

A Proper Subclass of the Separable Permutations Whose Generating Function Lies Outside $\mathbb{Q}(z, \sqrt{1-4z}, \sqrt{1-6z+5z^2})$

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Abstract

Question 5 of R. Brignall's abstract (joint work with M. Bóna, C. Defant, M. Opler and V. Vatter) in Oberwolfach Report 6/2024, p. 292, asks whether every proper subclass of the separable permutations $\text{Av}(2413, 3142)$ has a generating function in $K = \mathbb{Q}(z, \sqrt{1-4z}, \sqrt{1-6z+5z^2})$. The answer is no. The class $\mathcal{C} = \text{Av}(2341, 2413, 3142)$ is a proper subclass of the separable permutations which contains $\text{Av}(231)$. Its generating function $f = \sum_{n \geq 0} |\mathcal{C}_n| z^n$ satisfies $z(1-z+z^2)f^2 - (1-2z+2z^2)f + (1-z)^2 = 0$, so that $f = (1-2z+2z^2 - \sqrt{D})/(2z(1-z+z^2))$ with $D = 1-8z+20z^2-24z^3+16z^4-4z^5$. This enumeration is due to Callan, Mansour and Shat-tuck (Discrete Math. Theor. Comput. Sci. 19:1 (2017), #6, Theorem 11), and the same generating function is in the PermPAL database. The polynomial D is irreducible over $\mathbb{Q}$ and coprime to $(1-4z)(1-z)(1-5z)$, whereas a nonzero element of $\mathbb{Q}(z)$ is a square in K only if it lies in the square class of 1, $1-4z$, $(1-z)(1-5z)$ or $(1-4z)(1-z)(1-5z)$; hence $\sqrt{D} \notin K$ and $f \notin K$. The contribution of this note is the observation that this class answers Question 5 negatively, a self-contained proof of its enumeration by the substitution decomposition, and an exact certificate that its generating function is not in K . This is an unrefereed note.

1 Introduction

1.1 The question

A permutation $\pi = \pi(1) \cdots \pi(n)$ *contains* σ if some subsequence of π is order isomorphic to σ , and *avoids* σ otherwise; a *permutation class* is a set of permutations closed under containment, and $\text{Av}(B)$ is the class of permutations avoiding every element of B . The generating function of a class $\mathcal{D}$ is $\sum_{n \geq 0} |\mathcal{D}_n| z^n$, where $\mathcal{D}_n$ is the set of members of length n (the empty permutation counts once); the constant term plays no role below, since f , $f-1$ and $(f-1)/z$ generate the same field over $\mathbb{Q}(z)$.

The *separable permutations* form the class $\mathcal{S} = \text{Av}(2413, 3142)$; it is also the smallest class that contains the permutation 1 and is closed under direct and skew sums (Section 2). Brignall's abstract [Bri24, p. 290] writes their generating function as $(1-z-\sqrt{1-6z+z^2})/(2z) = 1+2z+6z^2+22z^3+90z^4+\cdots$; with this normalisation the coefficient of z^n is the number of separable permutations of length $n+1$ (the large Schröder numbers, A006318 in [OEI26]).

The abstract [Bri24] reports on work of M. Bóna, R. Brignall, C. Defant, M. Opler and V. Vatter at the Oberwolfach mini-workshop *Permutation Patterns* (28 January – 2 February 2024). It recalls the theorem of Albert, Atkinson and Vatter [AAV11, Theorem 4.1] that every subclass of the separable permutations that does not contain $\text{Av}(231)$ or a symmetry of this class has a rational generating

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function [Bri24, Theorem 2]. Its Section 2 reports that during the workshop it was noted that proper subclasses of the separable permutations containing $\text{Av}(231)$ or a symmetry of it had generating functions in $\mathbb{Q}(z, \sqrt{1-4z})$, the field of the Catalan generating function; it asks whether every proper subclass has its generating function in this field, and reports that the answer is negative: the class $\text{Av}(2143, 2413, 3142)$, enumerated by Brignall, Huczynska and Vatter [BHV08] (A033321 in [OEI26]), has the generating function $2/(1+z+\sqrt{1-6z+5z^2})$. With this example in mind, Question 5 of Brignall's abstract [Bri24, p. 292] asks whether “every proper subclass of the separable permutations has a generating function that belongs to” the field

$$K = \mathbb{Q}(z, \sqrt{1-4z}, \sqrt{1-6z+5z^2}).$$

The same report contains a different Question 5, in V. Vatter's abstract (p. 284), on labelled well-quasi-ordered classes; the present note concerns only Question 5 of Brignall's abstract (p. 292).

We write $a = 1 - 4z$ and $b = 1 - 6z + 5z^2 = (1-z)(1-5z)$; $\sqrt{a}$ and $\sqrt{b}$ are the power series with constant term 1 whose squares are a and b , and K is the subfield of $\mathbb{Q}((z))$ that they generate together with z . The proof in Section 4 uses only that K is generated over $\mathbb{Q}(z)$ by square roots of a and b , so the signs of the radicals do not matter.

1.2 The result

Theorem 1.1. *Let $\mathcal{C} = \text{Av}(2341, 2413, 3142)$ and $f = \sum_{n \geq 0} |\mathcal{C}_n| z^n$.*

(a) *$\mathcal{C}$ is a proper subclass of the separable permutations, and it contains $\text{Av}(231)$.*

(b) *The generating function f satisfies*

$$z(1-z+z^2)f^2 - (1-2z+2z^2)f + (1-z)^2 = 0. \quad (1)$$

Consequently

$$f = \frac{1-2z+2z^2-\sqrt{D}}{2z(1-z+z^2)}, \quad D = 1-8z+20z^2-24z^3+16z^4-4z^5, \quad (2)$$

where $\sqrt{D}$ is the power series with constant term 1 whose square is D , and $f = 1+z+2z^2+6z^3+21z^4+77z^5+290z^6+1118z^7+4398z^8+17595z^9+\dots$.

(c) *The polynomial D is irreducible over $\mathbb{Q}$ and coprime to $(1-4z)(1-z)(1-5z)$, and $f \notin K$.*

Corollary 1.2. *The answer to Question 5 of Brignall's abstract [Bri24, p. 292] is negative. It remains negative for the proper subclasses that contain $\text{Av}(231)$, and for the subclasses $\text{Av}(2413, 3142, \beta)$ with a single additional basis element β .*

Proof of Theorem 1.1(a). By definition $\mathcal{C} \subseteq \text{Av}(2413, 3142)$. The permutation 2341 has length 4 and differs from 2413 and 3142, so it is separable, and it is not in $\mathcal{C}$; hence the inclusion is proper. Each of 2341, 2413 and 3142 contains 231 (at the entries 2, 3, 1, at 2, 4, 1 and at 3, 4, 2), so a permutation avoiding 231 avoids all three, and $\text{Av}(231) \subseteq \mathcal{C}$. $\square$

Part (b) is proved in Section 3 and part (c) in Section 4. Corollary 1.2 follows from (a) and (c), since $\mathcal{C} = \text{Av}(2413, 3142, 2341)$ is a proper subclass that contains $\text{Av}(231)$.

1.3 What is new and what is not

The enumeration (b) is not new. It is Theorem 11 of Callan, Mansour and Shattuck [CMS17] (Theorem 41 of the full-length arXiv version [CMS16]), proved there by a decomposition according to left-to-right maxima, and the same generating function is listed in the PermPAL database [Per26] for the symmetric class $\text{Av}(1432, 2413, 3142)$, with a specification found by Combinatorial Exploration [ABC⁺22]. The papers [CMS17, CMS16] mention neither separable permutations nor the field of the generating function. The contribution of this note is the observation that this class answers Question 5 negatively; a self-contained proof of the enumeration through the substitution decomposition (Section 3), so that the answer does not rest on the cited proof; and an exact certificate that $f \notin K$, which rests on Kummer theory over $\mathbb{Q}(z)$ (Section 4). Section 5 contains remarks, and Section 6 describes the computations, the verification and the scope.

2 Sums, cuts and separable permutations

For permutations σ of length k and τ of length l , the *direct sum* and the *skew sum* are

$$\sigma \oplus \tau = \sigma(1) \cdots \sigma(k) (\tau(1) + k) \cdots (\tau(l) + k), \quad \sigma \ominus \tau = (\sigma(1) + l) \cdots (\sigma(k) + l) \tau(1) \cdots \tau(l).$$

Both operations are associative. A nonempty permutation is *sum-decomposable* if it equals $\sigma \oplus \tau$ with σ, τ nonempty, and *sum-indecomposable* otherwise; *skew-decomposable* and *skew-indecomposable* are defined in the same way with $\ominus$. We write $\delta_a = a(a-1) \cdots 1$ for the decreasing permutation of length $a \geq 1$.

Let π have length n . An integer c with $1 \leq c \leq n-1$ is a *sum cut* of π if $\{\pi(1), \dots, \pi(c)\} = \{1, \dots, c\}$, and a *skew cut* if $\{\pi(1), \dots, \pi(c)\} = \{n-c+1, \dots, n\}$. Clearly $\pi = \sigma \oplus \tau$ with $|\sigma| = c$ and $|\tau| = n-c \geq 1$ exactly when c is a sum cut, and then σ and τ are the patterns of the first c and of the last $n-c$ entries; the same holds for $\ominus$ and skew cuts.

Lemma 2.1. (a) *Every nonempty permutation π can be written as $\pi = \alpha_1 \oplus \cdots \oplus \alpha_k$ with $k \geq 1$ and all α_i sum-indecomposable, in exactly one way, and the numbers $|\alpha_1| + \cdots + |\alpha_j|$ with $1 \leq j < k$ are exactly the sum cuts of π . The same holds for $\ominus$ and skew cuts.*

(b) *No permutation of length at least 2 is both sum-decomposable and skew-decomposable.*

Proof. (a) Let $c_1 < \cdots < c_{k-1}$ be the sum cuts of π , and put $c_0 = 0$ and $c_k = n$. For $0 \leq i < j \leq k$ the entries in the positions $c_i + 1, \dots, c_j$ are exactly the values $c_i + 1, \dots, c_j$. Hence $\pi = \alpha_1 \oplus \cdots \oplus \alpha_k$, where α_i is the pattern of the entries in the positions $c_{i-1} + 1, \dots, c_i$. If α_i had a sum cut d , then $c_{i-1} + d$ would be a sum cut of π strictly between c_{i-1} and c_i ; so every α_i is sum-indecomposable. Conversely, let $\pi = \beta_1 \oplus \cdots \oplus \beta_l$ with sum-indecomposable β_j , and put $e_j = |\beta_1| + \cdots + |\beta_j|$. Each e_j with $1 \leq j < l$ is a sum cut of π . If a sum cut c of π satisfied $e_{j-1} < c < e_j$, then $c - e_{j-1}$ would be a sum cut of β_j , because the entries of π in the positions $e_{j-1} + 1, \dots, c$ are the values $e_{j-1} + 1, \dots, c$; this is impossible. So $e_1, \dots, e_{l-1}$ are the sum cuts, and the β_j are the α_i . For $\ominus$ the proof is the same, with the values counted from the top.

(b) If $\pi = \sigma \oplus \tau$ with σ, τ nonempty, then $\pi(1) \leq |\sigma| < \pi(n)$; if $\pi = \sigma \ominus \tau$ with σ, τ nonempty, then $\pi(1) > \pi(n)$. $\square$

The α_i in (a) are the *sum components* of π ; the *skew components* are defined in the same way. By (b), a sum-decomposable permutation is skew-indecomposable, and a skew-decomposable permutation is sum-indecomposable.

Lemma 2.2 (blocks). *Let $\pi = \tau_1 \ominus \cdots \ominus \tau_m$ with $m \geq 1$ and nonempty τ_i , and let ρ be a nonempty permutation with skew components $\rho_1, \dots, \rho_r$. If π contains ρ , then there are indices $1 \leq i_1 \leq i_2 \leq$*

$\dots \leq i_r \leq m$ such that for every i the block τ_i contains the skew sum of those ρ_j with $i_j = i$ (if there are any). In particular:

- (i) if ρ is skew-indecomposable, then some τ_i contains ρ ;
- (ii) if $r = 2$, then some τ_i contains ρ , or there are $i < i'$ such that τ_i contains ρ_1 and $\tau_{i'}$ contains ρ_2 .

The same holds with $\oplus$ and sum components in place of $\ominus$ and skew components.

Proof. Fix an occurrence of ρ in π , and let $S_1, \dots, S_t$ ($t \geq 1$) be its nonempty intersections with the blocks, from left to right, lying in the blocks $b_1 < \dots < b_t$. Every entry of an earlier block lies to the left of and above every entry of a later block. Hence $\rho = \sigma_1 \ominus \dots \ominus \sigma_t$, where σ_q is the pattern of the entries in S_q , and τ_{b_q} contains σ_q . Writing each σ_q as the skew sum of its skew components gives a decomposition of ρ into skew-indecomposable permutations, which by the uniqueness in Lemma 2.1(a) is $\rho_1, \dots, \rho_r$. So there are $0 = j_0 < j_1 < \dots < j_t = r$ with $\sigma_q = \rho_{j_{q-1}+1} \ominus \dots \ominus \rho_{j_q}$, and $i_j = b_q$ for $j_{q-1} < j \leq j_q$ has the required property. Parts (i) and (ii) are the cases $r = 1$ and $r = 2$. For $\oplus$, replace “above” by “below”. $\square$

Lemma 2.3. *Every separable permutation of length at least 2 is sum-decomposable or skew-decomposable.*

This is the equivalence of the two descriptions of the separable permutations recalled in [Bri24, p. 290]; we include the short proof.

Proof. By induction on the length n ; for $n = 2$ we have $12 = 1 \oplus 1$ and $21 = 1 \ominus 1$. Let $n \geq 3$, let $\pi \in \text{Av}(2413, 3142)$ have length n , and let π' be obtained from π by deleting the entry n . Then π' is separable of length $n - 1 \geq 2$, so $\pi' = \sigma \ominus \tau$ or $\pi' = \sigma \oplus \tau$ with σ and τ nonempty, and we regard the entries of σ and τ as entries of π .

Let $\pi' = \sigma \ominus \tau$. If n lies to the left of every entry of τ , then the first $|\sigma| + 1$ entries of π are n and the entries of σ , which are the $|\sigma| + 1$ largest values, so $|\sigma| + 1$ is a skew cut of π . Otherwise let T_1 and T_2 be the sets of entries of τ to the left and to the right of n , so that $T_1 \neq \emptyset$. If $T_2 = \emptyset$, then n is the last entry and $n - 1$ is a sum cut. If $T_2 \neq \emptyset$, then every $x \in T_1$ exceeds every $y \in T_2$: otherwise, for any entry s of σ , the entries s, x, n, y would form the pattern 3142 (they appear in this order, and $x < y < s < n$). Hence the entries to the right of n are the $|T_2|$ smallest values, and $n - |T_2|$ is a skew cut.

Let $\pi' = \sigma \oplus \tau$. If n lies to the right of every entry of σ , then $|\sigma|$ is a sum cut. Otherwise let S_1 and S_2 be the sets of entries of σ to the left and to the right of n , so that $S_2 \neq \emptyset$ and every entry of τ lies to the right of n . If $S_1 = \emptyset$, then n is the first entry and 1 is a skew cut. If $S_1 \neq \emptyset$, then every $x \in S_1$ is smaller than every $y \in S_2$: otherwise, for any entry t of τ , the entries x, n, y, t would form 2413 (in this order, with $y < x < t < n$). Hence the entries to the left of n are the $|S_1|$ smallest values, and $|S_1|$ is a sum cut. $\square$

Conversely, 2413 and 3142 are both sum- and skew-indecomposable (for $c = 1, 2, 3$ the sets $\{\pi(1), \dots, \pi(c)\}$ are $\{2\}, \{2, 4\}, \{1, 2, 4\}$ and $\{3\}, \{1, 3\}, \{1, 3, 4\}$, none of the form $\{1, \dots, c\}$ or $\{5 - c, \dots, 4\}$), so by Lemma 2.2(i) the direct and the skew sum of two separable permutations are separable. Together with Lemma 2.3 this shows that the separable permutations are exactly the permutations obtained from 1 by direct and skew sums.

3 The enumeration of $\mathcal{C}$

Throughout, $\mathcal{C} = \text{Av}(2341, 2413, 3142)$. As $\mathcal{C}$ is a class, every permutation contained in a member of $\mathcal{C}$ is a member; in particular the sum and skew components of a member are members. For $c = 1, 2, 3$ the sets $\{\pi(1), \dots, \pi(c)\}$ are $\{2\}, \{2, 3\}, \{2, 3, 4\}$ for $\pi = 2341$, and are listed after Lemma 2.3 for 2413 and 3142; none of them is $\{1, \dots, c\}$. So 2341, 2413 and 3142 are sum-indecomposable.

Lemma 3.1. *If $\sigma, \tau \in \mathcal{C}$, then $\sigma \oplus \tau \in \mathcal{C}$. Consequently a permutation lies in $\mathcal{C}$ if and only if all its sum components do.*

Proof. By Lemma 2.2(i), in its $\oplus$ form, an occurrence of one of the sum-indecomposable permutations 2341, 2413, 3142 in $\sigma \oplus \tau$ would lie in σ or in τ . The second statement follows by induction on the number of components. $\square$

Lemma 3.2. *Let $m \geq 2$ and let $\tau_1, \dots, \tau_m$ be skew-indecomposable. Then $\tau_1 \ominus \dots \ominus \tau_m \in \mathcal{C}$ if and only if $\tau_1, \dots, \tau_m \in \mathcal{C}$ and $\tau_1, \dots, \tau_{m-1}$ avoid 123.*

Proof. Let $\pi = \tau_1 \ominus \dots \ominus \tau_m$. Suppose that $\pi \in \mathcal{C}$. The τ_i are contained in π , so they lie in $\mathcal{C}$. If some τ_i with $i \leq m-1$ contained 123, at entries x, y, w , then for any entry u of τ_{i+1} the entries x, y, w, u would form 2341 in π , since u lies to the right of and below x, y and w .

Conversely, suppose that the conditions hold. As 2413 and 3142 are skew-indecomposable, Lemma 2.2(i) shows that an occurrence of one of them in π would lie in a single block $\tau_i \in \mathcal{C}$, which is impossible. The skew components of $2341 = 123 \ominus 1$ are 123 and 1 ($123 = 1 \oplus 1 \oplus 1$ is sum-decomposable, hence skew-indecomposable). By Lemma 2.2(ii), an occurrence of 2341 in π would lie in a single block, which is impossible, or there would be blocks $i < i'$ such that τ_i contains 123; then $i \leq m-1$, a contradiction. Hence $\pi \in \mathcal{C}$. $\square$

Lemma 3.3. *A permutation is separable, skew-indecomposable and avoids 123 if and only if it is 1 or $\delta_a \oplus \delta_b$ with $a, b \geq 1$. These permutations are pairwise distinct and lie in $\mathcal{C}$, and their generating function is*

$$J(z) = z + \sum_{a,b \geq 1} z^{a+b} = z + \frac{z^2}{(1-z)^2}.$$

Proof. Let τ be separable, skew-indecomposable and 123-avoiding, of length at least 2. By Lemma 2.3, τ is sum-decomposable, $\tau = \alpha_1 \oplus \dots \oplus \alpha_k$ with $k \geq 2$ sum components. One entry from each of three components would form 123, so $k = 2$. If α_1 contained 12, these two entries and any entry of α_2 would form 123; so α_1 is decreasing, $\alpha_1 = \delta_a$. Likewise any entry of α_1 followed by an occurrence of 12 in α_2 would form 123, so $\alpha_2 = \delta_b$.

Conversely, $\delta_a \oplus \delta_b$ is sum-decomposable, hence skew-indecomposable; an increasing subsequence has at most one entry in each of the two decreasing blocks, so it avoids 123; and it avoids the sum-indecomposable permutations 2413 and 3142 by Lemma 2.2(i), because the blocks are decreasing. The first a entries of $\delta_a \oplus \delta_b$ are $a, a-1, \dots, 1$, so the permutation determines a , and there are exactly $n-1$ such permutations of each length $n \geq 2$. Finally, 2341 contains 123, so a permutation avoiding 123 avoids 2341; hence these separable permutations lie in $\mathcal{C}$. $\square$

Proposition 3.4 (specification). *Let $E(z)$ and $I(z)$ be the generating functions of the sum-indecomposable and of the skew-indecomposable members of $\mathcal{C}$, respectively (all of length at least 1), and let J be as in Lemma 3.3. Then*

$$(S1) \quad f = \frac{1}{1-E}, \quad (S2) \quad I = z + \frac{E^2}{1-E}, \quad (S3) \quad E = z + \frac{J}{1-J} I.$$

Proof. (S1) By Lemma 3.1 and the uniqueness in Lemma 2.1(a), the map $(\alpha_1, \dots, \alpha_k) \mapsto \alpha_1 \oplus \dots \oplus \alpha_k$ is a bijection from the finite sequences ($k \geq 0$) of sum-indecomposable members of $\mathcal{C}$ onto $\mathcal{C}$, the empty sequence giving the empty permutation, and lengths add. Hence $f = \sum_{k \geq 0} E^k$.

(S2) A skew-indecomposable member of $\mathcal{C}$ of length at least 2 is sum-decomposable by Lemma 2.3, and every sum-decomposable permutation is skew-indecomposable by Lemma 2.1(b). So the skew-indecomposable members of $\mathcal{C}$ other than 1 are the sum-decomposable members of $\mathcal{C}$, and as in (S1) these correspond to the sequences of $k \geq 2$ sum-indecomposable members of $\mathcal{C}$; their generating function is $\sum_{k \geq 2} E^k = E^2/(1 - E)$.

(S3) In the same way, the sum-indecomposable members of $\mathcal{C}$ other than 1 are the skew-decomposable members of $\mathcal{C}$. By Lemma 2.1(a) and Lemma 3.2, these are exactly the permutations $\tau_1 \ominus \dots \ominus \tau_m$ with $m \geq 2$, where $\tau_1, \dots, \tau_{m-1}$ are skew-indecomposable members of $\mathcal{C}$ that avoid 123 and τ_m is any skew-indecomposable member of $\mathcal{C}$, each permutation arising once. Members of $\mathcal{C}$ are separable, so by Lemma 3.3 the possible $\tau_1, \dots, \tau_{m-1}$ are exactly the permutations counted by J . The generating function is $\sum_{m \geq 2} J^{m-1} I = JI/(1 - J)$. $\square$

Proof of Theorem 1.1(b). Substituting (S2) into (S3) and multiplying by $(1 - J)(1 - E)$ gives $E(1 - E)(1 - J) = z(1 - E) + JE^2$, that is, $E^2 - (1 + z - J)E + z = 0$. Since $1 + z - J = 1 - z^2/(1 - z)^2 = (1 - 2z)/(1 - z)^2$, this is

$$(1 - z)^2 E^2 - (1 - 2z)E + z(1 - z)^2 = 0. \quad (3)$$

By (S1), $E = 1 - 1/f$. Substituting this into (3) and multiplying by f^2 gives (1): the coefficient of f^2 is $(1 - z)^2 - (1 - 2z) + z(1 - z)^2 = z - z^2 + z^3$, that of f is $-2(1 - z)^2 + (1 - 2z) = -(1 - 2z + 2z^2)$, and the constant term is $(1 - z)^2$. Write

$$P = z(1 - z + z^2), \quad Q = 1 - 2z + 2z^2, \quad R = (1 - z)^2.$$

By (1),

$$(Q - 2Pf)^2 = Q^2 - 4P(Qf - Pf^2) = Q^2 - 4PR = D; \quad (4)$$

the last equality is an expansion, with $Q^2 = 1 - 4z + 8z^2 - 8z^3 + 4z^4$ and $4PR = 4z - 12z^2 + 16z^3 - 12z^4 + 4z^5$. Two power series with constant term 1 and the same square are equal (their difference times their sum, which has constant term 2, vanishes). As $Q - 2Pf$ has constant term 1, it is the power series $\sqrt{D}$ of Theorem 1.1, and (2) follows because P is invertible in $\mathbb{Q}((z))$. The coefficients of f , E and I can be computed from (S1)–(S3); the first ones are listed in Table 1. $\square$

n	1	2	3	4	5	6	7	8	9	10
$ \mathcal{C}_n $	1	2	6	21	77	290	1118	4398	17595	71385
sum-indecomposable	1	1	3	10	34	120	440	1665	6460	25570
skew-indecomposable	1	1	3	11	43	170	678	2733	11135	45815

Table 1: The numbers of members of $\mathcal{C}$ of length n , and of the sum- and the skew-indecomposable ones (the coefficients of f , E and I). They agree with a direct count over all permutations of length $n \leq 10$ (Section 6). The next values of $|\mathcal{C}_n|$, for $n = 11, \dots, 15$, are 293042, 1215035, 5081259, 21408350 and 90786332.

Remark 3.5. Callan, Mansour and Shattuck [CMS17, proof of Theorem 11] decompose a member of $\mathcal{C}$ according to its left-to-right maxima and obtain $F - 1 - xF = xF(F - 1) + \frac{x^2}{(1-x)^2}(F - 1)F$ for its generating function $F(x)$. Multiplied by $(1 - x)^2$, this equation becomes $PF^2 - QF + R = 0$ with x in place of z , because $x(1 - x)^2 + x^2 = P$ and $P + (1 - x)^3 = Q$. So the two derivations lead to the same quadratic equation (1).

4 The generating function is not in K

Let $F = \mathbb{Q}(z)$, $a = 1 - 4z$ and $b = 1 - 6z + 5z^2 = (1 - z)(1 - 5z)$, and let $L = F(\sqrt{b})$ and $K = L(\sqrt{a}) = F(\sqrt{a}, \sqrt{b})$, all inside $\mathbb{Q}((z))$. For $c \in F^\times$ we call $cF^{\times 2} = \{cs^2 : s \in F^\times\}$ the *square class* of c .

Lemma 4.1. *Let $c \in \mathbb{Q}[z]$ be nonzero. If some irreducible polynomial $p \in \mathbb{Q}[z]$ divides c with odd multiplicity, then c is not a square in F .*

Proof. Write $\nu_p(h)$ for the multiplicity of p in a nonzero $h \in \mathbb{Q}[z]$. If $c = (u/v)^2$ with $u, v \in \mathbb{Q}[z]$ and $v \neq 0$, then $cv^2 = u^2$, so $\nu_p(c) + 2\nu_p(v) = 2\nu_p(u)$ and $\nu_p(c)$ is even. $\square$

Lemma 4.2. *Let $N \subseteq M$ be fields of characteristic 0, let $d \in N$ be a non-square in N , and suppose $M = N(s)$ with $s^2 = d$. Then $1, s$ is a basis of M over N , and if $x \in M$ satisfies $x^2 \in N$, then $x \in N$ or $x \in Ns$.*

Proof. As $s \notin N$, the minimal polynomial of s over N is $t^2 - d$, so $[M : N] = 2$. Write $x = p + qs$ with $p, q \in N$. Then $x^2 = p^2 + q^2d + 2pq s$, so $x^2 \in N$ forces $pq = 0$. $\square$

Lemma 4.3. $[K : F] = 4$, and $1, \sqrt{a}, \sqrt{b}, \sqrt{a}\sqrt{b}$ is a basis of K over F .

Proof. By Lemma 4.1, b is not a square in F (its factor $1 - z$ has multiplicity 1), so $[L : F] = 2$. Suppose that $a = x^2$ with $x \in L$. By Lemma 4.2 with $N = F$ and $d = b$, either $x \in F$, which is impossible by Lemma 4.1, or $x = q\sqrt{b}$ with $q \in F$; then $ab = (qb)^2$ would be a square in F , which is impossible by Lemma 4.1, since the factor $1 - 4z$ of ab has multiplicity 1. Hence a is not a square in L , and $[K : L] = 2$. $\square$

Proposition 4.4 (Kummer). *If $c \in F^\times$ is a square in K , then c lies in one of the square classes $F^{\times 2}, aF^{\times 2}, bF^{\times 2}$ and $abF^{\times 2}$.*

Proof. Let $x \in K$ with $x^2 = c$. By Lemma 4.2 with $N = L$ and $d = a$ (legitimate by Lemma 4.3), either $x \in L$, or $x = y\sqrt{a}$ with $y \in L$ and $y^2 = c/a \in F$. In the first case, Lemma 4.2 with $N = F$ and $d = b$ gives $x \in F$, so that $c \in F^{\times 2}$, or $x = t\sqrt{b}$ with $t \in F$, so that $c = t^2b$. In the second case the same argument, applied to y , gives $c/a \in F^{\times 2}$ or $c/a \in bF^{\times 2}$. $\square$

Conversely, the elements of these four classes are squares in K ; in the language of Kummer theory, the quadratic extensions of F inside K are $F(\sqrt{a})$, $F(\sqrt{b})$ and $F(\sqrt{ab})$.

Corollary 4.5. *Let $g \in \mathbb{Q}((z))$ satisfy $Ag^2 + Bg + C = 0$ with $A, B, C \in \mathbb{Q}[z]$, $A \neq 0$, and let $\Delta = B^2 - 4AC \neq 0$. Then $g \in K$ if and only if Δ lies in one of the square classes of $1, a, b$ and ab .*

Proof. We have $(2Ag + B)^2 = 4A(Ag^2 + Bg) + B^2 = \Delta$. If $g \in K$, then Δ is a square in K , and Proposition 4.4 applies. Conversely, if $\Delta = cs^2$ with $c \in \{1, a, b, ab\}$ and $s \in F^\times$, then $x = s\sqrt{c} \in K$ (with $\sqrt{ab} = \sqrt{a}\sqrt{b}$) satisfies $x^2 = (2Ag + B)^2$; as $\mathbb{Q}((z))$ is a field, $2Ag + B = \pm x$, and $g = (\pm x - B)/(2A) \in K$. $\square$

Proposition 4.6. *The polynomial D is irreducible over $\mathbb{Q}$, $\gcd(D, ab) = 1$, and D lies in none of the square classes of $1, a, b$ and ab . Consequently D is not a square in K .*

Proof. The polynomial $D = 1 - 8z + 20z^2 - 24z^3 + 16z^4 - 4z^5$ has integer coefficients, and its reduction modulo 5 is

$$\bar{D} = z^5 + z^4 + z^3 + 2z + 1 \in \text{GF}(5)[z],$$

of degree 5, because 5 does not divide the leading coefficient -4 . The values of $\bar{D}$ at $0, 1, 2, 3, 4$ are $1, 1, 1, 3, 3$, so $\bar{D}$ has no root in $\text{GF}(5)$, and it is not divisible by any of the ten monic irreducible

quadratic polynomials over $\text{GF}(5)$ (the polynomials $z^2 + pz + q$ without a root): on division by $z^2 + 2$, $z^2 + 3$, $z^2 + z + 1$, $z^2 + z + 2$, $z^2 + 2z + 3$, $z^2 + 2z + 4$, $z^2 + 3z + 3$, $z^2 + 3z + 4$, $z^2 + 4z + 1$ and $z^2 + 4z + 2$ the remainders are $4z$, $3z$, $2z + 1$, $3z + 4$, $z + 2$, $4z + 2$, $3z + 4$, $3z$, 1 and $2z + 2$. A factorisation of a polynomial of degree 5 has a factor of degree 1 or 2, so $\bar{D}$ is irreducible over $\text{GF}(5)$. If $D = gh$ with $g, h \in \mathbb{Q}[z]$ of positive degree, then by Gauss's lemma we may take $g, h \in \mathbb{Z}[z]$; their leading coefficients divide -4 and are therefore prime to 5, and $\bar{g}\bar{h} = \bar{D}$ would be a factorisation of $\bar{D}$ into factors of positive degree. Hence D is irreducible over $\mathbb{Q}$.

Since $\deg D = 5$ exceeds the degrees of 1, a , b and ab , the irreducible polynomial D divides none of them; in particular $\gcd(D, ab) = 1$. (Directly: $D(1/4) = -17/256$, $D(1) = 1$ and $D(1/5) = 101/3125$.) If $D \in cF^{\times 2}$ for some $c \in \{1, a, b, ab\}$, then $Dc = c^2 \cdot (D/c)$ would be a square in F ; but D divides Dc with multiplicity exactly 1, contrary to Lemma 4.1. The last statement follows from Proposition 4.4. $\square$

Proof of Theorem 1.1(c). By Proposition 4.6, D is irreducible over $\mathbb{Q}$ and coprime to $ab = (1 - 4z)(1 - z)(1 - 5z)$. If f were in K , then $Q - 2Pf \in K$, and by (4) the element $D = (Q - 2Pf)^2$ would be a square in K , contrary to Proposition 4.6. Hence $f \notin K$. (Equivalently, Corollary 4.5 applies to (1).) As D is not a square in F either, $F(f) = F(\sqrt{D})$ is a quadratic extension of F not contained in K . $\square$

Remark 4.7. Only two properties of D were used: D is squarefree, and it is coprime to ab . The same argument works over $\mathbb{C}(z)$, since D has five distinct complex roots (being squarefree, D is coprime to its derivative), none of which is $1/4$, $1/5$ or 1 ; so $f \notin \mathbb{C}(z, \sqrt{1 - 4z}, \sqrt{1 - 6z + 5z^2})$ either.

5 Remarks

5.1 The example of the source, and the word “proper”

Corollary 4.5 decides membership in K for every power series that is algebraic of degree at most 2 over F . The Catalan generating function $u = (1 - \sqrt{1 - 4z})/(2z)$ of $\text{Av}(231)$ satisfies $zu^2 - u + 1 = 0$, with discriminant a . The generating function $g = 2/(1 + z + \sqrt{b})$ of the example $\text{Av}(2143, 2413, 3142)$ of the source satisfies

$$z(2 - z)g^2 - (1 + z)g + 1 = 0$$

(squaring $\sqrt{b} = 2/g - 1 - z$ gives $bg^2 = (2 - (1 + z)g)^2$, and $(1 + z)^2 - b = 4z(2 - z)$), with discriminant $(1 + z)^2 - 4z(2 - z) = b$. So g lies in K , in the class of b ; and as b lies neither in the class of 1 nor in that of a (Lemma 4.1), $g \notin \mathbb{Q}(z, \sqrt{1 - 4z})$ by Lemma 4.2, as the source says. This class contains $\text{Av}(213)$, a symmetry of $\text{Av}(231)$, since each of 2143, 2413 and 3142 contains 213 (at the entries 2, 1, 4, at 2, 1, 3 and at 3, 1, 4). A direct count reproduces its first terms 1, 1, 2, 6, 21, 79, 311, 1265, 5275, 22431, 96900 (Section 6).

The series $s = (1 - z - \sqrt{1 - 6z + z^2})/(2z)$ of [Bri24, p. 290] satisfies $zs^2 - (1 - z)s + 1 = 0$, with discriminant $(1 - z)^2 - 4z = 1 - 6z + z^2$. This polynomial is irreducible over $\mathbb{Q}$ (its roots $3 \pm 2\sqrt{2}$ are irrational) and coprime to ab , so by Lemma 4.1 and Corollary 4.5 the generating function of the separable permutations is not in K : the word “proper” in Question 5 cannot be dropped. For $\mathcal{C}$ the discriminant is D , which lies in none of the four classes; this is Theorem 1.1(c).

5.2 The observation of the source on classes containing $\text{Av}(231)$

Section 2 of [Bri24] reports that a property noted during the workshop, namely that the proper subclasses of the separable permutations which contain $\text{Av}(231)$ or a symmetry of it have generating functions in $\mathbb{Q}(z, \sqrt{1 - 4z})$, does not hold in general. The source asks whether it always holds, that is,

whether every proper subclass has its generating function in $\mathbb{Q}(z, \sqrt{1-4z})$ (by [AAV11, Theorem 4.1] the two forms are equivalent, since the other proper subclasses have rational generating functions), and reports that it does not, because of the class $\text{Av}(2143, 2413, 3142)$, which contains $\text{Av}(213)$. The class $\mathcal{C}$ is consistent with this report: it contains $\text{Av}(231)$ itself, and its generating function is not even in the larger field K . The noted property does hold in examples, such as $\text{Av}(1324, 2413, 3142)$, which contains $\text{Av}(132)$ and whose generating function, listed in [CMS17, Table 1, case 221], lies in $\mathbb{Q}(z, \sqrt{1-4z})$.

5.3 Consistency with known theorems

The rationality theorem of Albert, Atkinson and Vatter [AAV11, Theorem 4.1] ([Bri24, Theorem 2]) does not apply to $\mathcal{C}$, which contains $\text{Av}(231)$, and indeed f is not rational, since D is not a square in F . A permutation is *simple* if the only sets of consecutive positions on which it takes a set of consecutive values are the empty set, the singletons and the set of all positions; by Lemma 2.3 (one of the two blocks of a decomposition has size between 2 and $n-1$ if $n \geq 3$), the only simple separable permutations are 1, 12 and 21. Albert and Atkinson [AA05] proved that a permutation class with only finitely many simple permutations has an algebraic generating function, and Brignall, Huczynska and Vatter [BHV08] refined and extended this result; so every subclass of the separable permutations has an algebraic generating function, and f is algebraic of degree 2.

5.4 One additional basis element

No class $\text{Av}(2413, 3142, \beta)$ with $|\beta| \leq 3$ answers Question 5 negatively. If $|\beta| \leq 2$, the class has at most one permutation of each length, so its generating function is rational. If $\beta \in \{132, 213, 231, 312\}$, then β is contained in both 2413 and 3142 (in 2413 at the entries 2, 4, 3, at 2, 1, 3, at 2, 4, 1 and at 4, 1, 3, and in 3142 at 1, 4, 2, at 3, 1, 4, at 3, 4, 2 and at 3, 1, 2), so the class is $\text{Av}(\beta)$, whose generating function is the Catalan one, in $\mathbb{Q}(z, \sqrt{1-4z}) \subseteq K$. If $\beta \in \{123, 321\}$, then β avoids 132, 213, 231 and 312, so the class contains none of $\text{Av}(132)$, $\text{Av}(213)$, $\text{Av}(231)$ and $\text{Av}(312)$, which are $\text{Av}(231)$ and its symmetries, and its generating function is rational by [AAV11, Theorem 4.1]. Hence $\mathcal{C}$ answers Question 5 with a single additional basis element of the least possible length. For the other β of length 4, PermPAL lists equations for the generating functions of $\text{Av}(\beta, 2413, 3142)$; they are not re-proved or used here.

6 Computations and reproducibility

No proof depends on a computation. The program `check_note.py` of the reproducibility package (Python 3 with numpy and sympy; about ten seconds) tests the statements as printed; it writes only to standard output, and its output is recorded in the package. (i) For $n \leq 10$ it generates all permutations of length n and tests containment of a pattern of length k on every k -subset of positions. It counts the members of $\mathcal{C}$ and their sum- and skew-indecomposable members (Table 1), the separable permutations, $\text{Av}(231)$ (the Catalan numbers, as a control; all of them lie in $\mathcal{C}$), $\text{Av}(2143, 2413, 3142)$, and the permutations of Lemma 3.3. A generating tree, which inserts a new maximum and tests every occurrence of a basis element through it, extends the count of $\mathcal{C}$ to $n = 12$. (ii) It solves (S1)–(S3) as exact power series to order 15, compares them with these counts, and tests each equation separately on the counted series. With sympy it derives (3) and (1) by elimination, checks (1), (4) and the expansion of (2) to order 15, and computes the factorisation of D over $\mathbb{Q}$, the remainders modulo 5 listed in Proposition 4.6, $\gcd(D, ab) = 1$ and the factorisations of Dc for $c \in \{1, a, b, ab\}$; as controls, it places $1-4z$ and $1-6z+5z^2$ in the classes of a and b , and $1-6z+z^2$ in none. It also checks the normalisation stated in Section 1. (iii) It compares the counts of $\text{Av}(2143, 2413, 3142)$ for $n \leq 10$

with the series of $2/(1 + z + \sqrt{1 - 6z + 5z^2})$, and checks the equation and the discriminant given in Section 5. All checks pass.

Verification. The result was first obtained with proofs and with programs (an enumeration of $\mathcal{C}$ up to $n = 13$, a test over all permutations up to $n = 9$, an exact computation of the specification, of D and of its square classes). Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first run re-derived the statement and every proof of that version line by line, compared the cited statements with [Bri24, CMS17, CMS16] and with PermPAL, and checked with its programs the counts of $\mathcal{C}$ up to $n = 14$, the equations (S1)–(S3) separately up to $n = 14$, Lemmas 3.1 to 3.3 on all separable permutations of length at most 8, and the irreducibility of D modulo 5. The second run counted $\mathcal{C}$ with two programs of its own, over all permutations up to $n = 11$ and by a generating tree up to $n = 15$, solved the specification to 60 terms, checked the lemmas on all separable permutations of length at most 9 and the square-class criterion on controls, ran the original programs again (all recorded outputs were reproduced), repeated the literature search and checked the bibliographic data with Crossref. Both runs confirmed Theorem 1.1 and found no gap in the proofs that are given here; the one gap they found, a missing continuation step, was in an analytic second argument, which is omitted. All corrections they required were made: the question is identified as Question 5 of Brignall’s abstract (p. 292) and quoted with both radicals; the normalisation of the generating function of the separable permutations is stated correctly; the algebraicity theorem is attributed to Albert and Atkinson [AA05], with [BHV08] as the refinement; the source’s own report on the noted property is described accurately (Section 5.2); the list of checked sources was completed; and the analytic argument is omitted, together with an incorrect side remark on growth rates. When the note was written, the proof of Lemma 2.3 (which the first version took from the literature), Corollary 4.5, the present form of Section 5 and the program `check_note.py` were added after the two runs.

A third independent verification run, also AI-assisted, then examined the text of this note and the package as if nothing had been checked before, beginning with the additions just listed. It read the report again (Brignall’s abstract [Bri24] and Vatter’s abstract, pp. 282–284), the statement and the proof of [CMS17, Theorem 11] with Table 1, [CMS16, Theorem 41], the PermPAL entry, the abstract and Example 6.4 of [BHV08] and [AAV11, Theorem 4.1], and re-derived all proofs line by line. Its own programs, written independently of `check_note.py` and of the programs of the earlier runs, count $\mathcal{C}$ by a generating tree that adds a new first entry, up to $n = 16$ (they find $|\mathcal{C}_{16}| = 387212538$, the coefficient of z^{16} in f), and over all permutations up to $n = 11$, together with the sum- and skew-indecomposable members; they test (S1)–(S3) separately on these counts and (1), (3), (4) and (2) on exact series to order 40; they decide the square classes with their own exact polynomial arithmetic over $\mathbb{Q}$ and $\text{GF}(5)$, reproduce the data printed in Proposition 4.6 and test sixteen positive and negative controls; and they test Lemmas 2.1 to 2.3 and 3.1 to 3.3, the statements of Sections 1 and 5 and the printed occurrences of patterns on all permutations of length at most 8 and all separable permutations of length at most 9. The run also re-ran `check_note.py` from the source archive, with the recorded output reproduced apart from the running time, checked the bibliographic data with Crossref and the arXiv API, and repeated the literature search. It found no mathematical error. The corrections it required were made: the abstract speaks of a nonzero element of $\mathbb{Q}(z)$; the description of the first two runs above was made accurate; the rationality theorem is cited as [AAV11, Theorem 4.1]; two checks of `check_note.py` now test the square class itself, as described in this section; a phrase of Section 5 was made precise; and this run is recorded here and in the package.

Scope and priority. The note answers Question 5 of Brignall’s abstract [Bri24, p. 292] negatively, by one explicit class; it does not characterise the proper subclasses whose generating functions lie in

K , and it does not treat Question 1 and Conjecture 3 of Brignall’s abstract or the questions of Vatter’s abstract (pp. 283–284). The question is due to R. Brignall, with M. Bóna, C. Defant, M. Opler and V. Vatter [Bri24]. The enumeration of $\mathcal{C}$ is due to D. Callan, T. Mansour and M. Shattuck [CMS17, Theorem 11] ([CMS16, Theorem 41]), and it is in the PermPAL database [Per26]. The contribution of this note is the observation that this class answers Question 5 negatively, a self-contained proof of its enumeration, and an exact certificate that its generating function is not in K . No printed answer to Question 5 was found in the sources searched on 16 October 2026: the report [Bri24]; the works citing it (Semantic Scholar lists one, [HS25], on permuton limits; OpenAlex and Crossref list none); the paper [BOV26], which comes from the same workshop; the problem collection [Vat26]; [BHV08], whose Example 6.4 treats only the source’s example; [CMS17, CMS16]; the programme of the conference Permutation Patterns 2025; queries to the arXiv API, to zbMATH Open, to Crossref, to the OEIS (the counting sequence of $\mathcal{C}$ is not there) and to PermPAL; and nine web searches. The third verification run repeated the search on the same day (the arXiv API, Crossref, OpenAlex, zbMATH Open, Semantic Scholar, the OEIS and one of these web searches). A search that finds nothing is not a proof of novelty, and no priority is claimed. What was read: Brignall’s and Vatter’s abstracts in [Bri24]; Section 3.3 and Table 1 of [CMS17]; Section 3.12 of [CMS16]; the abstract and Example 6.4 of the arXiv version of [BHV08]; the abstract and Theorem 4.1 of the arXiv version of [AAV11]; the texts of [Vat26, HS25, BOV26], searched for the question. The paper [AA05] was not read; its theorem is used as stated in the abstract of [BHV08].

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AA05] M. H. Albert and M. D. Atkinson. Simple permutations and pattern restricted permutations. *Discrete Mathematics*, 300(1–3):1–15, 2005. doi:10.1016/j.disc.2005.06.016.
- [AAV11] Michael H. Albert, M. D. Atkinson, and Vincent Vatter. Subclasses of the separable permutations. *Bulletin of the London Mathematical Society*, 43(5):859–870, 2011. arXiv:1007.1014. doi:10.1112/blms/bdr022.
- [ABC⁺22] Michael H. Albert, Christian Bean, Anders Claesson, Émile Nadeau, Jay Pantone, and Henning Ulfarsson. Combinatorial Exploration: An algorithmic framework for enumeration, 2022. arXiv:2202.07715.
- [BHV08] Robert Brignall, Sophie Huczynska, and Vincent Vatter. Simple permutations and algebraic generating functions. *Journal of Combinatorial Theory, Series A*, 115(3):423–441, 2008. arXiv:math/0608391. doi:10.1016/j.jcta.2007.06.007.
- [BOV26] Robert Brignall, Michal Opler, and Vincent Vatter. Linear clique-width and modular decomposition, 2026. arXiv:2602.22089.
- [Bri24] Robert Brignall. Enumeration in subclasses of the separable permutations, and related questions. In *Mini-Workshop: Permutation Patterns (organised by Miklós Bóna, Mathilde Bouvel, Robert Brignall and Jay Pantone)*, Oberwolfach Report 6/2024, volume 21 of *Oberwolfach Reports*, pages 290–293. EMS Press, 2024. Joint work with M. Bóna, C. Defant, M. Opler and V. Vatter. doi:10.4171/OWR/2024/6.

- [CMS16] David Callan, Toufik Mansour, and Mark Shattuck. Wilf classification of triples of 4-letter patterns, 2016. Full-length version, v1 of 16 May 2016. [arXiv:1605.04969](#).
- [CMS17] David Callan, Toufik Mansour, and Mark Shattuck. Wilf classification of triples of 4-letter patterns II. *Discrete Mathematics & Theoretical Computer Science*, 19(1), 2017. Paper no. 6. doi:10.23638/DMTCS-19-1-6.
- [HS25] Kaitlyn Hohmeier and Erik Slivken. Permuton limits for some permutations avoiding a single pattern, 2025. [arXiv:2502.19541](#).
- [OEI26] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. <https://oeis.org>, 2026. Sequences A006318 and A033321; accessed 16 October 2026.
- [Per26] PermPAL. PermPAL: The Permutation Pattern Avoidance Library. <https://permpal.com>, 2026. Entry for $\text{Av}(1432, 2413, 3142)$, reached from https://permpal.com/perms/basis/2341_2413_3142/; accessed 16 October 2026.
- [Vat26] Vincent Vatter. An assortment of problems in permutation patterns: unimodality, equivalence, derangements, and sorting, 2026. v1 of 18 February 2026, v2 of 25 February 2026. [arXiv:2602.16355](#).