

Higher Koszul Homology of Square-Zero Algebras in Positive Characteristic

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Abstract

We compute the higher Koszul homology, defined by the Euler cap differential, of the connected graded algebra $A = k \oplus V$ with $V^2 = 0$ and $\dim_k V = d < \infty$. In positive total weight n , the calculation reduces to the natural map from invariants to coinvariants of a signed cyclic rotation on $V^{\otimes n}$. A word orbit contributes one class in each of bidegrees $(n, 0)$ and $(n - 1, 1)$ precisely when its signed rotation is compatible and its length is divisible by the characteristic. This gives an explicit necklace formula in every degree. In particular, $k[x, y]/(x, y)^2$ is Koszul but has nonzero higher Koszul homology over every field of positive characteristic. We give a three-term cycle in characteristic three. These examples rule out a characteristic-free Koszul-acyclicity equivalence; they do not address its characteristic-zero converse.

1 Definitions and scope

All tensor products are over the field k . For a quadratic algebra $A = T(V)/(R)$, set $W_0 = k$, $W_1 = V$, and

$$W_q = \bigcap_{i+j=q-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \quad (q \geq 2).$$

We use the Koszul calculus of Berger, Lambre and Solotar [1, §§2,5]. Its chain space is $C_q = A \otimes W_q$ and its boundary is

$$b(a \otimes v_1 \cdots v_q) = av_1 \otimes v_2 \cdots v_q + (-1)^q v_q a \otimes v_1 \cdots v_{q-1}. \quad (1)$$

Tensor words in this notation are not products in A . The higher homology is

$$\mathrm{HK}_q^{\mathrm{hi}}(A) = H_q(\mathrm{HK}_\bullet(A), \partial), \quad \mathrm{HK}_\bullet(A) = H_\bullet(C, b).$$

On b -cycles the Euler cap differential is

$$\partial[a \otimes v_1 \cdots v_q] = [av_1 \otimes v_2 \cdots v_q]. \quad (2)$$

We write $\mathrm{HK}_q^{\mathrm{hi}}(A)_m$ for coefficient weight m ; total weight is $q + m$. Both differentials decrease q and increase m by one.

Theorem 6.4 of [1] proves higher acyclicity for Koszul algebras in characteristic zero. The following Conjectures 6.5 and 6.6 propose, respectively, an acyclicity characterization and a comparison with the homology of the left Koszul complex. The report [2, p. 481] also asks for the acyclicity characterization. The field restriction matters: the results below disprove the unrestricted extension to all fields, including the corresponding unrestricted comparison with

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the left Koszul complex. They do not refute the characteristic-zero theorem, and give no answer to the characteristic-zero converse.

The cyclic-word description of Hochschild homology for algebras with radical square zero is classical; see Cibils [3]. We use the elementary two-term complex directly and compute its *additional Euler cap homology*. We make no claim that cyclic-word decomposition or the Koszulity of $k \oplus V$ is new.

2 The complete computation

For a nonnegative integer d and $\ell \geq 1$, put

$$L_d(\ell) = \frac{1}{\ell} \sum_{e|\ell} \mu(e) d^{\ell/e}.$$

This integer counts primitive cyclic word orbits of length ℓ on d letters. The division takes place in $\mathbb{Z}$, not in k .

Theorem 1. *Let $A = T(V)/(V^{\otimes 2}) = k \oplus V$, where $\dim_k V = d$. Then A is Koszul over every field and $\mathrm{HK}_0^{\mathrm{hi}}(A) = k$. In characteristic zero, all positive higher Koszul homology vanishes.*

Suppose $\mathrm{char} k = p > 0$. For $n \geq 1$ define

$$c_{n,p}(d) = \sum_{\substack{\ell|n, p|\ell \\ (-1)^{(n-1)\ell} = 1 \text{ in } k}} L_d(\ell).$$

Then

$$\dim_k \mathrm{HK}_n^{\mathrm{hi}}(A)_0 = \dim_k \mathrm{HK}_{n-1}^{\mathrm{hi}}(A)_1 = c_{n,p}(d). \quad (3)$$

Besides $\mathrm{HK}_0^{\mathrm{hi}}(A)_0 = k$ and the components in (3), all bigraded components vanish. For odd p , the sign condition is equivalent to n being odd or ℓ being even; for $p = 2$ it is automatic.

Proof. Here $W_q = V^{\otimes q}$. The left Koszul differential is $d_\ell(a \otimes w) = av_1 \otimes v_2 \cdots v_q$. In the splitting

$$A \otimes V^{\otimes q} = (k \otimes V^{\otimes q}) \oplus (V \otimes V^{\otimes q}),$$

it is zero on the second summand and maps the first isomorphically onto $V \otimes V^{\otimes(q-1)}$. Thus its kernel in every positive degree is exactly the image from the next degree. Together with the augmentation $A \rightarrow k$, it is a linear free resolution of k . This proves Koszulity.

Fix positive total weight n . Since $A_m = 0$ for $m \geq 2$, the Koszul chain complex in this weight has only two nonzero terms, both identifiable with $E_n = V^{\otimes n}$:

$$C_{n,0} = E_n \xrightarrow{B_n} E_n = C_{n-1,1}, \quad B_n = 1 - s_n T, \quad s_n = (-1)^{n-1}, \quad (4)$$

where $T(v_1 \cdots v_n) = v_n v_1 \cdots v_{n-1}$. Equation (1) proves (4), with no assumption on the characteristic. Equation (2) now identifies the only potentially nonzero higher differential with

$$j_n : \ker B_n \longrightarrow \mathrm{coker} B_n, \quad z \longmapsto [z]. \quad (5)$$

Consequently

$$\mathrm{HK}_n^{\mathrm{hi}}(A)_0 = \ker B_n \cap \mathrm{im} B_n, \quad (6)$$

$$\mathrm{HK}_{n-1}^{\mathrm{hi}}(A)_1 = E_n / (\ker B_n + \mathrm{im} B_n). \quad (7)$$

There are no hidden incoming maps: both differentials preserve total weight, and (4) lists all its chain groups.

Choose a basis of V and decompose E_n into spans of word orbits under T . On an orbit of length $\ell \mid n$, write $w_j = T^j w_0$ for $0 \leq j < \ell$ and $s = s_n$. If $s^\ell \neq 1$ in k , the map $1 - sT$ is invertible:

$$(1 - sT) \sum_{j=0}^{\ell-1} (sT)^j = (1 - s^\ell)1.$$

This orbit contributes no homology.

If $s^\ell = 1$, both the kernel and cokernel of $1 - sT$ have dimension one. A kernel generator is

$$z_{\mathcal{O}} = \sum_{j=0}^{\ell-1} s^j w_j.$$

In the cokernel, $[w_j] = s^{-j}[w_0]$, so

$$j_n(z_{\mathcal{O}}) = \ell[w_0]. \quad (8)$$

Therefore this orbit contributes one class to each of (6) and (7) if $\ell = 0$ in k , and contributes none otherwise. This also proves vanishing in characteristic zero.

An orbit of words of length n having minimal period ℓ is obtained by repeating a primitive word of length ℓ . Its number is $L_d(\ell)$: if $P_d(\ell)$ counts primitive words, then $d^r = \sum_{\ell \mid r} P_d(\ell)$ and Möbius inversion gives $P_d(\ell) = \sum_{e \mid \ell} \mu(e) d^{\ell/e}$; each orbit contains ℓ primitive words. Summing the preceding orbit calculation proves (3). At total weight zero the sole chain group is k , so it survives. For $n = 1$ every orbit has length one and the map (5) is an isomorphism, proving $\mathrm{HK}_0^{\mathrm{hi}}(A) = k$. $\square$

Corollary 2. *Suppose $\mathrm{char} k = p > 0$ and $d \geq 2$. The positive total weights supporting higher Koszul homology of $k \oplus V$ are exactly the positive multiples of p . The first nonzero positive homological degree is $p - 1$, and*

$$\dim_k \mathrm{HK}_{p-1}^{\mathrm{hi}}(A)_1 = \dim_k \mathrm{HK}_p^{\mathrm{hi}}(A)_0 = \frac{d^p - d}{p} > 0.$$

In particular, the three-dimensional commutative quadratic algebra $k[x, y]/(x, y)^2$ is a counterexample to characteristic-free Koszul acyclicity in every positive characteristic.

Proof. An orbit length dividing n can be divisible by p only if $p \mid n$. Conversely, for every such n the word $x^{n-1}y$ on two distinct letters has orbit length n . Its sign is compatible because $(-1)^{n(n-1)} = 1$. At $n = p$ only $\ell = p$ contributes and $L_d(p) = (d^p - d)/p$. The two-term description then gives the assertion about the first homological degree. The commutative presentation has vector-space basis $1, x, y$ and all products of two positive-degree elements zero, so it is the algebra of Theorem 1 with $d = 2$. $\square$

Remark 3. If $d = 0$ or $d = 1$, every nonempty word orbit has length one, so higher acyclicity holds in every characteristic. Thus within connected graded quadratic algebras of finite total dimension, dimension three is the smallest possible dimension for a Koszul algebra failing higher acyclicity. Indeed dimension at most two permits at most one degree-one generator; if present, its square must be zero.

3 A concrete certificate over $\mathbb{F}_3$

Let $A = \mathbb{F}_3[x, y]/(x, y)^2$. In C_3 put

$$z = 1 \otimes (xyx + yxx).$$

These are three distinct tensor words, hence $z \neq 0$. Equation (1) gives $bz = 0$ by cancellation. Moreover,

$$dz = x \otimes xy + x \otimes yx + y \otimes xx = b(1 \otimes (xyx - xxy))$$

in characteristic three. Hence $[z]$ is a higher cycle. It cannot be a b -boundary, since its coefficient weight is zero and b increases coefficient weight. For the same reason it cannot be a higher boundary. Therefore

$$0 \neq [[z]] \in \mathrm{HK}_3^{\mathrm{hi}}(A)_0.$$

For clarity, in the ordered bases xyx, yxy, yxx in both terms of (4), the boundary and witnesses are

$$B_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

They satisfy $B_3 z = 0$ and $B_3 q = z$ over $\mathbb{F}_3$. The latter equality is false over $\mathbb{Q}$; there the induced cap map on this orbit is multiplication by 3 and is invertible. There is another primitive orbit, represented by xyy . Consequently both $\mathrm{HK}_3^{\mathrm{hi}}(A)_0$ and $\mathrm{HK}_2^{\mathrm{hi}}(A)_1$ have dimension two.

The example also has $H_q(K_\ell(A)) = 0$ for $q > 0$ by the explicit resolution above. Its nonzero $\mathrm{HK}_3^{\mathrm{hi}}(A)$ therefore precludes an all-characteristic identification $\mathrm{HK}_q^{\mathrm{hi}}(A) \cong H_q(K_\ell(A))$.

4 Verification and interpretation

The accompanying Python checker forms the full word-basis matrix (4) and uses exact rational or finite-field row reduction. It compares the resulting cap ranks against separately enumerated word orbits in 105 cases: $d = 1$, $1 \leq n \leq 10$; $d = 2$, $1 \leq n \leq 7$; and $d = 3$, $1 \leq n \leq 4$, over $\mathbb{Q}, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7$. It also checks the displayed characteristic-three certificate.

As a calibration of conventions, the checker reconstructs the six-dimensional non-Koszul algebra $k\langle x, y \rangle / (x^2, y^2 - xy)$ of [1, §9]. Through homological degree seven over $\mathbb{Q}$, it recovers the published higher dimensions $1, 0, 2, 0, \dots$ and the left Koszul homology dimensions. It checks associativity of that multiplication table and the differential identities. These finite checks diagnose implementation errors; the all-degree assertions follow from Theorem 1.

The generalizations of Koszul calculus in [4, 5] provide further context but are not needed in the proof. The obstruction here is the orbit-length scalar in (8), which can vanish in positive characteristic. This identifies a field hypothesis needed for any unrestricted reading of the question in [2]. The characteristic-zero converse remains outside the scope of this note.

This manuscript was developed with AI assistance for mathematical exploration, drafting, and exact computational checks. The argument is self-contained and the accompanying code is reproducible; the note is unrefereed and makes no absolute historical-priority claim.

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