869	24486	-8863	24690	-8932	24723	-8929	24780	-8954	24817	-8947	24870	-8975	24901	-8974	25020	-9016
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