Shadowing and Generalized Hyperbolicity for Dissipative Composition Operators without Bounded Distortion

Alper Ferudun*

September 29, 2026

Abstract

Let $T_f\varphi = \varphi \circ f$ be a dissipative composition operator on $L^p(X, \mathfrak{B}, \mu)$, $1 \le p < \infty$, induced by a bimeasurable bijection f such that $\mu \circ f$ and $\mu \circ f^{-1}$ are bounded by a multiple of μ. D'Aniello, Darji and Maiuriello proved that if f has bounded distortion, then T_f has the shadowing property if and only if it is generalized hyperbolic, if and only if the masses $\mu(f^k(W))$ of the iterates of a wandering set satisfy one of three growth conditions. They asked, and Maiuriello asked again in Oberwolfach Report 19/2024, whether this remains true without bounded distortion. We show that shadowing and generalized hyperbolicity are equivalent for every dissipative composition operator, for all $1 \le p < \infty$, for real or complex scalars and without separability assumptions. Both are equivalent to the surjectivity of $I - T_f$, and to a uniform exponential "tent" condition on the Radon–Nikodym densities $d(\mu \circ f^k)/d\mu$ on the wandering set. The proof represents T_f as a field of weighted shifts and rests on a deterministic lemma about a single fibre. For $p = 2$, complex scalars and separable $L^2(\mu)$ the equivalence also follows from a recent theorem of Pituk on separable Hilbert spaces. The characterization by masses fails without bounded distortion in both directions: Bernardes, D'Aniello and Maiuriello recently gave an example satisfying the contraction condition $(\mathcal{HC})$ without shadowing, and we record an elementary hyperbolic example that satisfies none of the three conditions. This is an unrefereed note.

1 Introduction

Throughout, $(X, \mathfrak{B}, \mu)$ is a σ-finite measure space and $f\colon X \to X$ is a bijection with $f(B), f^{-1}(B) \in \mathfrak{B}$ for all $B \in \mathfrak{B}$, such that for some constant $c \ge 1$

$$\mu(f^{-1}(B)) \le c\,\mu(B) \quad \text{and} \quad \mu(f(B)) \le c\,\mu(B) \qquad \text{for all } B \in \mathfrak{B}. \tag{$\star$}$$

Fix $1 \le p < \infty$ and a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The composition operator $T_f\varphi = \varphi \circ f$ is then a bounded invertible operator on $L^p(\mu)$ with $T_f^{-1} = T_{f^{-1}}$. Following [DDM21, Mai24], T_f is *dissipative* with *wandering set* $W \in \mathfrak{B}$ if $0 < \mu(W) < \infty$ and X is the disjoint union of the sets $f^k(W)$, $k \in \mathbb{Z}$. It has *bounded distortion* on W if there is $K_0 > 0$ with

$$K_0^{-1}\,\mu(f^k(W))\,\mu(B) \;\le\; \mu(f^k(B))\,\mu(W) \;\le\; K_0\,\mu(f^k(W))\,\mu(B)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`.

for all $k \in \mathbb{Z}$ and all measurable $B \subseteq W$. Write $m_k = \mu(f^k(W))$. The following conditions on the masses m_k are those of [DDM21]:

$$(\mathcal{HC}) \quad \limsup_{n\to\infty} \sup_{k\in\mathbb{Z}} \left(\frac{m_k}{m_{k+n}}\right)^{1/n} < 1; \qquad (\mathcal{HD}) \quad \liminf_{n\to\infty} \inf_{k\in\mathbb{Z}} \left(\frac{m_k}{m_{k+n}}\right)^{1/n} > 1;$$

$$(\mathcal{GH}) \quad \limsup_{n\to\infty} \sup_{k\le 0} \left(\frac{m_{k-n}}{m_k}\right)^{1/n} < 1 \quad \text{and} \quad \liminf_{n\to\infty} \inf_{k\ge 0} \left(\frac{m_k}{m_{k+n}}\right)^{1/n} > 1.$$

D'Aniello, Darji and Maiuriello proved [DDM21, Corollaries SC and GH] that if T_f is dissipative of bounded distortion, then T_f has the shadowing property if and only if one of $(\mathcal{HC})$, $(\mathcal{HD})$, $(\mathcal{GH})$ holds, and if and only if T_f is generalized hyperbolic. (The notions are recalled in Section 2.) They asked whether bounded distortion is needed [DDM21, Problem 5.0.1], and Maiuriello repeated the question in the report of an Oberwolfach mini-workshop [Mai24, Open Problem (1), p. 1080].

Problem 1.1. *Let T_f be dissipative, without the hypothesis of bounded distortion.*

(a) Is T_f generalized hyperbolic if and only if it has the shadowing property?

(b) Does the characterization of the shadowing property by $(\mathcal{HC})$, $(\mathcal{HD})$, $(\mathcal{GH})$ still hold?

The problem is record OWR-14298367-003 of the UnsolvedMath dataset [ula26]. Part (b) was recently answered in the negative in one direction by Bernardes, D'Aniello and Maiuriello [BDM26, Example 6.2]. They gave a dissipative T_f that satisfies $(\mathcal{HC})$ but does not have the shadowing property. We answer (a) affirmatively and complete the answer to (b).

The relevant data are densities rather than masses. By $(\star)$, for each $k \in \mathbb{Z}$ the measure $B \mapsto \mu(f^k(B))$ on the measurable subsets of W is equivalent to μ on W; let

$$h_k = \frac{d(\mu \circ f^k)}{d\mu} \colon W \to (0, \infty)$$

be its Radon–Nikodym derivative (see Section 2.2). Then $h_0 = 1$ and $c^{-1} h_k \le h_{k+1} \le c\, h_k$. For $x \in W$ let B_x be the bilateral weighted backward shift on $\ell^p(\mathbb{Z})$ given by

$$(B_x u)_k = a_k(x)\, u_{k+1}, \qquad a_k = (h_k/h_{k+1})^{1/p} \in [c^{-1/p}, c^{1/p}].$$

By Lemma 2.2 below, T_f is isometrically conjugate to the operator that acts as B_x on the fibre over each $x \in W$.

Theorem 1.2. *Let T_f be a dissipative composition operator on $L^p(\mu)$, $1 \le p < \infty$, with wandering set W, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Bounded distortion is not assumed. The following are equivalent.*

(i) T_f has the shadowing property.

(ii) $I - T_f$ is surjective.

(iii) T_f is generalized hyperbolic.

(iv) (Uniform tent condition.) *There are constants $K \ge 1$ and $0 < r < 1$ and a measurable function $J \colon W \to \mathbb{Z} \cup \{-\infty, +\infty\}$ such that for almost every $x \in W$ and all integers $m \le k$,*

$$h_m(x) \le K\, r^{k-m}\, h_k(x) \quad \text{if } k \le J(x), \qquad h_k(x) \le K\, r^{k-m}\, h_m(x) \quad \text{if } m \ge J(x).$$

(v) (Equi-shadowing of the fibres.) *There is $C > 0$ such that for almost every $x \in W$ and every bounded sequence $(z_n)_{n\in\mathbb{Z}}$ in $\ell^p(\mathbb{Z})$ there is a sequence $(y_n)_{n\in\mathbb{Z}}$ in $\ell^p(\mathbb{Z})$ with $y_{n+1} = B_x y_n + z_n$ for all n and $\sup_n \|y_n\| \le C \sup_n \|z_n\|$.*

Moreover, let $0 < \delta \le 1$ be such that $\|(I - T_f)^*\varphi\| \ge \delta\|\varphi\|$ for all $\varphi \in L^p(\mu)^*$. (Such a δ exists under (ii). Under (i) one may take $\delta = \min\{1, \delta_1\}$, where δ_1 is the shadowing constant for $\varepsilon = 1$.) Then (iv) holds with

$$K = \left(2e^{H_2}\right)^p, \qquad r = 2^{-p/(2R)}, \qquad \text{where } H_2 = \log\frac{2}{\delta} + 1, \quad R = \left\lfloor \frac{16\, e^{2H_2}}{\delta} \right\rfloor + 1.$$

These constants depend only on p and δ, and the constants $K^{1/p}$ and $r^{1/p}$ that control the fibre shifts depend only on δ.

Condition (iv) says that for almost every x the sequence $k \mapsto h_k(x)$ increases exponentially up to a peak at $J(x)$ and decreases exponentially after it, with rates and constants independent of x. If $J(x) = +\infty$ the sequence increases throughout, and if $J(x) = -\infty$ it decreases throughout.

Remark 1.3. For complex scalars, Lemma 3.2 below shows that $I - T_f$ is surjective if and only if $\lambda I - T_f$ is surjective for every $\lambda \in \mathbb{T}$. So (ii) says that the surjective spectrum $\sigma_s(T_f)$ does not meet the unit circle. For an arbitrary invertible operator on a complex Banach space, Dragičević and Pituk proved that shadowing is equivalent to $\sigma_s(T) \cap \mathbb{T} = \emptyset$ [DP26, Theorem 2.2] (we know this result through [Pit26, MTST26]). So for complex scalars the equivalence (i)⇔(ii) also follows from [DP26]. Our proof of (i)⇒(ii) is an elementary averaging argument (Lemma 3.1); the main point of Theorem 1.2 is (ii)⇒(iii).

Corollary 1.4. *Under the hypotheses of Theorem 1.2, T_f is hyperbolic if and only if (iv) holds with $J(x) \in \{-\infty, +\infty\}$ for almost every x. If (iv) holds with a function J that is finite on a set of positive measure, then 1 is an eigenvalue of T_f, and for complex scalars every $\lambda \in \mathbb{T}$ is an eigenvalue.*

Proposition 1.5. *Without bounded distortion, neither implication between "T_f has the shadowing property" and "one of $(\mathcal{HC})$, $(\mathcal{HD})$, $(\mathcal{GH})$ holds" is true in general.*

- *(a)* (Bernardes, D'Aniello, Maiuriello [BDM26, Example 6.2].) *There is a dissipative T_f that satisfies $(\mathcal{HC})$ and does not have the shadowing property; by Theorem 1.2 it is not generalized hyperbolic either.*
- *(b) There is a hyperbolic dissipative T_f, so one that has the shadowing property and is generalized hyperbolic, for which none of $(\mathcal{HC})$, $(\mathcal{HD})$, $(\mathcal{GH})$ holds.*

Part (b) is the elementary Example 6.1. Under bounded distortion, condition (iv) reduces to $(\mathcal{HC})$, $(\mathcal{HD})$ or $(\mathcal{GH})$ (Remark 6.4). So Theorem 1.2 contains [DDM21, Corollaries SC and GH], and it gives the corrected form of the characterization asked for in Problem 1.1(b): the growth condition has to be imposed on the densities $h_k(x)$, uniformly in x, and not on the masses m_k.

Relation to other work. If W is a single atom, then T_f is isometrically conjugate to a bilateral weighted backward shift on $\ell^p(\mathbb{Z})$, and every such shift with positive weights bounded away from 0 and ∞ arises in this way. In this case Theorem 1.2, with Remark 6.4, reduces to the characterization of the shadowing property of these shifts by Bernardes and Messaoudi [BM21, Theorem 18], and Lemmas 2.1, 3.1 and 4.1 give another proof of it for $\ell^p(\mathbb{Z})$ ($1 \le p < \infty$; [BM21] also covers $c_0(\mathbb{Z})$). The results of [DDM21] build on this characterization.

Generalized hyperbolic operators have the shadowing property [BCD+18, Theorem A]; see also [CGP21, Theorem 1] and [BCD+25]. Whether the converse holds for all invertible operators on Banach spaces was a central question, see [DDM21, Problem 5.0.3]. Two recent preprints settle it negatively. Pituk [Pit26, Theorem A] and, independently, Messaoudi, Tofanin Neto, Saavedra and Tsokanos [MTST26, Theorem 3.9] constructed operators with the shadowing property that are not generalized hyperbolic. The construction of [MTST26] works on $\ell^p(\mathbb{N})$ for every $1 < p < \infty$ with $p \neq 2$. Pituk also proved that the two properties are equivalent on separable complex Hilbert spaces [Pit26, Theorem B]. Consequently, for $p = 2$, complex scalars and separable $L^2(\mu)$, the equivalence (i)⇔(iii) of Theorem 1.2 is a special case of [Pit26, Theorem B]. For $1 < p < \infty$ with $p \neq 2$ the implication from shadowing to generalized hyperbolicity fails for some operators on $\ell^p(\mathbb{N})$, which is itself an L^p space; so Theorem 1.2 depends on the structure of composition operators and not only on the geometry of the space. The cases $p \neq 2$, real scalars and non-separable $L^2(\mu)$ are not covered by [Pit26, Theorem B]. For complex scalars, (i)⇔(ii) also follows from [DP26, Theorem 2.2] (Remark 1.3). We have not found the characterizations (ii), (iv) and (v) in this setting in the literature; see the comparison with the work of Antonevich below.

Carvalho, Darji and Varandas [CDV24, Theorem 2.4] showed that a dissipative composition operator is conjugate to a shift operator with identity weights on a weighted space of $L^p(\mu|_W)$-valued sequences. Our Lemma 2.2 is the same conjugacy, normalized so that the weight is moved into the operator. They proved that shadowing and generalized hyperbolicity coincide for shift operators over finite-dimensional spaces with bounded projections [CDV24, Corollary 2.16], introduced equi-shadowing [CDV24, Definition 2.13], and observed after [CDV24, Corollary 2.15] that in infinite dimensions shadowing of each coordinate shift does not imply shadowing; they asked whether the equivalence holds for general shift operators [CDV24, Question 5.1]. Theorem 1.2 settles this question for the shift operators that come from dissipative composition operators, and Example 6.3 is the composition-operator form of their diagonal example. The results on shadowing and hyperbolicity of composition operators in the follow-up works [DDM22, Mai22, AEH24, BBP26] are proved under the bounded distortion hypothesis (for [AEH24] we checked the arXiv version).

Antonevich's survey [Ant21] (English translation [Ant24]) studies one-sided invertibility of weighted shift operators $B - I$, $B = aT_\alpha$, on $L^2(X, \mu)$ for a homeomorphism α of a compact space X; it is remarked there that the results carry over to $L^p(X, \mu)$. A *graded dichotomy* is a splitting given by a measurable projector-valued function whose rank may vary from point to point, and it yields a right resolvent of B, in particular right invertibility of $B - I$ [Ant21, Theorem 3.4]. For scalar coefficients and a map α with an attractor, the right resolvent is built with the multiplication by the indicator function of a neighbourhood of the attractor (see [Ant21, Theorem 5.2] and the comment after it). The splitting by multiplication with an indicator function in the proof of (iv)⇒(iii), and the bounded right inverse of $I - T_f$ that it yields (see the paragraph after Lemma 2.1), parallel this graded dichotomy. Lemma 3.2 is a form of the classical rotation invariance of the spectrum of weighted shift operators over aperiodic maps [Ant21, Lemma 3.2]. The necessity direction (ii)⇒(iv), on which (ii)⇒(iii) rests, and the shadowing property are not treated in [Ant21] in the dissipative measurable setting considered here.

2 Preliminaries

2.1 Shadowing and generalized hyperbolicity

Let E be a Banach space over $\mathbb{K}$ and let T be an invertible bounded operator on E. A sequence $(x_n)_{n\in\mathbb{Z}}$ is a *δ-pseudotrajectory* of T if $\|Tx_n - x_{n+1}\| \le \delta$ for all n. The operator T has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-pseudotrajectory (x_n) there is $v \in E$ with $\|T^n v - x_n\| < \varepsilon$ for all $n \in \mathbb{Z}$. For a bounded operator S let $r(S) = \lim_n \|S^n\|^{1/n}$ be its spectral radius. The operator T is *hyperbolic* if $\sigma(T) \cap \mathbb{T} = \emptyset$ (for $\mathbb{K} = \mathbb{R}$, $\sigma(T)$ is the spectrum of the complexification). It is *generalized hyperbolic* if $E = M \oplus N$ with closed subspaces M, N such that

$$T(M) \subseteq M, \qquad T^{-1}(N) \subseteq N, \qquad r(T|_M) < 1, \qquad r(T^{-1}|_N) < 1.$$

This is the definition used in [BCD⁺18, Mai24, Pit26]. In [DDM21, CGP21] the restrictions are required to be proper contractions. This is the same notion after passing to an equivalent norm, as noted in [CGP21], and the proofs in [DDM21] establish the spectral radius bounds. The shadowing property does not change under an equivalent norm. The renorming cannot be dropped. Let T be a bilateral weighted shift with $r(T) < 1 < \|T\|$, for instance with one weight 2 and all other weights $\frac{1}{2}$. Up to isometric conjugacy it is a dissipative composition operator whose wandering set is one atom, so it has bounded distortion. If $T|_M$ and $T^{-1}|_N$ had to be proper contractions in the given norm, then $\|x\| \le \|T^n\|\|T^{-n}x\| \le \|T^n\|\|x\| \to 0$ for $x \in N$ would force $N = \{0\}$, and then $\|T\| < 1$. So under that reading T, which is hyperbolic, would not be generalized hyperbolic, and [DDM21, Corollary GH] would fail.

Lemma 2.1 ([BCD⁺18, Theorem A])**.** *Let $E = M \oplus N$ be as in the definition of generalized hyperbolicity, let P be the projection onto M along N, and let $Q = I - P$. Suppose that $\|T^n|_M\| \le Cs^n$ and $\|T^{-n}|_N\| \le Cs^n$ for all $n \ge 0$, where $C \ge 1$ and $0 < s < 1$. For every bounded sequence $(z_n)_{n\in\mathbb{Z}}$ in E the sequence*

$$y_n = \sum_{j\ge1} T^{j-1}Pz_{n-j} - \sum_{j\ge0} T^{-j-1}Qz_{n+j} \tag{1}$$

satisfies $y_{n+1} = Ty_n + z_n$ for all n and $\sup_n\|y_n\| \le C' \sup_n\|z_n\|$, where $C' = C(\|P\| + s\|Q\|)/(1-s)$. Consequently T has the shadowing property.

Proof. The series converge absolutely, since $T^{j-1}Pz \in M$ and $T^{-j-1}Qz \in N$ with norms at most $Cs^{j-1}\|P\|\|z\|$ and $Cs^{j+1}\|Q\|\|z\|$. This also gives the bound on $\|y_n\|$. Shifting the summation indices in y_{n+1} gives

$$\begin{aligned} y_{n+1} &= Pz_n + \sum_{j\ge1} T^jPz_{n-j} - \sum_{j\ge1} T^{-j}Qz_{n+j}, \\ Ty_n &= \sum_{j\ge1} T^jPz_{n-j} - Qz_n - \sum_{j\ge1} T^{-j}Qz_{n+j}, \end{aligned}$$

so $y_{n+1} - Ty_n = Pz_n + Qz_n = z_n$. Given $\varepsilon > 0$, let $\delta = \varepsilon/(2C')$ and let (x_n) be a δ-pseudotrajectory. Put $z_n = x_{n+1} - Tx_n$ and take (y_n) as above. Then $w_n = x_n - y_n$ satisfies $w_{n+1} = Tw_n$, so $w_n = T^n w_0$, and $\|T^n w_0 - x_n\| = \|y_n\| \le C'\delta < \varepsilon$. □

If $r(T|_M) < 1$ and $r(T^{-1}|_N) < 1$, constants C and s as in Lemma 2.1 exist by the spectral radius formula. The easy direction of Pilyugin's criterion [Pil99] (bounded solutions of $y_{n+1} = Ty_n + z_n$ give shadowing) is the last step of the proof above. With $z_n = z$ for all n, (1) gives the same vector $y_n = Rz$ for every n, where

$$R = \sum_{j\ge0} T^jP - \sum_{j\ge1} T^{-j}Q,$$

and $(I - T)Rz = z$. So a generalized hyperbolic T has the bounded right inverse R of $I - T$, with $\|R\| \le C'$. By Theorem 1.2, for a dissipative composition operator T_f the surjectivity of $I - T_f$ is therefore equivalent to its right invertibility.

2.2 Densities and the fibre model

Iterating ($\star$) gives $\mu(f^k(B)) \le c^{|k|}\mu(B)$ and $\mu(B) = \mu(f^{-k}(f^k(B))) \le c^{|k|}\mu(f^k(B))$ for all $k \in \mathbb{Z}$ and $B \in \mathfrak{B}$. Hence for each k the finite measure $\mu_k(B) = \mu(f^k(B))$, $B \subseteq W$ measurable, satisfies $c^{-|k|}\mu(B) \le \mu_k(B) \le c^{|k|}\mu(B)$, and $c^{-1}\mu_k(B) \le \mu_{k+1}(B) \le c\,\mu_k(B)$. Let $h_k = d\mu_k/d\mu$ on W. After changing each h_k on a null set we may assume that $h_0 = 1$, $c^{-|k|} \le h_k \le c^{|k|}$ and $c^{-1}h_k \le h_{k+1} \le c\, h_k$ everywhere on W, for all k. Put

$$\rho_k = h_k^{-1/p}, \qquad a_k = (h_k/h_{k+1})^{1/p} = \rho_{k+1}/\rho_k.$$

Let $Y = W \times \mathbb{Z}$ with the σ-finite measure $\nu = \mu|_W \otimes$ (counting measure), and let $q \in (1, \infty]$ be the conjugate exponent of p.

Lemma 2.2. *Define $(U\varphi)(x,k) = h_k(x)^{1/p}\, \varphi(f^k(x))$ for $x \in W$, $k \in \mathbb{Z}$.*

(a) U is an isometric isomorphism of $L^p(\mu)$ onto $L^p(\nu)$.

(b) $T := UT_fU^{-1}$ is given by $(Tu)(x,k) = a_k(x)\, u(x,k+1)$, and for every $n \in \mathbb{Z}$,

$$(T^n u)(x,k) = \big(h_k(x)/h_{k+n}(x)\big)^{1/p}\, u(x,k+n).$$

(c) Identify $L^p(\nu)^*$ with $L^q(\nu)$ through $\langle u, \psi\rangle = \int_Y u\psi\, d\nu$. Then $(T^*\psi)(x,k) = a_{k-1}(x)\, \psi(x,k-1)$.

(d) If $\psi(x,k) = \rho_k(x)\, \theta(x,k)$, then $\big((I-T)^*\psi\big)(x,k) = \rho_k(x)\big(\theta(x,k) - \theta(x,k-1)\big)$.

Proof. (a) For measurable $g \ge 0$ on X and $k \in \mathbb{Z}$, $\int_{f^k(W)} g\, d\mu = \int_W (g \circ f^k)\, h_k\, d\mu$. For $g = 1_A$ with $A \subseteq f^k(W)$ this reads $\mu(A) = \mu_k(f^{-k}(A))$; the general case follows by linearity and monotone convergence. With $g = |\varphi|^p$ and a sum over k this gives $\|U\varphi\|_p = \|\varphi\|_p$. Since f^k maps null sets to null sets, U is well defined on equivalence classes. It is onto: for $u \in L^p(\nu)$, the function defined on each $f^k(W)$ by $\varphi(y) = h_k(f^{-k}(y))^{-1/p}\, u(f^{-k}(y), k)$ is measurable and $U\varphi = u$.

(b) $(UT_f\varphi)(x,k) = h_k(x)^{1/p}\varphi(f^{k+1}(x)) = a_k(x)(U\varphi)(x,k+1)$. The formula for T^n follows by induction, because $a_k a_{k+1} \cdots a_{k+n-1} = (h_k/h_{k+n})^{1/p}$ and $(T^{-1}u)(x,k) = u(x,k-1)/a_{k-1}(x)$.

(c) Since ν is σ-finite, $L^p(\nu)^* = L^q(\nu)$, also for $p = 1$. Shifting the index, $\sum_k \int_W a_k\, u(x,k+1)\, \psi(x,k)\, d\mu = \sum_k \int_W u(x,k)\, a_{k-1}\psi(x,k-1)\, d\mu$.

(d) By (c), $((I-T)^*\psi)(x,k) = \rho_k\theta(x,k) - a_{k-1}\rho_{k-1}\theta(x,k-1)$, and $a_{k-1}\rho_{k-1} = \rho_k$. ☐

On the fibre over x, T acts as the shift B_x of the introduction. For a sequence $\theta = (\theta_k)_{k\in\mathbb{Z}}$ write $(D\theta)_k = \theta_k - \theta_{k-1}$. By Lemma 2.2(d), applied on a single fibre, $(I - B_x)^*(\rho(x)\theta) = \rho(x)\, D\theta$ for every finitely supported θ, where $\rho(x) = (\rho_k(x))_{k\in\mathbb{Z}}$ and $(I - B_x)^*$ acts on $\ell^q(\mathbb{Z}) = \ell^p(\mathbb{Z})^*$.

3 From shadowing to a fibrewise inequality

Lemma 3.1. *Let T be an invertible operator on a Banach space E and $C > 0$. Suppose that for every $x \in E$ there is a sequence $(y_n)_{n\ge0}$ with $y_{n+1} = Ty_n + x$ for all $n \ge 0$ and $\sup_n \|y_n\| \le C\|x\|$. Then $\|(I-T)^*\varphi\| \ge C^{-1}\|\varphi\|$ for all $\varphi \in E^*$, and $I - T$ is surjective. If T has the shadowing property and δ_1 is the shadowing constant for $\varepsilon = 1$, the hypothesis holds with $C = 1/\delta_1$.*

Proof. Fix x and (y_n), and let $V_N = N^{-1}\sum_{n=0}^{N-1} y_n$. Then $\|V_N\| \le C\|x\|$ and

$$(I-T)V_N = \frac{1}{N}\sum_{n=0}^{N-1}\big(y_n - y_{n+1} + x\big) = x + \frac{y_0 - y_N}{N}.$$

For $\varphi \in E^*$ this gives $\varphi(x) = ((I-T)^*\varphi)(V_N) - \varphi(y_0 - y_N)/N$, so $|\varphi(x)| \le C\|x\|\, \|(I-T)^*\varphi\| + 2C\|x\|\|\varphi\|/N$. Letting $N \to \infty$ and taking the supremum over $\|x\| \le 1$ gives $\|\varphi\| \le C\|(I-T)^*\varphi\|$. A bounded operator between Banach spaces whose adjoint is bounded below is surjective [Bre11, Theorem 2.20].

Now let T have the shadowing property. Given $x \neq 0$, put $x' = \delta_1 x/\|x\|$ and define $(x_n)_{n\in\mathbb{Z}}$ by $x_0 = 0$ and $x_{n+1} = Tx_n + x'$ for all $n \in \mathbb{Z}$ (for $n < 0$, $x_n = T^{-1}(x_{n+1} - x')$). This is a δ_1-pseudotrajectory, so there is v with $\|T^n v - x_n\| < 1$ for all n. The sequence $y_n = (\|x\|/\delta_1)(x_n - T^n v)$ satisfies $y_{n+1} = Ty_n + x$ and $\|y_n\| \leq \|x\|/\delta_1$. □

Lemma 3.2. *For $\lambda \in \mathbb{K}$ with $|\lambda| = 1$, the operator $(V_\lambda u)(x,k) = \lambda^k u(x,k)$ is an isometric isomorphism of $L^p(\nu)$ and $V_\lambda^{-1} T V_\lambda = \lambda T$. Hence $I - T$ is surjective if and only if $\bar{\lambda} I - T$ is surjective.*

Proof. $(V_\lambda^{-1} T V_\lambda u)(x,k) = \lambda^{-k} a_k(x) \lambda^{k+1} u(x,k+1) = \lambda (Tu)(x,k)$. So $I - \lambda T = V_\lambda^{-1}(I - T)V_\lambda$, and $\bar{\lambda} I - T = \bar{\lambda}(I - \lambda T)$. □

The *test family* Θ_0 is the countable set of the following finitely supported sequences on $\mathbb{Z}$: the boxes $1_{[a,d]}$ for integers $a \leq d$, and the tents

$$\tau_{b,R}(k) = \max\{0,\, 1 - |k-b|/R\}, \qquad b \in \mathbb{Z},\ R \in \mathbb{N}.$$

Norms of sequences are taken in $\ell^q(\mathbb{Z})$.

Lemma 3.3. *Let $\delta > 0$ and suppose that $\|(I - T)^* \psi\|_q \geq \delta \|\psi\|_q$ for all $\psi \in L^q(\nu)$. Then there is a null set $\mathcal{N} \subseteq W$ such that for every $x \in W \setminus \mathcal{N}$ and every $\theta \in \Theta_0$,*

$$\|\rho(x)\, D\theta\|_q \geq \delta\, \|\rho(x)\, \theta\|_q. \tag{2}$$

Proof. Fix $\theta \in \Theta_0$, supported in $[-L, L]$. The functions $A(x) = \|\rho(x) D\theta\|_q$ and $G(x) = \|\rho(x)\theta\|_q$ are continuous functions of the finitely many measurable functions ρ_k with $|k| \leq L+1$. So they are measurable, $G > 0$, and $G \leq c^{L/p} \sum_k |\theta_k|$. For $0 < \eta < \delta$ let $E_\eta = \{x \in W :\ A(x) \leq (\delta - \eta) G(x)\}$, and suppose $\mu(E_\eta) > 0$. The function $\psi(x,k) = 1_{E_\eta}(x) \rho_k(x) \theta_k$ is bounded, vanishes outside $E_\eta \times [-L, L]$, and $\mu(E_\eta) < \infty$; so $\psi \in L^q(\nu)$. By Lemma 2.2(d), $((I-T)^*\psi)(x,k) = 1_{E_\eta}(x)\rho_k(x)(D\theta)_k$. For a measurable function g on Y we have $\|g\|_{L^q(\nu)} = \|x \mapsto \|g(x,\cdot)\|_{\ell^q}\|_{L^q(\mu|_W)}$: for $q < \infty$ by Tonelli's theorem, and for $q = \infty$ because a set $Z \subseteq Y$ is ν-null if and only if every slice $\{x : (x,k) \in Z\}$ is μ-null. Hence, by monotonicity of the $L^q(\mu|_W)$-norm,

$$\|(I-T)^*\psi\|_q = \|1_{E_\eta} A\|_{L^q(\mu|_W)} \leq (\delta - \eta)\|1_{E_\eta} G\|_{L^q(\mu|_W)} = (\delta - \eta)\|\psi\|_q < \delta \|\psi\|_q,$$

since $\|\psi\|_q > 0$. This contradicts the hypothesis. So $\mu(E_{\delta/j}) = 0$ for every $j \geq 2$, and (2) holds for this θ outside the null set $\bigcup_j E_{\delta/j}$. Let $\mathcal{N}$ be the union of these null sets over the countable family Θ_0. □

4 A single fibre

Lemma 4.1. *Let $S\colon \mathbb{Z} \to \mathbb{R}$, $\rho_k = e^{S_k}$, $q \in [1,\infty]$ and $0 < \delta \leq 1$. Suppose that $\|\rho\, D\theta\|_q \geq \delta \|\rho\, \theta\|_q$ for all $\theta \in \Theta_0$. Put*

$$H_1 = \log\frac{2}{\delta}, \qquad H_2 = H_1 + 1, \qquad H = 2H_2 + \log 2, \qquad R = \lfloor 8e^H/\delta \rfloor + 1.$$

(a) No humps. *$S_b \leq \max\{S_a, S_d\} + H_1$ for all integers $a \leq b \leq d$.*

(b) No long flat stretches. *For every $b \in \mathbb{Z}$ there are $i, k \in [b - R + 1, b + R]$ with $|S_i - S_k| > H$.*

(c) V-shape. *Define $J \in \mathbb{Z} \cup \{\pm\infty\}$ as follows. If $\inf_{k \geq 0} S_k = -\infty$, let $J = +\infty$. If $\inf_{k \leq 0} S_k = -\infty$, let $J = -\infty$. (These cases exclude each other.) Otherwise $m_* = \inf_{\mathbb{Z}} S$ is finite; let J be the first integer in the list $0, 1, -1, 2, -2, \ldots$ with $S_J \leq m_* + 1$. Then*

$$(Q^-)\ \ S_b \leq S_a + H_2 \ \text{ for } a \leq b \leq J, \qquad (Q^+)\ \ S_a \leq S_b + H_2 \ \text{ for } J \leq a \leq b.$$

(d) Exponential rates. *For all integers $a \le b$,*

$$S_b - S_a \le H_2 + \log 2 - \frac{(b-a)\log 2}{2R} \quad \text{if } b \le J, \qquad S_a - S_b \le H_2 + \log 2 - \frac{(b-a)\log 2}{2R} \quad \text{if } a \ge J.$$

Proof. Note that $H_1 \ge \log 2 > 0$ because $\delta \le 1$.

(a) If $b = d$ there is nothing to prove. Otherwise let $\theta = 1_{[a,d-1]}$. Then $D\theta = e_a - e_d$, so $\|\rho D\theta\|_q \le 2e^{\max\{S_a,S_d\}}$, while $\|\rho\theta\|_q \ge e^{S_b}$ because $a \le b \le d-1$. The hypothesis gives $\delta e^{S_b} \le 2e^{\max\{S_a,S_d\}}$.

(b) Let $\theta = \tau_{b,R}$. Then $(D\theta)_k = 1/R$ for $b - R + 1 \le k \le b$, $(D\theta)_k = -1/R$ for $b+1 \le k \le b+R$, and $(D\theta)_k = 0$ otherwise. Let M be the maximum of S on $[b-R+1, b+R]$ and m' its minimum on $\{k : |k-b| \le R/2\}$, a set of at least $R/2$ integers on which $\theta \ge 1/2$. Then (reading $t^{1/q} = 1$ for $q = \infty$)

$$\|\rho D\theta\|_q \le R^{-1}(2R)^{1/q} e^M, \qquad \|\rho\theta\|_q \ge \tfrac{1}{2}(R/2)^{1/q} e^{m'},$$

so $e^{M-m'} \ge \delta R\, 2^{-1}4^{-1/q} \ge \delta R/8 > e^H$. Take i, k where the maximum and the minimum are attained.

(c) *Case $J = +\infty$.* Let $a \le b$. Since $\inf_{k\ge b} S_k = -\infty$, there is $d \ge b$ with $S_d \le S_a$, and (a) gives $S_b \le S_a + H_1$. So (Q^-) holds, and (Q^+) is empty. *Case $J = -\infty$ is symmetric:* for $a \le b$ pick $e \le a$ with $S_e \le S_b$ and apply (a) to $e \le a \le b$. *Exclusion.* If both infima were $-\infty$, then for any b we could pick $a \le b \le d$ with $S_a, S_d \le S_b - H_1 - 1$, and (a) would give $S_b \le S_b - 1$. *Finite case.* For $a \le b \le J$, (a) gives $S_b \le \max\{S_a, S_J\} + H_1 \le \max\{S_a, m_* + 1\} + H_1 \le S_a + 1 + H_1$, since $S_a \ge m_*$. Symmetrically, for $J \le a \le b$ we get $S_a \le \max\{S_J, S_b\} + H_1 \le S_b + H_2$.

(d) Let a' be an integer with $a' + 2R \le J$. By (b) with $b = a' + R - 1$ there are $i < k$ in $[a', a'+2R-1]$ with $|S_i - S_k| > H$. Since $H > H_2$, (Q^-) excludes $S_k > S_i + H$; hence $S_i > S_k + H$. Using (Q^-) twice,

$$S_{a'+2R} \le S_k + H_2 < S_i - H + H_2 \le S_{a'} + 2H_2 - H = S_{a'} - \log 2.$$

Now let $a \le b \le J$ and $t = \lfloor (b-a)/(2R) \rfloor$. Applying this to $a' = a, a+2R, \ldots, a+2R(t-1)$ gives $S_{a+2Rt} \le S_a - t\log 2$, and ($\mathrm{Q}^-$) gives $S_b \le S_{a+2Rt} + H_2$. So $S_b - S_a \le H_2 - t\log 2$, and $t > (b-a)/(2R) - 1$. For $J \le a \le b$ the argument is symmetric: in a window $[a', a'+2R-1]$ with $a' \ge J$, (Q^+) excludes $S_i > S_k + H$ for $i < k$, hence $S_k > S_i + H$, and $S_{a'} \le S_i + H_2 < S_k - H + H_2 \le S_{a'+2R} + 2H_2 - H$. So $S_{a'} \le S_{a'+2R} - \log 2$, and we conclude as before. □

5 Proofs of Theorem 1.2 and Corollary 1.4

Let $T = UT_fU^{-1}$ be as in Lemma 2.2. Since U is an isometric isomorphism, the conditions (i)–(iii) hold for T_f if and only if they hold for T, and $\|(I - T_f)^*\varphi\| \ge \delta\|\varphi\|$ for all φ if and only if $\|(I-T)^*\psi\| \ge \delta\|\psi\|$ for all ψ, because $(I-T)^* = (U^{-1})^*(I-T_f)^*U^*$ and U^* is an isometric isomorphism. For $x \in W$ put $S_k(x) = \log \rho_k(x) = -\frac{1}{p}\log h_k(x)$, so that

$$S_k(x) - S_m(x) = \tfrac{1}{p}\log\big(h_m(x)/h_k(x)\big). \tag{3}$$

(i)⇒(ii). This is Lemma 3.1, which also gives $\|(I-T_f)^*\varphi\| \ge \delta_1\|\varphi\|$.

(ii)⇒(iv), with the stated constants. If $I - T_f$ is surjective, then $(I-T_f)^*$ is bounded below [Bre11, Theorem 2.20]; if the bound δ exceeds 1 we replace it by 1. So $\|(I-T)^*\psi\|_q \ge \delta\|\psi\|_q$ for all $\psi \in L^q(\nu)$ with $0 < \delta \le 1$. Let $\mathcal{N}$ be the null set of Lemma 3.3. For $x \in W \setminus \mathcal{N}$ the sequence $S(x)$ satisfies the hypothesis of Lemma 4.1, and we let $J(x)$ be given by the rule of Lemma 4.1(c). This rule defines a measurable function on all of W. Indeed, the sets $E_+ = \{x : \inf_{k\ge 0} S_k(x) = -\infty\}$ and $E_- = \{x : \inf_{k\le 0} S_k(x) = -\infty\}$ and the function $m_*(x) = \inf_k S_k(x)$ are measurable, since countable

infima of measurable functions are measurable. Put $J = +\infty$ on E_+ and $J = -\infty$ on $E_- \setminus E_+$ (outside $\mathcal{N}$ the sets E_+ and E_- are disjoint). If $k_0, k_1, \dots$ enumerates $\mathbb{Z}$ in the order $0, 1, -1, 2, -2, \dots$, then

$$\{J = k_j\} = \big(W \setminus (E_+ \cup E_-)\big) \cap \{S_{k_j} \le m_* + 1\} \cap \bigcap_{i<j} \{S_{k_i} > m_* + 1\}$$

is measurable. By Lemma 4.1(d) and (3), for $x \notin \mathcal{N}$ and $m \le k$ we get $h_m(x)/h_k(x) \le \exp\big(p(H_2 + \log 2)\big)\, 2^{-p(k-m)/(2R)} = K\, r^{k-m}$ if $k \le J(x)$. Likewise $h_k(x)/h_m(x) \le K\, r^{k-m}$ if $m \ge J(x)$. Here $e^H = 2e^{2H_2}$, so $R = \lfloor 16 e^{2H_2}/\delta \rfloor + 1$ as stated.

(iv)⇒(iii). The set $\Gamma = \{(x,k) \in Y : k \le J(x)\} = \bigcup_{k \in \mathbb{Z}} \{x : J(x) \ge k\} \times \{k\}$ is measurable. Let P be multiplication by 1_Γ, a projection of norm at most 1 on $L^p(\nu)$, and let $M = \operatorname{ran} P$ and $N = \ker P$. If $u \in M$ and $k > J(x)$, then $k+1 > J(x)$, so $(Tu)(x,k) = a_k(x) u(x, k+1) = 0$; thus $T(M) \subseteq M$. If $u \in N$ and $k \le J(x)$, then $k - 1 \le J(x)$, so $(T^{-1}u)(x,k) = u(x, k-1)/a_{k-1}(x) = 0$; thus $T^{-1}(N) \subseteq N$. Let $n \ge 0$. For $u \in M$, by Lemma 2.2(b), $(T^n u)(x,k) = (h_k(x)/h_{k+n}(x))^{1/p} u(x, k+n)$ vanishes unless $k \le k + n \le J(x)$, and then the factor is at most $K^{1/p} r^{n/p}$ by (iv). For $u \in N$, $(T^{-n}u)(x,k) = (h_k(x)/h_{k-n}(x))^{1/p} u(x, k-n)$ vanishes unless $J(x) < k - n \le k$, and then the factor is at most $K^{1/p} r^{n/p}$. Hence

$$\|T^n|_M\| \le K^{1/p} r^{n/p}, \qquad \|T^{-n}|_N\| \le K^{1/p} r^{n/p} \qquad (n \ge 0),$$

so $r(T|_M) \le r^{1/p} < 1$ and $r(T^{-1}|_N) \le r^{1/p} < 1$. Thus T, and therefore T_f, is generalized hyperbolic.

(iii)⇒(i). This is Lemma 2.1.

(iv)⇒(v). Let x be outside the null set where (iv) fails, and split $\ell^p(\mathbb{Z}) = M_x \oplus N_x$ with $M_x = \{u : u_k = 0 \text{ for } k > J(x)\}$ and $N_x = \{u : u_k = 0 \text{ for } k \le J(x)\}$. The computation above, on one fibre, shows that $B_x(M_x) \subseteq M_x$, $B_x^{-1}(N_x) \subseteq N_x$, and $\|B_x^n|_{M_x}\|, \|B_x^{-n}|_{N_x}\| \le K^{1/p} r^{n/p}$. The coordinate projections have norm at most 1. By Lemma 2.1, (v) holds with $C = K^{1/p}(1 + r^{1/p})/(1 - r^{1/p})$, which does not depend on x.

(v)⇒(iv). For almost every x, Lemma 3.1 applied to B_x on $\ell^p(\mathbb{Z})$ (with constant forcing $z_n = w$, $w \in \ell^p(\mathbb{Z})$) gives $\|(I - B_x)^* \phi\|_q \ge C^{-1} \|\phi\|_q$ for $\phi \in \ell^q(\mathbb{Z})$. With $\phi = \rho(x)\theta$ and the identity $(I - B_x)^*(\rho(x)\theta) = \rho(x) D\theta$ from Section 2.2, the sequence $S(x)$ satisfies the hypothesis of Lemma 4.1 with $\delta = \min\{1, 1/C\}$. We conclude as in the proof of (ii)⇒(iv). This completes the proof of Theorem 1.2. □

Proof of Corollary 1.4. Suppose that (iv) holds with $J(x) \in \{\pm\infty\}$ for almost every x. Then $\Gamma = E \times \mathbb{Z}$ with $E = \{J = +\infty\}$ up to a null set, so M and N above are unions of whole fibres and $T(M) = M$, $T(N) = N$. Hence $T = T|_M \oplus T|_N$, $\sigma(T) = \sigma(T|_M) \cup \sigma(T|_N)$, and the bounds above give $\sigma(T|_M) \subseteq \{|z| \le r^{1/p}\}$ and $\sigma(T|_N) \subseteq \{|z| \ge r^{-1/p}\}$. So T_f is hyperbolic.

Now suppose that (iv) holds and that J is finite on a set E of positive measure. For $x \in E$ outside the exceptional null set, (iv) with the peak $j = J(x)$ gives $h_k(x) \le K\, r^{|k-j|} h_j(x)$ for all k. Hence $\sum_k h_k(x) < \infty$, and there are $L < \infty$ and a set $E' \subseteq E$ of positive finite measure with $\sum_k h_k \le L$ on E'. For $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ let $u(x,k) = 1_{E'}(x) \lambda^{-k} h_k(x)^{1/p}$. Then $\|u\|_p^p = \int_{E'} \sum_k h_k \, d\mu \le L \mu(E')$, $u \ne 0$, and

$$(Tu)(x,k) = a_k(x)\, 1_{E'}(x)\, \lambda^{-k-1} h_{k+1}(x)^{1/p} = \lambda^{-1} u(x,k).$$

So λ^{-1} is an eigenvalue of T, hence of T_f. In particular T_f is not hyperbolic. Finally, if T_f is hyperbolic, then it has the shadowing property [BCD$^+$18, Theorem A], so (iv) holds by Theorem 1.2, and by what we just proved $J \in \{\pm\infty\}$ almost everywhere. □

6 The mass conditions

Example 6.1. *A hyperbolic T_f for which none of $(\mathcal{HC})$, $(\mathcal{HD})$, $(\mathcal{GH})$ holds.* Let $X = \{\alpha_j, \beta_j : j \in \mathbb{Z}\}$ with $\mathfrak{B}$ the power set, $\mu(\{\alpha_j\}) = 2^j$, $\mu(\{\beta_j\}) = 2^{-j}$, $f(\alpha_j) = \alpha_{j+1}$, $f(\beta_j) = \beta_{j+1}$, and $W = \{\alpha_0, \beta_0\}$. Then $(\star)$ holds with $c = 2$ and T_f is dissipative. Bounded distortion fails: $\mu(f^k(\{\beta_0\}))/\mu(\{\beta_0\}) = 2^{-k}$, whereas $\mu(f^k(W))/\mu(W) = (2^k + 2^{-k})/2$. The space $L^p(\mu)$ is the direct sum of the ℓ^p spaces over the two orbits, and $\|T_f\varphi\|_p^p = \sum_j |\varphi(\alpha_{j+1})|^p 2^j = \frac{1}{2}\|\varphi\|_p^p$ for φ supported on the α-orbit, while $\|T_f\varphi\|_p^p = 2\|\varphi\|_p^p$ on the β-orbit. So T_f is the direct sum of $2^{-1/p}$ times a surjective isometry and $2^{1/p}$ times a surjective isometry. Hence $\sigma(T_f) \subseteq \{|z| = 2^{-1/p}\} \cup \{|z| = 2^{1/p}\}$ and T_f is hyperbolic. (In Theorem 1.2(iv) one can take $K = 1$, $r = 1/2$, $J(\alpha_0) = +\infty$, $J(\beta_0) = -\infty$.) On the other hand $m_k = 2^k + 2^{-k}$, so for all $n \geq 1$

$$\frac{m_{-n}}{m_0} = \frac{2^n + 2^{-n}}{2} \geq 2^{n-1}, \qquad \frac{m_0}{m_n} = \frac{2}{2^n + 2^{-n}} \leq 2^{1-n}.$$

The first inequality (with $k = -n$ in $(\mathcal{HC})$, and $k = 0$ in the first half of $(\mathcal{GH})$) shows that the two suprema are at least $2^{(n-1)/n} \to 2$, so $(\mathcal{HC})$ and the first half of $(\mathcal{GH})$ fail. The second inequality (with $k = 0$) shows that the two infima are at most $2^{(1-n)/n} \to \frac{1}{2}$, so $(\mathcal{HD})$ and the second half of $(\mathcal{GH})$ fail. This proves Proposition 1.5(b). The same works for a non-atomic space, for instance $X = [0, 1) \times \mathbb{Z}$ with $f(t, k) = (t, k + 1)$, $W = [0, 1) \times \{0\}$ and density 2^k on $[0, \frac{1}{2}) \times \{k\}$ and 2^{-k} on $[\frac{1}{2}, 1) \times \{k\}$ with respect to Lebesgue measure.

Example 6.2 ([BDM26, Example 6.2])**.** Let $X = \{\alpha_j, \beta_j : j \in \mathbb{Z}\}$, $f(\alpha_j) = \alpha_{j+1}$, $f(\beta_j) = \beta_{j+1}$, $W = \{\alpha_0, \beta_0\}$, with $\mu(\{\alpha_j\}) = \mu(\{\beta_j\}) = 2^{j-1}$ for $j \leq 0$, and $\mu(\{\alpha_j\}) = 2^j - 1$, $\mu(\{\beta_j\}) = 1$ for $j \geq 1$. Then $(\star)$ holds with $c = 3$, and $m_j = 2^j$ for all j, so $(\mathcal{HC})$ holds. Bernardes, D'Aniello and Maiuriello showed directly that T_f does not have the shadowing property, which proves Proposition 1.5(a). In terms of Theorem 1.2: $h_k(\beta_0) = 2$ for all $k \geq 1$, so $S(\beta_0)$ is constant on $[1, \infty)$. For $b = R + 1$ the function $\psi = 1_{\{\beta_0\}} \otimes \rho\,\tau_{b,R}$ satisfies $\|(I - T)^*\psi\|_q / \|\psi\|_q = \|D\tau_{b,R}\|_q / \|\tau_{b,R}\|_q \leq 8/R$, by the estimates in the proof of Lemma 4.1(b). So $(I - T)^*$ is not bounded below and $I - T_f$ is not surjective. By Lemma 3.1 T_f does not have the shadowing property, as [BDM26] showed directly, and by Lemma 2.1 it is not generalized hyperbolic either.

Example 6.3. *Every fibre is hyperbolic, but T_f has neither property.* Let $X = \{\xi_{n,k} : n \geq 1,\ k \in \mathbb{Z}\}$ with $\mu(\{\xi_{n,k}\}) = 2^{-n}(1 + 1/n)^{pk}$, $f(\xi_{n,k}) = \xi_{n,k+1}$ and $W = \{\xi_{n,0} : n \geq 1\}$, so $\mu(W) = 1$ and $(\star)$ holds with $c = 2^p$. On the fibre over $\xi_{n,0}$ we have $h_k = (1 + 1/n)^{pk}$ and $a_k = n/(n + 1)$, so $B_{\xi_{n,0}}$ is $n/(n + 1)$ times the unweighted shift and is hyperbolic. But for $\psi = 1_{\{\xi_{n,0}\}} \otimes \tau_{0,R}$ we get $(I - T)^*\psi = 1_{\{\xi_{n,0}\}} \otimes \big(D\tau_{0,R} + \frac{1}{n+1}\tau_{0,R}(\cdot - 1)\big)$, hence $\|(I - T)^*\psi\|_q \leq (8/R + 1/(n + 1))\|\psi\|_q$. Letting $n, R \to \infty$ shows that $(I - T)^*$ is not bounded below. So $I - T_f$ is not surjective, and T_f has neither the shadowing property nor is it generalized hyperbolic. This is the composition-operator form of the diagonal example given after [CDV24, Corollary 2.15]. It shows that the uniformity in Theorem 1.2(iv) and (v) cannot be dropped.

Remark 6.4 (Bounded distortion). Suppose T_f has bounded distortion with constant K_0, and put $\bar{m}_k = m_k/\mu(W)$. If $h_k > K_0\bar{m}_k$ on a set $B \subseteq W$ of positive measure, then $\mu(f^k(B)) = \int_B h_k\,d\mu > K_0\bar{m}_k\mu(B)$, contrary to the definition; similarly for the lower bound. So $K_0^{-1}\bar{m}_k \leq h_k \leq K_0\bar{m}_k$ almost everywhere, for all k. Consequently (iv) holds if and only if the sequence $(\bar{m}_k)$ satisfies the tent condition for some $J_0 \in \mathbb{Z} \cup \{\pm\infty\}$ (take $J_0 = J(x_0)$ for a suitable x_0, respectively $J \equiv J_0$), with K replaced by KK_0^2. Since $\bar{m}_{k+1}/\bar{m}_k \in [c^{-1}, c]$, a routine argument with n-th roots shows that the tent condition for $(\bar{m}_k)$ with $J_0 = +\infty$, $J_0 = -\infty$ or J_0 finite is equivalent to $(\mathcal{HC})$, $(\mathcal{HD})$ or $(\mathcal{GH})$, respectively. For instance, $(\mathcal{HC})$ gives n_0 and $s < 1$ with $m_k/m_{k+n} \leq s^n$ for $n \geq n_0$, and

$m_k/m_{k+n} \le c^n$ for $n < n_0$, so $m_k/m_{k+n} \le (c/s)^{n_0} s^n$ for all $n \ge 0$. A finite peak J_0 can be moved to 0 at the cost of replacing K by $K(c/r)^{|J_0|}$. Thus Theorem 1.2 contains [DDM21, Corollaries SC and GH]. We include this only as a consistency check.

7 Concluding remarks

Theorem 1.2 answers Problem 1.1(a) affirmatively for every dissipative composition operator. It answers Problem 1.1(b) negatively in its literal form, as [BDM26] had already shown for one direction, and it replaces the conditions on the masses m_k by the uniform tent condition on the densities h_k. The method needs only a lower bound for $(I - T_f)^*$ on the countable test family Θ_0, fibre by fibre. The conservative case [DDM21, Problem 5.0.2] and the question on structural stability in [Mai24, Open Problem (2)] are not addressed here.

Verification. The claims were checked as follows. The proofs are meant to be checked by hand. An independent referee checked every step: the fibre model, Lemmas 3.1, 3.3 and 4.1, the splitting in the proof of (iv)⇒(iii), the formula (1), Corollary 1.4 and the three examples. A second independent referee checked all proofs again line by line, including Lemma 3.2 and Remark 6.4. Numerical sanity checks were also run; the proofs do not depend on them.

- `verify_model.py` checks Lemma 2.2 on 20 random atomic systems for each of $p = 1$ (exact rational arithmetic), $p = 2$ and $p = 3$.
- `fibre_lemma_check.py` checks Lemma 4.1(a),(b) for $q \in \{1, 1.5, 2, 3, \infty\}$ on 120 random windows each, compares the test family with the true lower bound of the fibre operator in 210 cases (an eigenvalue problem for $q = 2$, a linear program for $q = \infty$), and checks (c),(d) with the explicit constants on 12 windows of length 3–$6 \cdot 10^4$. `examples.py` computes the masses of Examples 6.1 and 6.2 exactly and the fibre lower bounds of Examples 6.1 to 6.3.
- The referee's script `indep_adversarial.py` maximizes the violation of Lemma 4.1(a),(b) by hill climbing against the true window lower bound for $q = 2$ and $q = \infty$; the maxima stayed negative. It finds no violation of (d) on 40 windows of length 2–$8 \cdot 10^4$.
- `lead_check.py` checks (1) and its bound for the splitting of (iv)⇒(v) on 30 random fibres, the tent estimates of Examples 6.2 and 6.3, and the inequalities of Example 6.1 for $n \le 200$ in exact arithmetic.
- The second referee wrote four scripts from the text of the paper, before reading the others. `r2_model_check.py` checks Lemma 2.2, the formula for T^n with $|n| \le 5$, and Lemma 3.2 on 25 random atomic systems in each of five settings ($p = 1$ in exact arithmetic, $p = 1.5$, $p = 2$, and $p = 2, 3$ with complex scalars). `r2_fibre_check.py` checks window versions of Lemma 4.1(a)–(d) with the explicit constants, for $q \in \{1, 1.5, 2, 3, \infty\}$, on 36 windows of length up to 9000. `r2_examples_check.py` checks Examples 6.1 to 6.3 and the tent bound $\|D\tau_{b,R}\|_q \le (8/R)\|\tau_{b,R}\|_q$. `r2_shift_check.py` checks (1) and its constant for the splitting of (iv)⇒(v), and the eigenvectors of Corollary 1.4. The second referee also re-ran the five scripts above; their outputs were byte-identical to the recorded ones.

No violation was found.

Scope and priority. Theorem 1.2 answers Problem 1.1(a) as posed in [DDM21, Mai24] and in their setting: a bimeasurable bijection f satisfying $(\star)$, a wandering set of finite positive measure, and generalized hyperbolicity in the spectral form of [Mai24]. For $p = 2$, complex scalars and separable $L^2(\mu)$ the equivalence of shadowing and generalized hyperbolicity is a special case of Pituk's theorem [Pit26, Theorem B]. The cases $p \neq 2$, real scalars and non-separable $L^2(\mu)$ are not covered by [Pit26, Theorem B]. For complex scalars, (i)⇔(ii) also follows from [DP26, Theorem 2.2] (Remark 1.3). We have not found the characterizations (ii), (iv) and (v) in this setting in the literature. For a single atom W, Theorem 1.2 reduces to the theorem of Bernardes and Messaoudi [BM21, Theorem 18]. The splitting in the proof of (iv)⇒(iii) parallels Antonevich's graded dichotomy [Ant21], which is set on compact spaces and does not treat the necessity direction in our setting. Problem 1.1(b) had already been answered negatively in one direction by [BDM26, Example 6.2]; our Example 6.1 for the other direction is elementary, and Example 6.3 is the composition-operator form of an example in [CDV24]. We searched arXiv (through September 29, 2026), OpenAlex (including all 25 works it lists as citing [DDM21]), zbMATH Open and Crossref, including queries on one-sided invertibility of weighted shift operators, and made two general web searches. The works on composition operators that we found assume bounded distortion, and we found no other treatment of the problem. The Russian-language literature on functional operators was not checked systematically. The area is active (four relevant preprints appeared in July and August 2026), so concurrent work is possible. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to develop the proofs and the examples, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

[AEH24] M. S. Al Ghafri, Y. Estaremi, and Z. Huang. Orlicz spaces and their hyperbolic composition operators. *Mathematics*, 12(18):Paper No. 2809, 2024. Published version of the preprint by Y. Estaremi, Hyperbolic composition operators on Orlicz spaces, arXiv:2209.11930 (2022), as far as the titles and abstracts show; we read the arXiv version. `doi:10.3390/math12182809`.

[Ant21] A. B. Antonevich. Right-sided invertibility of binomial functional operators and graded dichotomy. *Sovremennaya Matematika. Fundamental'nye Napravleniya (Contemporary Mathematics. Fundamental Directions)*, 67(2):208–236, 2021. In Russian; English translation [Ant24]. `doi:10.22363/2413-3639-2021-67-2-208-236`.

[Ant24] A. B. Antonevich. Right-sided invertibility of binomial functional operators and graded dichotomy. *Journal of Mathematical Sciences*, 278(1):12–38, 2024. English translation of [Ant21]; not seen by us, we cite [Ant21]. `doi:10.1007/s10958-024-06903-w`.

[BBP26] N. C. Bernardes Jr., A. Bonilla, and J. V. A. Pinto. On the dynamics of weighted composition operators II. *Mediterranean Journal of Mathematics*, 23(5):Paper No. 146, 2026. arXiv:2512.06425. `doi:10.1007/s00009-026-03136-w`.

[BCD+18] N. C. Bernardes Jr., P. R. Cirilo, U. B. Darji, A. Messaoudi, and E. R. Pujals. Expansivity and shadowing in linear dynamics. *Journal of Mathematical Analysis and Applications*, 461(1):796–816, 2018. arXiv:1612.02921. The result cited as Theorem A follows the numbering used in [Pit26] and in the introduction of [DDM21]; [DDM21] also cites it as

Corollary 8, and it is Theorem 33 of arXiv v1. We could not check the numbering of the published version. `doi:10.1016/j.jmaa.2017.11.059`.

[BCD+25] N. C. Bernardes Jr., B. M. Caraballo, U. B. Darji, V. V. Fávaro, and A. Peris. Generalized hyperbolicity, stability and expansivity for operators on locally convex spaces. *Journal of Functional Analysis*, 288:110696, 2025. `doi:10.1016/j.jfa.2024.110696`.

[BDM26] N. C. Bernardes Jr., E. D'Aniello, and M. Maiuriello. The specification property for composition operators. Preprint, arXiv:2607.15831, 2026. July 2026; numbering of v1.

[BM21] N. C. Bernardes Jr. and A. Messaoudi. Shadowing and structural stability for operators. *Ergodic Theory and Dynamical Systems*, 41(4):961–980, 2021. `doi:10.1017/etds.2019.107`.

[Bre11] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Universitext. Springer, New York, 2011. `doi:10.1007/978-0-387-70914-7`.

[CDV24] M. Carvalho, U. B. Darji, and P. Varandas. Shift operators and their classification. Preprint, arXiv:2407.20890, 2024. Numbering of v1.

[CGP21] P. R. Cirilo, B. Gollobit, and E. Pujals. Dynamics of generalized hyperbolic linear operators. *Advances in Mathematics*, 387:107830, 2021. arXiv:1907.01146; theorem numbers cited from the arXiv version. `doi:10.1016/j.aim.2021.107830`.

[DDM21] E. D'Aniello, U. B. Darji, and M. Maiuriello. Generalized hyperbolicity and shadowing in L^p spaces. *Journal of Differential Equations*, 298:68–94, 2021. arXiv:2009.11526. Problem numbers follow arXiv v1, as in [Pit26]. `doi:10.1016/j.jde.2021.06.038`.

[DDM22] E. D'Aniello, U. B. Darji, and M. Maiuriello. Shift-like operators on $L^p(X)$. *Journal of Mathematical Analysis and Applications*, 515(1):126393, 2022. arXiv:2107.12103. `doi:10.1016/j.jmaa.2022.126393`.

[DP26] D. Dragičević and M. Pituk. Duality between shadowing and uniform expansivity in linear dynamics. *Transactions of the American Mathematical Society*, 2026. Published online 2026; not seen by us, quoted from [Pit26, MTST26]. `doi:10.1090/tran/9879`.

[Mai22] M. Maiuriello. Expansivity and strong structural stability for composition operators on L^p spaces. *Banach Journal of Mathematical Analysis*, 16(4):Paper No. 51, 2022. arXiv:2206.00353. `doi:10.1007/s43037-022-00196-4`.

[Mai24] M. Maiuriello. Preserved dynamics: visualizing the connection between composition operators and weighted shifts. In *Mini-Workshop: New Horizons in Linear Dynamics, Universality, and the Invariant Subspace Problem*, number 19/2024 in Oberwolfach Reports, pages 1078–1081. EMS Press, 2024. Joint work with E. D'Aniello and U. B. Darji; Open Problem (1) on p. 1080. `doi:10.4171/OWR/2024/19`.

[MTST26] A. Messaoudi, J. Tofanin Neto, M. Saavedra, and I. Tsokanos. On generalized hyperbolicity, stability, and shadowing for linear operators. Preprint, arXiv:2608.17021, 2026. August 2026; numbering of v1.

[Pil99] S. Yu. Pilyugin. *Shadowing in Dynamical Systems*, volume 1706 of *Lecture Notes in Mathematics*. Springer, Berlin, 1999. `doi:10.1007/BFb0093184`.

[Pit26] M. Pituk. Resolving the generalized hyperbolicity conjecture for shadowing. Preprint, arXiv:2608.19499, 2026. August 2026.

[ula26] ulamai. UnsolvedMath. Hugging Face dataset, `https://huggingface.co/datasets/ulamai/UnsolvedMath`, 2026. Version 1.6.0; record OWR-14298367-003.