

Sharp Growth Exponents for Positive Robin Eigenvalues at Fixed Volume

Alper Ferudun*

26 October 2026

Abstract

Fix a positive Robin parameter α and a volume M . We prove that the k th Robin eigenvalue of every bounded convex domain in $\mathbb{R}^n$ with volume M is at least $ak^{2/(2n-1)}$, with $a > 0$ depending only on n, α, M . The proof combines a one-dimensional trace estimate, a John ellipsoid, and a finite-dimensional counting argument using the convex-domain Poincaré inequality. For general bounded Lipschitz sets, the bound $bk^{1/n}$ follows directly from the established heat-trace estimate of van den Berg and Bucur; we record the constants and this attribution explicitly. Thin boxes and unions of equal balls show that neither exponent can be increased uniformly over its respective class. These statements answer the Euclidean Robin conjecture printed on p. 2094 of Oberwolfach Report 36/2023 in the standard bounded-domain setting. The separate question about manifolds on the preceding page is not addressed. No optimal-constant or historical-priority claim is made.

1 The question and the result

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz open set, possibly disconnected, and let $\alpha > 0$. Write

$$q_{\Omega, \alpha}[u] = \int_{\Omega} |\nabla u|^2 dx + \alpha \int_{\partial\Omega} |u|^2 dS, \quad u \in H^1(\Omega),$$

and let $0 < \rho_1(\Omega, \alpha) \leq \rho_2(\Omega, \alpha) \leq \dots$ be the associated eigenvalues, counted with multiplicity. The outward-normal boundary condition is $\partial_\nu u + \alpha u = 0$. A bounded open convex set with nonempty interior is Lipschitz and is included in this framework. Set $\omega_n = |B_1(0)|$.

Freitas's problem-session contribution in [1, p. 2094], motivated by the rectangle and union-of-rectangles results of Freitas–Kennedy [2], asks for positive constants a, b , depending only on n, α, M , such that

$$\rho_k(\Omega, \alpha) \geq ak^{2/(2n-1)} \quad \text{for convex } \Omega, \quad \rho_k(\Omega, \alpha) \geq bk^{1/n} \quad \text{in general,} \quad |\Omega| = M. \quad (1)$$

The frozen UnsolvedMath v1.6.0 entry `OWR-14297736-026` merges this conjecture with the preceding, separate question about Pólya-type inequalities on manifolds. We treat precisely (1), not that additional question.

Theorem 1. *For every $n \geq 2$, $\alpha > 0$ and $M > 0$, both inequalities (1) hold for all $k \geq 1$, with the general-domain inequality understood for bounded Lipschitz open sets. Both exponents are optimal for these classes. The constants may be chosen explicitly as in (8) and (10) below. The corresponding assertions for $n = 1$ also hold.*

*Mercury Software GmbH. alper@mercurycodelab.com; github.com/AlperTheKing.

The new argument supplied here for the convex estimate uses classical ingredients. The grid and Poincaré counting method is standard; compare Gittins–Léna [4, Appendix A]. The role of the Robin boundary term is to prevent every short John axis from being too small at the spectral scale. Keeping one long axis distinct gives the exponent $n - \frac{1}{2}$ for the counting function. The general-domain half is an explicit consequence of van den Berg–Bucur [3, Lemmas 3–5 and Proposition 7], not a claim of a new heat-kernel bound. We determine orders of growth, not extremizing shapes or leading asymptotic constants. All assertions about sharpness concern domains which are allowed to vary with k .

2 A width estimate and a counting lemma

Lemma 2 (A slab estimate). *If a bounded open convex set Ω is contained between two parallel hyperplanes a distance $w > 0$ apart, then*

$$\rho_1(\Omega, \alpha) \geq \min\{w^{-2}, \alpha/w\}. \quad (2)$$

Proof. For $v \in H^1(0, \ell)$, the fundamental theorem of calculus and Cauchy–Schwarz give

$$\int_0^\ell |v(x)|^2 dx \leq 2\ell |v(0)|^2 + \ell^2 \int_0^\ell |v'(x)|^2 dx.$$

Applying the same argument from the other endpoint and averaging yields

$$\int_0^\ell |v|^2 \leq \ell(|v(0)|^2 + |v(\ell)|^2) + \ell^2 \int_0^\ell |v'|^2. \quad (3)$$

Slice Ω along lines normal to the two hyperplanes. Almost every nonempty slice is an interval of length $\ell \leq w$. Integrating (3), the endpoint terms become $\int_{\partial\Omega} |u|^2 |\nu \cdot e| dS$, where e is the slicing direction. Consequently,

$$\|u\|_2^2 \leq w^2 \int_\Omega |\nabla u|^2 + w \int_{\partial\Omega} |u|^2 \leq \max\{w^2, w/\alpha\} q_{\Omega, \alpha}[u].$$

The slicing calculation first applies to smooth functions and extends to $H^1(\Omega)$ by density and continuity of the trace. The variational principle proves (2). $\square$

Lemma 3 (Counting inside a box). *Suppose that a bounded open convex set Ω is contained in an orthogonal box with side lengths $L_1, \dots, L_n$. For $\lambda > 0$, put*

$$N_{\Omega, \alpha}(\lambda) = \#\{j : \rho_j(\Omega, \alpha) \leq \lambda\}, \quad h = \frac{\pi}{2\sqrt{n}\sqrt{\lambda}}.$$

Then

$$N_{\Omega, \alpha}(\lambda) \leq \prod_{i=1}^n \left(\frac{L_i}{h} + 2 \right). \quad (4)$$

Proof. Use a grid of cubes of side h parallel to the box axes, and retain the cubes whose open interiors meet Ω . Their intersections $U_1, \dots, U_m$ with Ω are convex, have diameter at most $\sqrt{n}h$, and partition Ω up to a null set. Each coordinate interval of length L_i meets at most $L_i/h + 2$ grid intervals, so m is no greater than the product in (4).

If $N_{\Omega, \alpha}(\lambda) > m$, there is a nonzero function u in the span of the Robin eigenfunctions with eigenvalue at most λ such that $\int_{U_j} u = 0$ for every j . The Payne–Weinberger inequality [5] on each U_j gives

$$\int_{U_j} |u|^2 \leq \frac{\text{diam}(U_j)^2}{\pi^2} \int_{U_j} |\nabla u|^2 \leq \frac{1}{4\lambda} \int_{U_j} |\nabla u|^2.$$

Summing and using positivity of the Robin boundary term gives

$$\|u\|_2^2 \leq \frac{q_{\Omega,\alpha}[u]}{4\lambda} \leq \frac{1}{4} \|u\|_2^2,$$

a contradiction. This argument counts multiplicities and also works when λ is itself an eigenvalue. $\square$

3 The convex-domain bound

Translate and rotate a John ellipsoid of $\bar{\Omega}$ to have semiaxes $0 < a_1 \leq \dots \leq a_n$ and center zero. John's theorem [6, 7] gives

$$E \subset \bar{\Omega} \subset nE.$$

Only an orthogonal change of coordinates is used in the spectral argument; no anisotropic change of the Laplacian is made. Put

$$R = (M/\omega_n)^{1/n}, \quad \lambda_0 = \min\{(2nR)^{-2}, \alpha/(2nR)\}.$$

The volume inclusions imply

$$R^n/n^n \leq \prod_{i=1}^n a_i \leq R^n, \quad a_1 \leq R, \quad a_n \geq R/n. \quad (5)$$

In particular, Ω lies in a slab of width $2na_1 \leq 2nR$, so Lemma 2 gives $\rho_1 \geq \lambda_0$.

Fix $k \geq 1$ and set $\lambda = \rho_k(\Omega, \alpha)$. Applying Lemma 2 to the slab of width $2na_1$ gives

$$\lambda \geq \min\{(2na_1)^{-2}, \alpha/(2na_1)\},$$

and therefore

$$a_1 \geq \frac{1}{2n} \min\{\lambda^{-1/2}, \alpha/\lambda\}. \quad (6)$$

Indeed, if $2na_1$ were smaller than both quantities in this minimum, both terms in the preceding lower bound would exceed λ .

Define

$$\begin{aligned} B &= \frac{4n\sqrt{n}}{\pi}, & D &= \max\{\lambda_0^{-1/2}, \alpha^{-1}\}, \\ C_1 &= B + \frac{2n}{R\sqrt{\lambda_0}}, & C_2 &= \frac{B}{\sqrt{\lambda_0}} + 4nD, & A &= R^n C_1 C_2^{n-1}. \end{aligned} \quad (7)$$

Since $\lambda \geq \lambda_0$, (6) implies, for every i ,

$$a_i^{-1} \leq 2n \max\{\sqrt{\lambda}, \lambda/\alpha\} \leq 2nD\lambda.$$

The box circumscribed about nE has sides $L_i = 2na_i$. Lemma 3 and (5) now give

$$\begin{aligned} k &\leq \prod_{i=1}^n (2na_i/h + 2) = \left(\prod_{i=1}^n a_i \right) \prod_{i=1}^n (B\sqrt{\lambda} + 2/a_i) \\ &\leq R^n \left(B + \frac{2n}{R\sqrt{\lambda_0}} \right) \sqrt{\lambda} \left(\frac{B}{\sqrt{\lambda_0}} + 4nD \right)^{n-1} \lambda^{n-1} = A\lambda^{n-1/2}. \end{aligned}$$

The factor estimated at order $\sqrt{\lambda}$ is the one corresponding to a_n ; it uses $a_n \geq R/n$, not a lower bound on all axes by R/n . We conclude that

$$\rho_k(\Omega, \alpha) \geq A^{-2/(2n-1)} k^{2/(2n-1)}. \quad (8)$$

Thus $a = A^{-2/(2n-1)}$ is an explicit choice depending only on the three fixed parameters.

4 The general-domain bound from a known heat-trace theorem

Here $n \geq 2$, and Ω is any bounded Lipschitz open set with volume M . Write

$$c_n = \frac{1}{n\omega_n^{1/n}}, \quad s = \frac{M^{-1/n}}{c_n}, \quad \gamma = \left(\sqrt{\alpha^2 + \alpha s} - \alpha\right)^2 > 0.$$

The BV Sobolev inequality applied to the zero extension of $|u|^2$ is [3, Lemma 3]

$$\|u\|_{2n/(n-1)}^2 \leq c_n \left(2\|u\|_2 \|\nabla u\|_2 + \int_{\partial\Omega} |u|^2 dS \right). \quad (9)$$

Hölder's inequality bounds the left side from below by $M^{-1/n} \|u\|_2^2$. For any nonzero u , let $t = q_{\Omega,\alpha}[u] / \|u\|_2^2$. It follows that

$$s \leq 2\sqrt{t} + t/\alpha.$$

Solving this quadratic inequality in $\sqrt{t}$ shows $\rho_1(\Omega, \alpha) \geq \gamma$, without a shape-dependent constant. Set

$$N_0 = c_n \left(\alpha^{-1} + \frac{\alpha}{\gamma} \right).$$

Van den Berg–Bucur's Proposition 7, with constants from their equations (7) and (15), states

$$\sum_{j=1}^{\infty} e^{-\tau\rho_j} \leq (192n)^n N_\alpha^n M \tau^{-n}, \quad \tau > 0, \quad N_\alpha = c_n(\alpha^{-1} + \alpha/\rho_1).$$

Thus $N_\alpha \leq N_0$. For $\lambda = \rho_k$, use $ke^{-\tau\lambda} \leq \sum_j e^{-\tau\rho_j}$ and choose $\tau = n/\lambda$. This yields

$$k \leq (192eN_0)^n M \lambda^n$$

and hence

$$\rho_k(\Omega, \alpha) \geq \frac{k^{1/n}}{192eN_0 M^{1/n}}. \quad (10)$$

All trace and Sobolev operations above are standard on bounded Lipschitz sets. More singular open sets require specifying a Robin form domain; no convention about traces on cracks is implicit here.

5 Optimality of the exponents

Proposition 4. *For $n \geq 2$, fixed $\alpha, M > 0$ and each $k \geq 1$, there is a rectangular box Q_k of volume M such that*

$$\rho_k(Q_k, \alpha) \leq (2n-1)\pi^{2/(2n-1)} \alpha^{(2n-2)/(2n-1)} M^{-2/(2n-1)} k^{2/(2n-1)}. \quad (11)$$

There is also a disjoint union B_k of k equal balls with total volume M such that

$$\rho_k(B_k, \alpha) \leq n\alpha(\omega_n/M)^{1/n} k^{1/n}. \quad (12)$$

Proof. Let $Q = (0, L) \times (0, t)^{n-1}$, with $L = M/t^{n-1}$. Use the k -dimensional trial space of functions constant in the last $n-1$ variables and spanned in the first by $\sin(j\pi x_1/L)$, $1 \leq j \leq k$. The two end faces contribute zero boundary energy, and the other faces give $2(n-1)\alpha/t$ times the squared L^2 norm. Therefore

$$\rho_k(Q, \alpha) \leq \frac{\pi^2 k^2 t^{2n-2}}{M^2} + \frac{2(n-1)\alpha}{t}.$$

Choosing $t = (\alpha M^2/(\pi^2 k^2))^{1/(2n-1)}$ proves (11). It is not necessary to assert that this trial space consists of Robin eigenfunctions or that Q_k is a minimizer.

For the union of balls, use functions constant on each component. Each ball has radius $r = (M/(k\omega_n))^{1/n}$ and the entire k -dimensional trial space has Rayleigh quotient $n\alpha/r$. This proves (12), a standard comparison already used in Antunes–Freitas–Kennedy [8]. For either class, a uniform lower bound with a larger power of k would contradict these upper bounds as $k \rightarrow \infty$. $\square$

6 The one-dimensional case

In dimension one a bounded open convex set is an interval of length M . Lemma 2, or directly (3), gives $\rho_1 \geq \eta := \min\{M^{-2}, \alpha/M\}$. For $k \geq 2$, comparison with Neumann eigenvalues gives $\rho_k \geq \pi^2(k-1)^2/M^2 \geq \pi^2 k^2/(4M^2)$. Thus $\min\{\eta, \pi^2/(4M^2)\}k^2$ is a uniform lower bound.

For a finite union of intervals with total length M , a component of length ℓ can contribute to the spectrum below λ only if $\ell \geq \min\{\lambda^{-1/2}, \alpha/\lambda\}$, by (3). Neumann comparison bounds the number contributed by that component by $1 + \ell\sqrt{\lambda}/\pi$. Also every component has length at most M , so $\rho_1 \geq \eta$. For $\lambda = \rho_k \geq \eta$ we obtain

$$k \leq M \max\{\sqrt{\lambda}, \lambda/\alpha\} + \frac{M\sqrt{\lambda}}{\pi} \leq M \left((1 + \pi^{-1})\eta^{-1/2} + \alpha^{-1} \right) \lambda.$$

This proves the general exponent 1. Dirichlet comparison on one interval gives the matching k^2 upper order, while constants on k equal intervals give the upper bound $2\alpha k/M$. This completes Theorem 1.

The manuscript and supplementary checks were developed with AI assistance and checked by the originating authoring system. The written analytic argument is the proof; the numerical and exact arithmetic diagnostics are supplementary. This preprint has not undergone independent peer review or formal verification.

References

- [1] P. Freitas, Pólya-type inequalities, in *Geometric Spectral Theory*, Oberwolfach Reports **20** (2023), Report 36, 2017–2106, problem-session contribution pp. 2093–2094. doi:10.4171/owr/2023/36.
- [2] P. Freitas and J. B. Kennedy, Extremal domains and Pólya-type inequalities for the Robin Laplacian on rectangles and unions of rectangles, *Int. Math. Res. Not. IMRN* (2021), no. 18, 13730–13782. doi:10.1093/imrn/rnz204.
- [3] M. van den Berg and D. Bucur, On the torsion function with Robin or Dirichlet boundary conditions, *J. Funct. Anal.* **266** (2014), 1647–1666. doi:10.1016/j.jfa.2013.07.007.
- [4] K. Gittins and C. Léna, Upper bounds for Courant-sharp Neumann and Robin eigenvalues, arXiv:1810.09950, Appendix A; the updated version includes a corrigendum in Appendix B.
- [5] L. E. Payne and H. F. Weinberger, An optimal Poincaré inequality for convex domains, *Arch. Rational Mech. Anal.* **5** (1960), 286–292. doi:10.1007/BF00252910.
- [6] F. John, Extremum problems with inequalities as subsidiary conditions, in *Studies and Essays Presented to R. Courant on his 60th Birthday*, Interscience, New York, 1948, 187–204.
- [7] R. Howard, *The John Ellipsoid Theorem*, lecture notes, University of South Carolina, November 1997, Theorem 1. <https://people.math.sc.edu/howard/Notes/john.pdf>.
- [8] P. R. S. Antunes, P. Freitas and J. B. Kennedy, Asymptotic behaviour and numerical approximation of optimal eigenvalues of the Robin Laplacian, *ESAIM Control Optim. Calc. Var.* **19** (2013), 438–459. doi:10.1051/cocv/2012016.