

Brooms Can Be Easier than Stars in Tree-Flow Least Squares

Alper Ferudun*

21 October 2026

Abstract

For Gaussian denoising of nonnegative leaking flows, consider least squares over the full flow cone, with a fixed bound on the true root flow. We construct an N -vertex broom whose worst-case normalized risk is $O(\sqrt{\log N}/N)$, while the N -vertex rooted star has risk $\Omega(\log N/N)$ already at the zero signal. The ratio of the two worst-case risks tends to zero. Thus the star is not the easiest tree for this estimator when logarithmic factors are retained. The proof combines a deterministic cone-projection identity with an elementary bound for the maximum of Gaussian leaf noise. For context, we also reconstruct the known parallel-arm obstruction to the path being hardest, using a nonnegative witness. The comparisons concern the full-cone estimator, not a budget-constrained projection or minimax risk over all estimators.

1 The model and the comparison

Let T be a finite rooted tree on N vertices. Its cone of nonnegative leaking flows is

$$C_T = \left\{ \mu \in \mathbb{R}^N : \mu_v \geq \sum_{w \in \text{ch}(v)} \mu_w \text{ for every } v \right\}.$$

The empty sum is zero, so the leaf values, and then all values, are nonnegative. We observe $Y = \mu + \sigma G$, where $G \sim N(0, I_N)$ and $\sigma > 0$. The least-squares estimator is

$$\hat{\mu}_T(Y) = \Pi_{C_T} Y.$$

For $V > 0$ define its worst-case normalized risk by

$$\mathcal{R}_T(V, \sigma) = \sup_{\substack{\mu \in C_T \\ \mu_{\text{root}} \leq V}} \frac{1}{N} \mathbb{E}_\mu \|\hat{\mu}_T(Y) - \mu\|^2. \quad (1)$$

Crucially, the constraint $\mu_{\text{root}} \leq V$ restricts the truth, not the projection. This is the cone estimator of Chatterjee and Lafferty [1, eqs. (1.4)–(1.6)].

Lafferty’s report [2, p. 766] asks whether the path is the hardest tree-flow estimation problem and the star the easiest. The parallel-arm family of [1, Theorem 2.12] rules out the path endpoint. Our main comparison concerns the star endpoint.

Write $B(h, m)$ for a broom with a trunk of h vertices, rooted at one end, and m additional leaves attached to the other end. Thus $B(h, m)$ has $h + m$ vertices. Let S_N be the star on N vertices, rooted at its hub. All logarithms are natural.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing). Unrefereed preprint. CC BY 4.0.

Theorem 1. For every pair of positive integers h, m ,

$$\mathcal{R}_{B(h,m)}(V, \sigma) \leq \frac{(h+1)V^2 + 2\sigma^2 h + 4\sigma^2(\log m + 1)/h}{h+m}. \quad (2)$$

For $N \geq 3$, put

$$h_N = \lceil \sqrt{\log N} \rceil, \quad B_N = B(h_N, N - h_N).$$

Then

$$\mathcal{R}_{B_N}(V, \sigma) \leq (3V^2 + 12\sigma^2) \frac{\sqrt{\log N}}{N}. \quad (3)$$

On the other hand, for all sufficiently large N ,

$$\mathcal{R}_{S_N}(V, \sigma) \geq \frac{\sigma^2 \log(N-1)}{8N}. \quad (4)$$

Consequently, for every fixed $V, \sigma > 0$,

$$\lim_{N \rightarrow \infty} \frac{\mathcal{R}_{B_N}(V, \sigma)}{\mathcal{R}_{S_N}(V, \sigma)} = 0. \quad (5)$$

The trees in the comparison have exactly the same number of vertices. The upper bound is uniform over every allowed signal on the broom. The lower bound uses one allowed signal on the star, which suffices for its worst-case risk. The separation is logarithmic, not a change in the leading polynomial exponent. It therefore makes no distinction at a resolution that discards all logarithmic factors.

2 Two cone-projection facts

We give the elementary ingredients to keep the star comparison self-contained. For a closed convex cone $C \subset \mathbb{R}^N$, let

$$\delta(C) = \mathbb{E} \|\Pi_C G\|^2, \quad G \sim N(0, I_N).$$

Lemma 2. For every $z \in \mathbb{R}^N$,

$$\|\Pi_C z\| = \sup_{\substack{v \in C \\ \|v\| \leq 1}} \langle z, v \rangle, \quad (6)$$

$$\sup_{v \in C} \{2\langle z, v \rangle - \|v\|^2\} = \|\Pi_C z\|^2. \quad (7)$$

For any $\mu, z \in \mathbb{R}^N$,

$$\|\Pi_C(\mu + z) - \mu\|^2 \leq \|\mu\|^2 + \|\Pi_C z\|^2. \quad (8)$$

Proof. If $p = \Pi_C y$, the projection variational inequality, tested at 0 and $2p$, gives $\langle y - p, p \rangle = 0$. Testing at any $v \in C$ then gives $\langle y - p, v \rangle \leq 0$. Hence $\langle y, v \rangle \leq \langle p, v \rangle \leq \|p\| \|v\|$. Equality in (6) is achieved by $v = p/\|p\|$ when $p \neq 0$, and by $v = 0$ when $p = 0$. Optimizing over the length of v proves (7). Now take $p = \Pi_C(\mu + z)$. Orthogonality gives

$$\|p - \mu\|^2 = \|\mu\|^2 + 2\langle z, p \rangle - \|p\|^2,$$

and (7) proves (8). □

Lemma 3. *If $g_1, \dots, g_m$ are independent standard normal variables and $M_+ = \max(0, g_1, \dots, g_m)$, then*

$$\mathbb{E}M_+^2 \leq 2 \log m + 2. \quad (9)$$

Proof. The Gaussian Chernoff inequality and a union bound give $\mathbb{P}(M_+^2 > u) \leq \min(1, me^{-u/2})$ for $u \geq 0$. Integration, splitting at $2 \log m$, proves the claim. This also covers $m = 1$, when the splitting point is zero. $\square$

3 The broom upper bound

A flow on $B(h, m)$ has trunk coordinates $t_1, \dots, t_h$ and leaf coordinates $x_1, \dots, x_m$ satisfying

$$t_1 \geq \dots \geq t_h \geq s := \sum_{j=1}^m x_j, \quad x_j \geq 0. \quad (10)$$

In particular,

$$\|\mu\|^2 \leq (h+1)V^2 \quad \text{if } \mu \in C_{B(h,m)}, \quad \mu_{\text{root}} \leq V, \quad (11)$$

because the trunk contributes at most hV^2 and the leaves contribute at most $(\sum_j x_j)^2 \leq V^2$.

Write a standard Gaussian vector as $G = (a, b) \in \mathbb{R}^h \times \mathbb{R}^m$ and put $M_+ = \max(0, b_1, \dots, b_m)$. For any flow (t, x) of norm at most one, (10) implies $hs^2 \leq \|t\|^2$, and therefore

$$\langle G, (t, x) \rangle \leq \|a\| \|t\| + M_+ s \leq \left(\|a\| + \frac{M_+}{\sqrt{h}} \right) \|t\| \leq \|a\| + \frac{M_+}{\sqrt{h}}.$$

Applying (6), then $(u+v)^2 \leq 2u^2 + 2v^2$ and Lemma 3, yields

$$\delta(C_{B(h,m)}) \leq 2h + \frac{2}{h} \mathbb{E}M_+^2 \leq 2h + \frac{4(\log m + 1)}{h}. \quad (12)$$

Since projection onto a cone is positively homogeneous, (8), (11), and (12) prove (2).

For $N \geq 3$ the stated choice of h_N satisfies $1 \leq h_N < N$ and

$$h_N + 1 \leq 3\sqrt{\log N}, \quad 2h_N \leq 4\sqrt{\log N}, \quad \frac{\log(N - h_N) + 1}{h_N} \leq 2\sqrt{\log N}.$$

Substitution proves (3). This deliberately coarse estimate is sufficient for the comparison; no optimal broom rate is asserted.

4 The star lower bound

Let $q = N - 1$ and $M = \max_{1 \leq j \leq q} G_j$, with coordinate zero reserved for the root. Each vector $(e_0 + e_j)/\sqrt{2}$ is a unit-norm star flow. Including zero in the support supremum gives

$$\|\Pi_{C_{S_N}} G\| \geq \frac{(G_0 + M)_+}{\sqrt{2}}.$$

For $q \geq 2$, the independent events $G_0 \geq 0$ and $M \geq \sqrt{\log q}$ therefore imply

$$\delta(C_{S_N}) \geq \frac{\log q}{4} \left(1 - \Phi(\sqrt{\log q})^q \right), \quad (13)$$

where Φ is the standard Gaussian distribution function.

For completeness, the parenthesis tends to one. If $t \geq 1$, integration of the Gaussian density on $[t, t + 1/t]$ gives

$$1 - \Phi(t) \geq \frac{e^{-3/2}}{\sqrt{2\pi}t} e^{-t^2/2}.$$

With $t = \sqrt{\log q}$ this implies $q(1 - \Phi(t)) \rightarrow \infty$, and hence $\Phi(t)^q \leq \exp(-q(1 - \Phi(t))) \rightarrow 0$. Thus (13) is at least $(\log q)/8$ for all sufficiently large q .

The zero signal is admissible for every $V > 0$. At zero the normalized risk is exactly $\sigma^2 \delta(C_{S_N})/N$, proving (4). Together with (3), this proves (5) and completes the proof of Theorem 1.

5 The known path obstruction

We recall the parallel-arm mechanism of Chatterjee and Lafferty [1, Theorem 2.12 and Section 3.10] with a nonnegative witness and explicit finite dimensions. This section is a reconstruction of a known rate, not a separate novelty claim.

Let $A(m, \ell)$ have one root and m disjoint arms, each containing ℓ vertices, so $N = 1 + m\ell$. Let $Z = \sigma G$ and let S_i be its coordinate sum on arm i . Set $a = (m^2 + m\ell)^{-1/2}$. Define a random vector v by

$$v_{\text{root}} = ma, \quad v_{i,j} = a \mathbf{1}_{\{S_i \geq 0\}} \quad (1 \leq i \leq m, 1 \leq j \leq \ell).$$

It is a nonnegative flow and $\|v\|^2 \leq a^2(m^2 + m\ell) = 1$. Since $S_i \sim N(0, \ell\sigma^2)$,

$$\mathbb{E}\langle Z, v \rangle = am \mathbb{E}(S_1)_+ = am\sigma \sqrt{\frac{\ell}{2\pi}}.$$

The root term has expectation zero. At zero truth, (6) and Jensen's inequality give

$$\mathcal{R}_{A(m,\ell)}(V, \sigma) \geq \frac{\sigma^2 m\ell}{2\pi(m + \ell)(1 + m\ell)}. \quad (14)$$

For $m = \ell = k$ this is $\sigma^2 k/[4\pi(1 + k^2)]$, of order $N^{-1/2}$. The usual isotonic cone LSE bound, recalled in [1, eq. (1.8)], gives for the N -vertex path P_N

$$\mathcal{R}_{P_N}(V, \sigma) \leq C \left[\left(\frac{\sigma^2 V}{N} \right)^{2/3} + \frac{\sigma^2 \log N}{N} \right].$$

Thus the path is not hardest. The witness v is used in the full-cone support function; it is not required to have root bounded by V . The true signal in (14) is zero.

6 Scope, related work, and reproducibility

The star lower bound and the parallel-arm phenomenon belong to the established cone-risk picture of [1]. The broom upper bound above is elementary and self-contained. It supplies a tree family with strictly smaller worst-case risk than the star, while the parallel-arm family supplies trees with larger risk than the path. No ranking of all trees, exact finite- N extremizer, or optimal broom rate is obtained.

Brooms also occur in Wang's recent preprint [3, Section 5], but its projection comparison is over the fixed-root slice $\{\mu \in C_T : \mu_{\text{root}} = V\}$ and compares that estimator with minimax risk.

Here the projection is onto C_T without a budget constraint, V and σ are fixed as N grows, and the comparison is between two trees. No theorem of [3] is used in our proof. The distinction is essential: constraining the estimated root changes the zero-signal overfitting that underlies the star bound.

The supplementary standard-library Python script performs 90,995 exact rational checks, including 492 small-dimensional cone-projection cases certified by active-set equations, 20,460 non-negative arm witnesses, and 22,500 coefficient comparisons. Normal and optimized Python runs agree. The finite checks do not establish Gaussian expectations or the asymptotic argument; those are proved above.

This note makes no absolute priority claim. AI tools assisted the author with source research, derivations, exact checks, and drafting. The manuscript is self-audited and unrefereed; no independent expert review or proof-assistant verification is claimed.

References

- [1] S. Chatterjee and J. Lafferty, *Denoising flows on trees*, IEEE Trans. Inform. Theory **64**(3) (2018), 1767–1783. <https://doi.org/10.1109/TIT.2017.2782369>. Equation and theorem numbers cited here refer to the retained preprint, <https://arxiv.org/abs/1602.08048v2>.
- [2] J. Lafferty, joint work with M. Xu and S. Chatterjee, *Shape constrained estimation in low and high dimensions*, in *Computationally and Efficient Inference for Complex Large-scale Data*, Oberwolfach Reports **13**(1) (2016), report 16/2016, 765–767. <https://doi.org/10.4171/OWR/2016/16>.
- [3] S. Wang, *The minimax rate of denoising flows on trees*, arXiv:2609.14091v1, 12 September 2026. <https://arxiv.org/abs/2609.14091v1>.