

Karpenko’s Conjecture on the Special Schubert Class in the Gamma Filtration Fails for $n = 8, 9$ and 11 : A Short Consequence of Printed Results

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October 20, 2026

Abstract

Let φ be a non-degenerate quadratic form of dimension $2n+1$ over a field F whose even Clifford algebra $C_0(\varphi)$ is a division algebra, let X be its maximal orthogonal Grassmannian and S the Severi–Brauer variety of $C_0(\varphi)$. N. Karpenko conjectured (Math. Z. 285 (2017), Conjecture 4.7, also Conjecture 3 of his abstract in Oberwolfach Report 5/2016) that for $n \geq 8$ the class $e_n \in K(X_{F(S)})$ of the special Schubert variety of lowest dimension does not lie in the term number $n+1 - \lfloor (n+1)/2 \rfloor$ of the gamma filtration. We observe that the conjecture is false: over every field of characteristic 0, for $n = 8, 9$ and 11 and for every such φ , the class e_n lies in this term (term 5 for $n = 8, 9$, term 6 for $n = 11$), and such forms exist. This is a short consequence of printed results, not found stated in print. The proof is Karpenko’s own printed argument that Conjecture 4.7 implies his Conjecture 3.3 on indexes of orthogonal Grassmannians, written out in a general form with two implicit steps filled in, combined with Totaro’s computation of the torsion indexes of $\text{Spin}(20)$ and $\text{Spin}(24)$ (for $n = 9, 11$) and with Karpenko’s theorem on $\text{Spin}(18)$ -torsors (for $n = 8$). A computer-assisted supplement, kept separate from the proof, finds by an exact computation in the Grothendieck ring of the split Grassmannian that e_8 lies in the seventh but not in the eighth gamma term, and e_9 in the eighth but not in the ninth. The cases $n = 10$ and $n = 12$ are not decided here, nor is any $n \geq 13$. No printed answer to the question was found in the sources searched; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The conjecture

We use the setting of [Kar17, §4]. Let $n \geq 1$, let F be a field and let φ' be a non-degenerate quadratic form of dimension $2n+1$ over F . Let X' be its maximal orthogonal Grassmannian, the variety of n -dimensional totally isotropic subspaces, and $K(X')$ its Grothendieck ring. Assume that the even Clifford algebra $C_0(\varphi')$, a central simple algebra of degree 2^n , is a division algebra, that is $\text{ind } C_0(\varphi') = 2^n$, and let S be its Severi–Brauer variety. Under this hypothesis Karpenko proved that the topological filtration on $K(X'_{F(S)})$ coincides with the gamma filtration [Kar17, Theorem 4.3]. Let $e_n \in K(X'_{F(S)})$ be the class of the special Schubert variety of lowest dimension, “corresponding to the class of a rational point on the quadric” [Kar17, §4]; see Section 2 for the precise definition. Write $[x]$ for the integer part of x .

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Conjecture 1.1 (Karpenko [Kar17, Conjecture 4.7], [Kar16, Conjecture 3]). *For $n \geq 8$, the special Schubert class $e_n \in K(X'_{F(S)})$ does not belong to the term number $n + 1 - \lfloor (n + 1)/2 \rfloor$ of the gamma filtration.*

The two printed statements are word for word the same; the second is in Karpenko’s abstract in the report of the Oberwolfach workshop “Algebraic Cobordism and Projective Homogeneous Varieties” [Kar16, pp. 226–227]. The conjecture is record OWR-14213-008 of the UnsolvedMath collection [ula26], where it is listed as open. The term numbers are 5 for $n = 8$ and 9, 6 for $n = 10$ and 11, and 7 for $n = 12$. Karpenko’s interest in the conjecture comes from the fact that it implies his Conjecture 3.3 [Kar17] (Conjecture 4 of [Kar16]): *for any even $m \geq 18$, the index of the $\lfloor m/4 \rfloor$ -th orthogonal Grassmannian of any generic m -dimensional quadratic form in I_q^3 is equal to $2^{\lfloor m/4 \rfloor}$.* The implication is proved in print, at the end of [Kar17, §4]. Karpenko also showed that the number $n + 1 - \lfloor (n + 1)/2 \rfloor$ cannot be lowered for $n = 8$: e_8 lies in the term number 4 [Kar17, Remark 4.8].

1.2 Result

Theorem 1.2. *Let F be a field of characteristic 0, let $n \in \{8, 9, 11\}$, and let φ' be a non-degenerate quadratic form of dimension $2n + 1$ over F such that $C_0(\varphi')$ is a division algebra. Let S be the Severi–Brauer variety of $C_0(\varphi')$ and X' the maximal orthogonal Grassmannian of φ' . Then e_n lies in the term number $n + 1 - \lfloor (n + 1)/2 \rfloor$ of the gamma filtration on $K(X'_{F(S)})$, that is, in the fifth term for $n = 8, 9$ and in the sixth term for $n = 11$. Forms φ' with these properties exist over $F = \mathbb{Q}((t_1)) \cdots ((t_{2n}))$ (Lemma 4.1). In particular Conjecture 1.1 is false.*

The theorem concerns every admissible form, not a particular one, so no reading of the word “generic” in [Kar17] can save the conjecture. The proof (Sections 3 and 4) needs no computation. It rests on the following general form of Karpenko’s argument (Proposition 3.2): if some finite extension L of $F(S)$ with $v_2([L : F(S)]) \leq j - 1$ gives the $(2n + 2)$ -dimensional form $\varphi = \varphi' \perp \langle d \rangle$ of trivial discriminant a Witt index at least j , then e_n lies in the term number $n + 1 - j$ of the topological filtration on $K(X'_{F(S)})$. The inputs are Totaro’s theorem on the torsion index of the spin groups [Tot05, Theorem 0.1] for $n = 9$ and 11, and Karpenko’s theorem on 18-dimensional forms [Kar25, Abstract, Corollary 4.4] for $n = 8$. For $n = 9$ a computer-assisted printed result [Kar23, §8] gives more, namely the sixth term (Remark 4.3).

Section 5 contains a computer-assisted supplement, kept apart from the theorem: an exact computation in the Grothendieck ring of the split Grassmannian $\text{OGr}(n, 2n + 1)$ gives that e_8 lies in the seventh but not in the eighth gamma term, and e_9 in the eighth but not in the ninth (Theorem 5.1). This holds at the level of a tested computation, not of a proof in the strict sense.

1.3 What is new and what is not

Every ingredient of the proof was in print by 2025, and for $n = 9$ and 11 the disproof was available as soon as [Kar17] appeared: the table after [Tot05, Theorem 0.1] gives $u(9) = 4$, so for $m = 20$ the index of the fifth orthogonal Grassmannian of a generic form in I_q^3 divides 2^4 , and Conjecture 3.3 of [Kar17] at $m = 20$ is incompatible with [Tot05, Theorem 0.1], a result that [Kar17] cites. The same holds at $m = 24$. At $m = 18$, Conjecture 3.3 is contradicted by [Kar25, Corollary 4.4] and, with computer assistance, by [Kar23, §6]; neither of these two papers mentions Conjecture 3.3 or Conjecture 4.7. The failure at $m = 18$ was reported earlier in the follow-up report FU119 to the UnsolvedMath collection, on its record OWR-14213-009. On the other hand, Devyatov, Karpenko and Merkurjev proved that Conjecture 3.3 holds for $m \geq 34$ [DKM21, Theorem 7.2 and the remark after it]. Since [Kar17] proves only that Conjecture 4.7 implies Conjecture 3.3, this route cannot refute Conjecture 1.1 for large n , and the truth of Conjecture 3.3 for large m says nothing about Conjecture 1.1.

What this note adds is the statement itself, with all quantifiers: the written-out argument with its two implicit steps filled in (the passage from $[L : F(S)]l_{j-1}$ to $2^{j-1}l_{j-1}$, and the bookkeeping of the base change), the general form Proposition 3.2, and the computational supplement of Section 5, whose exact gamma levels of e_8 and e_9 we did not find in print. We claim no priority beyond that. The cases $n = 10$ and $n = 12$ are not decided here: Totaro's exponents are $u(10) = 5 = [(10 + 1)/2]$ and $u(12) = 6 = [(12 + 1)/2]$, so the argument does not apply; the same holds for $n = 13$, where $u(13) = 7 = [(13 + 1)/2]$, and for $n \geq 14$ one has $u(n) > [(n + 1)/2]$ (Remark 4.5).

2 Setting and conventions

Filtrations. For a smooth variety V over a field, $K(V)$ is the Grothendieck group of vector bundles, identified with that of coherent sheaves. The *topological filtration* $T^c K(V)$ is generated by the classes of coherent sheaves whose support has codimension at least c . The *gamma filtration* $\gamma^c K(V)$ is defined from the λ -ring structure [FL85, Ch. III and V]: $\gamma^0 = K(V)$, γ^1 is the kernel of the rank (for connected V), and γ^c is generated by the products $\gamma^{c_1}(x_1) \cdots \gamma^{c_r}(x_r)$ with $x_i \in \gamma^1$ and $c_1 + \cdots + c_r \geq c$. One has $\gamma^c \subset T^c$ (see the proof of [Kar17, Proposition 4.2]). The *term number* c of a filtration is its c -th term; this is the numbering of [Kar17, Theorem 4.3, Remarks 4.4 and 4.8]. For a vector bundle V we write $c_j^K(V) = \gamma^j(V - \text{rk } V) \in \gamma^j$ for its gamma-Chern classes. For a projective variety V , $\text{ind } V$ is the g.c.d. of the degrees of its closed points, and i_W denotes the Witt index of a quadratic form.

Quadrics and Grassmannians. Let E be a field and φ a non-degenerate quadratic form on a vector space V of dimension $m = 2n + 2$ over E , of trivial discriminant. Let Y be its projective quadric, of dimension $D = 2n$; let X be one of the two components of its maximal orthogonal Grassmannian, whose points are totally isotropic subspaces of dimension $n + 1$; let $Z \subset Y \times X$ be the incidence variety of pairs (p, W) with $p \in \mathbb{P}(W)$, and $f: Z \rightarrow X$, $g: Z \rightarrow Y$ the projections. The map f is the projective bundle of the tautological bundle, of relative dimension n . Every fibre of g is a component of the maximal orthogonal Grassmannian of a form of dimension $2n$, of dimension $n(n - 1)/2 = \dim Z - \dim Y$; since Z and Y are smooth, g is flat. If φ' is the restriction of φ to a non-degenerate subspace $V' \subset V$ of dimension $2n + 1$, then $W \mapsto W \cap V'$ is an isomorphism from X onto the maximal orthogonal Grassmannian X' of φ' [EKM08, Propositions 85.2 and 86.17], as used in [Kar17, §1], and $C^+(\varphi) \simeq C_0(\varphi')$ [Kar17, §1].

The classes l_i , h and e_n . Let $\bar{E}$ be an algebraic closure of E ; over $\bar{E}$ the form is hyperbolic, and we write $\bar{Y}$, $\bar{X}$ for $Y_{\bar{E}}$, $X_{\bar{E}}$. For $0 \leq i \leq n$ let $l_i \in K(\bar{Y})$ be the class of the structure sheaf of an i -dimensional linear subspace of $\bar{Y}$. For $i \leq n - 1$ it does not depend on the subspace: the Chow class does not [Kar17, §1], and by the Grothendieck–Riemann–Roch theorem the Chern character of the structure sheaf of a linear subspace P depends only on the classes of P and of its linear sections in $\text{CH}(\bar{Y})$, while ch is injective on the free group $K(\bar{Y})$. For $i = n$ there are two classes, which we do not use. Let $h = [\mathcal{O}_Y] - [\mathcal{O}_Y(-1)]$, the class of a hyperplane section; it is defined over E . For an isotropic line p in $V_{\bar{E}}$ let $\Sigma_p = \{W \in \bar{X} : p \subset W\}$, and define $e_n = [\mathcal{O}_{\Sigma_p}] \in K(\bar{X})$. Then $\Sigma_p = f(g^{-1}(p))$, the map f restricted to $g^{-1}(p)$ is an isomorphism onto Σ_p , and $g^*[\mathcal{O}_p] = [\mathcal{O}_{g^{-1}(p)}]$ since g is flat, so

$$e_n = f_* g^*(l_0) \quad \text{in } K(\bar{X}), \quad (1)$$

the K-theoretic form of [Kar17, (1.2)]; Karpenko notes that it can be shown using [BR12, Lemma 2.1] [Kar17, Remark 4.6]. In particular e_n does not depend on p . Under $X \simeq X'$, with $p \subset V'_{\bar{E}}$, the variety Σ_p corresponds to $\Sigma'_p = \{U \in \bar{X}' : p \subset U\}$: if $p \subset W$ then $p \subset W \cap V'$, and if $p \subset U \subset W$ then $p \subset W$. So e_n corresponds to the special Schubert class of lowest dimension of X' , which has codimension n .

Descent to E . If $C_0(\varphi'_E)$ is split, then the change of field map $K(X'_E) \rightarrow K(\bar{X}')$ is an isomorphism, by Panin's computation in the form of [Kar17, Lemma 4.1]; we then also write e_n for the element of

$K(X'_E)$ mapping to e_n . This is the meaning of $e_n \in K(X'_{F(S)})$ in Conjecture 1.1, where $E = F(S)$ splits $C_0(\varphi')$. Likewise, if φ has trivial discriminant and $C^+(\varphi)$ is split, then $K(Y_E) \rightarrow K(\bar{Y})$ is an isomorphism by Swan's computation [Swa85]; we shall not need this.

Remark 2.1 (Change of field). For every field extension $L/F(S)$, the change of field map $K(X'_{F(S)}) \rightarrow K(X'_L)$ is an isomorphism preserving the gamma filtration, whereas the topological filtration is not preserved [Kar17, Remark 4.6] (this is the wording of the final version of [Kar17]). The proof of Theorem 1.2 does not use this remark: it uses only push-forward along finite morphisms and flat base change on Y and X , and [Kar17, Theorem 4.3] over $F(S)$ itself. The computational supplement of Section 5 uses it, to replace $F(S)$ by an algebraically closed field.

3 Karpenko's argument in a general form

The following identity is the one that [Kar17] quotes from [Kar90, §3.2] in the form $2^{r-1}l_{r-1} = l_0 + h^{m-3} + 2h^{m-4} + \dots + 2^{r-2}h^{m-r-1}$. We include a proof.

Lemma 3.1. *In the notation of Section 2, over $\bar{E}$ one has $h^{D-i} = 2l_i - l_{i-1}$ in $K(\bar{Y})$ for $1 \leq i \leq n-1$. Consequently, for $1 \leq j \leq n$,*

$$2^{j-1}l_{j-1} = l_0 + \sum_{i=1}^{j-1} 2^{i-1}h^{D-i}. \quad (2)$$

Proof. Let $R \subset V_{\bar{E}}$ be a totally isotropic subspace of dimension i . Since $i \leq n-1$, the orthogonal complement of $R+R^*$ for a dual totally isotropic subspace R^* is hyperbolic of dimension $2n+2-2i \geq 4$, so it contains a hyperbolic plane $H = \langle e, f \rangle$ with $\varphi(e) = \varphi(f) = 0$ and polar form 1 on (e, f) . On $W = R \oplus H$ the form is $r + xe + yf \mapsto xy$. Hence $\mathbb{P}(W) \cap \bar{Y}$ is the quadric $\{xy = 0\}$ in $\mathbb{P}(W) \simeq \mathbb{P}^{i+1}$, the union of the two linear subspaces $A = \mathbb{P}(R + \langle e \rangle)$ and $B = \mathbb{P}(R + \langle f \rangle)$ of dimension i , which meet in $\mathbb{P}(R) \simeq \mathbb{P}^{i-1}$. The intersection has dimension $i = D - (D - i)$, the expected one, and $\bar{Y}$ is Cohen–Macaulay, so the $D - i$ linear forms cutting out $\mathbb{P}(W)$ form a regular sequence on $\bar{Y}$ and the Koszul complex gives $[\mathcal{O}_{\bar{Y} \cap \mathbb{P}(W)}] = h^{D-i}$. The ideals (x) and (y) of A and B in $\mathbb{P}(W)$ satisfy $(x) \cap (y) = (xy)$, so the sequence $0 \rightarrow \mathcal{O}_{A \cup B} \rightarrow \mathcal{O}_A \oplus \mathcal{O}_B \rightarrow \mathcal{O}_{A \cap B} \rightarrow 0$ is exact and $h^{D-i} = 2l_i - l_{i-1}$. Multiplying $2l_i = h^{D-i} + l_{i-1}$ by 2^{i-1} and summing over $i = 1, \dots, j-1$ gives (2). For $j = r = \lfloor m/4 \rfloor$ and $D = m - 2$ this is the formula of [Kar17] quoted above. $\square$

The program of this note checks the identity of Lemma 3.1 independently through Chern characters for $n \leq 12$ (Section 6.1).

Proposition 3.2. *Let K be a field of characteristic 0 and ψ a non-degenerate quadratic form of dimension $m = 2n + 2$ over K of trivial discriminant, with Y, X as in Section 2 and with $K(X) \rightarrow K(\bar{X})$ an isomorphism. Let $1 \leq j \leq n$, and suppose that there is a finite field extension L/K with $v_2([L : K]) \leq j - 1$ and $i_W(\psi_L) \geq j$. Then $e_n \in T^{n+1-j}K(X)$.*

For $j = r = \lfloor m/4 \rfloor$ this is the argument printed at the end of [Kar17, §4]; there $n + 1 - r = n + 1 - \lfloor (n + 1)/2 \rfloor$, because $[(2n + 2)/4] = \lfloor (n + 1)/2 \rfloor$.

Proof. Write $c = m - j - 1 = D - (j - 1)$. In (i) and (ii) we show that $l_0 \in K(\bar{Y})$ is the image of an element of $T^c K(Y)$.

(i) *The class $2^{j-1}l_{j-1}$.* Over L the form has a totally isotropic subspace of dimension j , so Y_L contains a linear subspace $P \simeq \mathbb{P}^{j-1}$, and $[\mathcal{O}_P] \in T^c K(Y_L)$. Let $\pi : Y_L \rightarrow Y$ be the projection. It is finite, and $\dim Y_L = \dim Y$, so π_* maps $T^c K(Y_L)$ into $T^c K(Y)$. Since L/K is separable (characteristic 0), $Y_L \times_K \bar{K}$ is the disjoint union of $[L : K]$ copies of $\bar{Y}$, indexed by the embeddings $L \rightarrow \bar{K}$, and P

gives a linear subspace of dimension $j - 1$ in each copy. Since $j - 1 \leq n - 1$, each of them has class l_{j-1} (Section 2). By flat base change, the image of $\pi_*[\mathcal{O}_P]$ in $K(\bar{Y})$ is therefore $[L : K]l_{j-1}$.

The same reasoning applies to a second extension. A non-degenerate form of even dimension which is not hyperbolic becomes isotropic over a quadratic extension (adjoin a square root of $-ab$ for two entries a, b of a diagonalization of its anisotropic part), and its Witt index grows by at least one. Adjoining at most j square roots therefore gives a finite extension L'/K of degree 2^a with $a \leq j$ and $i_W(\psi_{L'}) \geq j$, and $2^a l_{j-1}$ is the image of an element of $T^c K(Y)$. Now $\gcd([L : K], 2^a) = 2^{\min(v_2[L:K], a)}$ divides 2^{j-1} , so an integral combination of the two elements is an element of $T^c K(Y)$ with image $2^{j-1}l_{j-1}$.

(ii) *The class l_0 .* Let $1 \leq i \leq j - 1$. Since K is infinite, a general linear subspace of codimension $D - i$ of the ambient projective space, defined over K , meets Y in codimension $D - i$, and by the Koszul complex its intersection with Y has class h^{D-i} ; so $h^{D-i} \in T^{D-i} K(Y) \subset T^c K(Y)$, as $D - i \geq D - (j - 1) = c$. By (2), $l_0 = 2^{j-1}l_{j-1} - \sum_{i=1}^{j-1} 2^{i-1}h^{D-i}$ is the image of an element $y \in T^c K(Y)$.

(iii) *The class e_n .* Since g is flat, g^* maps $T^c K(Y)$ into $T^c K(Z)$: the inverse image of a closed subset of codimension at least c has codimension at least c . Since f is proper of relative dimension n , f_* maps $T^c K(Z)$ into $T^{c-n} K(X)$: the higher direct images of a sheaf are supported on the image of its support, of dimension at most that of the support. Both maps commute with the flat base change from K to $\bar{K}$. By (1), the image of $f_*g^*(y) \in T^{c-n} K(X)$ in $K(\bar{X})$ is $f_*g^*(l_0) = e_n$. As $K(X) \rightarrow K(\bar{X})$ is an isomorphism, $e_n = f_*g^*(y) \in T^{c-n} K(X)$, and $c - n = 2n + 2 - j - 1 - n = n + 1 - j$. $\square$

Remark 3.3. Two steps are implicit in [Kar17]. First, the norm map is said there to act as multiplication by $[L : F(S)]$, which gives $[L : F(S)]l_{r-1}$; the passage to $2^{r-1}l_{r-1}$ needs a second extension of 2-power degree, as in (i). Second, the computation of π_* uses that L/K is separable, so that the base change of Y_L to $\bar{K}$ consists of $[L : K]$ copies of $\bar{Y}$, and that l_{j-1} does not depend on the subspace, which holds because $j - 1 \leq n - 1$. Neither step changes the argument.

4 Proof of the theorem

Lemma 4.1. *For every $n \geq 1$ there are a field F of characteristic 0 and a non-degenerate form φ' of dimension $2n + 1$ over F such that $C_0(\varphi')$ is a division algebra.*

Karpenko states this over any base field [Kar17, after Corollary 1.6]. For completeness we give an explicit example.

Proof. Let $F = \mathbb{Q}((t_1))((t_2)) \cdots ((t_{2n}))$ and $D = (t_1, t_2) \otimes (t_3, t_4) \otimes \cdots \otimes (t_{2n-1}, t_{2n})$, a tensor product of quaternion algebras. D is a division algebra, by induction on n with the following classical fact: if B is a division algebra over a field E of characteristic $\neq 2$, $u \in E^\times$ is not a square, and $B \otimes E(\sqrt{u})$ is a division algebra, then $B \otimes (u, t)$ is a division algebra over $E((t))$ (it is a ring of twisted Laurent series in an element j with $j^2 = t$ over $B \otimes E(\sqrt{u})$, j acting by the conjugation of $E(\sqrt{u})/E$, and its non-zero elements have invertible leading coefficients). Apply it with $E = \mathbb{Q}((t_1)) \cdots ((t_{2k+1}))$, $u = t_{2k+1}$, $t = t_{2k+2}$ and B the product of the first k factors over E ; here $B \otimes E(\sqrt{t_{2k+1}})$ is again of the same form over a field of iterated Laurent series, hence division by induction. The formulas $C(\langle a, b \rangle \perp \psi) \simeq (a, b) \otimes C(-ab\psi)$ and $C_0(\langle 1 \rangle \perp \chi) \simeq C(-\chi)$ for the Clifford algebras of diagonal forms [Lam05, Ch. V] give, by induction, a $2n$ -dimensional diagonal form χ with $C(\chi) \simeq D$, namely $\chi_1 = \langle t_1, t_2 \rangle$ and $\chi_k = \langle t_{2k-1}, t_{2k} \rangle \perp (-t_{2k-1}t_{2k})\chi_{k-1}$. Then $\varphi' = \langle 1 \rangle \perp (-\chi_n)$ has $C_0(\varphi') \simeq D$. $\square$

Proof of Theorem 1.2. Let F , $n \in \{8, 9, 11\}$ and φ' be as in the theorem, and put $r = [(n + 1)/2]$ and $m = 2n + 2$, so $r = 4, 5, 6$ and $m = 18, 20, 24$.

Step 1: the even-dimensional form. Let $d = (-1)^{n+1} \det \varphi'$ and $\varphi = \varphi' \perp \langle d \rangle$, a form of dimension m of trivial discriminant. Then $C^+(\varphi) \simeq C_0(\varphi')$ is a division algebra and S is its Severi–Brauer variety [Kar17, §1]. Let $K = F(S)$ and $\psi = \varphi_K$. Since K splits $C^+(\varphi)$, the form ψ has trivial discriminant and trivial Clifford invariant, so $[\psi] \in I_q^3(K)$ [Kar17, §1]; in the terminology of [Kar17, after Corollary 1.6] it is a generic m -dimensional form in I_q^3 . By [Kar17, Lemma 4.1], applied to φ'_K , the map $K(X) \rightarrow K(\overline{X})$ is an isomorphism, where X is a component of the maximal orthogonal Grassmannian of ψ .

Step 2: the index input. We need a finite extension L/K with $v_2([L : K]) \leq r - 1$ and $i_W(\psi_L) \geq r$.

(a) $n = 9$ and $n = 11$. Totaro’s theorem [Tot05, Theorem 0.1] gives the torsion index $2^{u(l)}$ of $\text{Spin}(2l + 2)$, with $u(9) = 4$ and $u(11) = 5$ (the table after Theorem 0.1 there). In the introduction of [Tot05] this is stated for forms, through Grothendieck’s theorem [Tot05, Theorem 1.1]: every $(2l + 2)$ -dimensional form in I^3 splits over a field extension of degree an odd multiple of 2^i for some $i \leq u(l)$. Applied to ψ with $l = n$, it gives L/K with $v_2([L : K]) \leq u(n) = r - 1$ over which ψ is hyperbolic, so $i_W(\psi_L) = n + 1 \geq r$. The surjectivity of the map from $\text{Spin}(m)$ -torsors to forms of trivial discriminant and Clifford invariant [Kar25, §1] is not needed here.

(b) $n = 8$. Karpenko proved that every non-degenerate 18-dimensional quadratic form of trivial discriminant and Clifford invariant acquires Witt index at least 5 over some finite extension of degree not divisible by 2^4 [Kar25, Abstract and the end of §4, after Corollary 4.4]; the proof there uses no computer. Applied to ψ , it gives L/K with $v_2([L : K]) \leq 3 = r - 1$ and $i_W(\psi_L) \geq 5 \geq r$.

Step 3: the topological filtration. By Proposition 3.2 with $j = r$, which satisfies $1 \leq r \leq n$, we get $e_n \in T^{n+1-r} K(X)$.

Step 4: the gamma filtration. Identify X with the maximal orthogonal Grassmannian X'_K of φ'_K as in Section 2; this identification carries e_n to the special Schubert class e_n of X' . Since $\text{ind } C_0(\varphi') = 2^n$ and $K = F(S)$, [Kar17, Theorem 4.3] gives $T^c K(X'_{F(S)}) = \gamma^c K(X'_{F(S)})$ for all c . Hence $e_n \in \gamma^{n+1-r} K(X'_{F(S)})$, and $n + 1 - r = n + 1 - [(n + 1)/2]$ is 5, 5, 6 for $n = 8, 9, 11$. Existence of such φ' is Lemma 4.1. $\square$

Remark 4.2 (Where the hypotheses are used). The hypothesis $\text{ind } C_0(\varphi') = 2^n$ is used only in Step 4, for [Kar17, Theorem 4.3]; Steps 1–3 use only that ψ has trivial discriminant and Clifford invariant. Characteristic 0 is used in Proposition 3.2(i), for the separability of L/K and for the diagonalisation and the square roots in the second extension; we have not tried to remove it.

Remark 4.3 (The case $n = 9$, computer-assisted). For $n = 9$ one gets the sixth term. Karpenko showed with computer assistance that for a generic $\text{Spin}(20)$ -torsor the fourth orthogonal Grassmannian has index 2^3 [Kar23, §8]; Karpenko’s later paper [Kar25, §1] notes that the proofs in [Kar23] rely on computer computations. Every 20-dimensional form of trivial discriminant and Clifford invariant comes from a $\text{Spin}(20)$ -torsor [Kar25, §1], and the index of the corresponding variety divides that of the generic torsor [Kar25, §1], [KM06, Theorem 6.4]. So there is L/K with $v_2([L : K]) \leq 3$ and $i_W(\psi_L) \geq 4$, and Proposition 3.2 with $j = 4$ gives $e_9 \in \gamma^6 K(X'_{F(S)})$. This rests on a computer-assisted printed result; the computer-free bound proved in Theorem 1.2 is the fifth term. By the same specialisation argument, the computer-assisted results of [Kar23, §§5–6] give a second route to Step 2(b).

Remark 4.4 (The case $n = 8$ and Karpenko’s Remark 4.8). Karpenko showed that e_8 lies in the fourth term, so that the number $n + 1 - [(n + 1)/2]$ in Conjecture 1.1 cannot be lowered for $n = 8$ [Kar17, Remark 4.8]. Theorem 1.2 places e_8 in the fifth term, and the computation of Section 5 in the seventh.

Remark 4.5 (Small n and the limits of the method). For $n = 6$ and $n = 7$, outside the range of Conjecture 1.1, Totaro’s exponents $u(6) = 2$ and $u(7) = 3$ give, as in Step 2(a), the hypothesis of Proposition 3.2 with $j = u(n) + 1 = [(n + 1)/2]$, that is $j = 3$ and $j = 4$. So e_6 and e_7 lie in the fourth term, which is again $n + 1 - [(n + 1)/2]$; this is consistent with the restriction $n \geq 8$ in the

conjecture. For $n \geq 8$, the inequality $u(n) < [(n+1)/2]$ holds only for $n = 9$ and $n = 11$. For $8 \leq n \leq 12$ this is the table of [Tot05]: $u(8), \dots, u(12) = 4, 4, 5, 5, 6$. For $n \geq 13$, Totaro's theorem gives $u(n) \geq n - \lfloor \log_2(\binom{n+1}{2} + 1) \rfloor$, and $\binom{n+1}{2} + 1 < 2^{\lfloor n/2 \rfloor + 1}$ (true for $n = 13, 14$, and the left side grows by a factor less than 2 when n increases by 2, since $n^2 + 5n + 8 < 2(n^2 + n + 2)$ for $n > 4$), so $u(n) \geq n - \lfloor n/2 \rfloor = [(n+1)/2]$. In the same way $\binom{n+1}{2} + 1 < 2^{\lfloor n/2 \rfloor}$ for $n = 14, 15$, hence for all $n \geq 14$, and then $u(n) > [(n+1)/2]$. For $n = 10, 12$ and 13 one has $u(n) = [(n+1)/2]$. For all these n a proof along these lines would need a bound $i_j < j$ with $j \leq [(n+1)/2]$ for the exponent indexes i_j of generic forms of dimension $2n+2$ (for $n = 10, 12, 13$, for instance, that the $u(n)$ -th orthogonal Grassmannian has index $2^{u(n)-1}$, as [Kar25] proves for $n = 8$), and we use no such statement. The program of this note checks the list for $n \leq 400$.

5 Computational supplement (computer-assisted)

This section is not used in the proof of Theorem 1.2. Its result is computer-assisted and has the status of a tested computation in exact integer and 2-adic arithmetic, not of a proof in the strict sense. By Remark 2.1, the gamma filtration on $K(X'_{F(S)})$ is the gamma filtration of the λ -ring $K(\overline{X})$, where $\overline{X} = \text{OGr}(n, 2n+1)$ is the split maximal orthogonal Grassmannian over an algebraically closed field of characteristic 0. So Conjecture 1.1 is a statement about the number $\lambda(n) = \max\{k : e_n \in \gamma^k K(\overline{X})\}$, which can be computed.

Theorem 5.1 (computer-assisted). $\lambda(8) = 7$ and $\lambda(9) = 8$. Hence, for every admissible φ' as in Theorem 1.2, e_8 lies in $\gamma^7 K(X'_{F(S)}) = T^7 K(X'_{F(S)})$ but not in γ^8 , and e_9 lies in γ^8 but not in γ^9 .

The model. We write T for the tautological bundle of rank n on $\overline{X}$. (M1) $\text{CH}(\overline{X})$ is free with basis the products $e_I = \prod_{i \in I} e_i$, $I \subset \{1, \dots, n\}$, subject to the relations [Kar17, (1.1)], and $\deg e_{\{1, \dots, n\}} = 1$ [EKM08, Theorem 86.12, Corollary 86.10]. (M2) $c(T^\vee) = 1 + 2 \sum_i e_i$, since $(-1)^i 2e_i$ is the i -th Chern class of T [EKM08, Proposition 86.13], as used in the proof of [Kar17, Proposition 4.2]; the ample generator L of the Picard group has $c_1(L) = e_1$ and $L^{\otimes 2} \simeq \det T^\vee$; and $T_{\overline{X}} = \Lambda^2 T^\vee + T^\vee \otimes (T^\perp/T)$ in $K(\overline{X})$, where the line bundle $T^\perp/T$ has $c_1 = 0$, so that $\text{td}(T_{\overline{X}}) = \text{td}(\Lambda^2 T^\vee) \text{td}(T^\vee)$. (M3) $\text{ch}(e_n) = e_n \cdot \text{td}(T^\vee)^{-1}$. Indeed $\Sigma_p \simeq \text{OGr}(n-1, 2n-1)$ is smooth, its normal bundle is $p^\vee \otimes ((U/p)^\vee \oplus U^\perp/U)$ in $K(\Sigma_p)$ with $p^\vee$ trivial, so $\text{td}(N) = \text{td}(T^\vee)|_{\Sigma_p}$, and the Grothendieck–Riemann–Roch theorem gives the formula. (M4) The following lemma.

Lemma 5.2. Let $G = \{c_1^K(L), c_1^K(T^\vee), \dots, c_n^K(T^\vee)\}$, with weights $1, 1, 2, \dots, n$, and let $M_k \subset K(\overline{X})$ be the $\mathbb{Z}$ -span of the monomials in G of weight at least k . Then $M_0 = K(\overline{X})$ and $\gamma^k K(\overline{X}) = M_k$ for all k .

Proof. Since $\text{Spin}(2n+1)$ is simply connected, the map from the representation ring $R(P)$ of the parabolic subgroup P to $K(\overline{X})$ is onto [Pit72, Ste75, Pan94]. As $R(P)$ is the representation ring of a Levi subgroup, a double cover of GL_n , the ring $K(\overline{X})$ is generated by the exterior powers of $T^\vee$ and by $L^{\pm 1}$. These are integral polynomials in G : $\lambda^i(T^\vee) = \sum_j \binom{n-j}{i-j} c_j^K(T^\vee)$, $L = 1 + c_1^K(L)$, and $L^{-1} = \sum_a (-c_1^K(L))^a$, a finite sum. So $M_0 = K(\overline{X})$. Clearly $M_k \subset \gamma^k$ and $M_a M_b \subset M_{a+b}$. Conversely $\gamma^1 = M_1$, and γ^k is spanned by products of $\gamma^{k_i}(y_i)$ with $y_i \in M_1$ and $\sum k_i \geq k$; as $\gamma_t = \sum \gamma^j t^j$ is additive-to-multiplicative, it suffices to show $\gamma^j(\pm \mu) \in M_j$ for every monomial $\mu \neq 1$ in G . There is a λ -ring map ρ from $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n} \otimes \mathbb{Z}[z^{\pm 1}]$ to $K(\overline{X})$ sending the i -th elementary symmetric function of the x_i to $\lambda^i(T^\vee)$ and z to L (it comes from the frame bundles of $T^\vee$ and L). With $u_i = x_i - 1$ and $v = z - 1$, we have $\rho(e_j(u)) = c_j^K(T^\vee)$ and $\rho(v) = c_1^K(L)$, so $\mu = \rho(\tilde{\mu})$ with $\tilde{\mu}$ the same monomial in the $e_j(u)$ and v , a polynomial of order at least one in (u, v) . Hence $\tilde{\mu} = \sum_\chi n_\chi (\chi - 1)$ over monomials χ in the x_i and z with non-negative exponents, and $\gamma_t(\pm \tilde{\mu}) = \prod_\chi (1 + (\chi - 1)t)^{\pm n_\chi}$. Its coefficient of

t^j is symmetric in u and of order at least j in (u, v) , hence an integral polynomial of weight at least j in the $e_i(u)$ and v . Applying ρ gives $\gamma^j(\pm\mu) \in M_j$. $\square$

Algorithm. All classes are represented by their Chern characters in $\text{CH}(\overline{X}) \otimes \mathbb{Q}$, in the basis e_I ordered by degree. The lattice $K(\overline{X})$ is computed as the closure of $\mathbb{Z} \cdot 1$ under multiplication by G ; it has rank 2^n and all pivots of its echelon form equal 1, that is, its graded pieces for the topological filtration are all of $\text{CH}(\overline{X})$. This confirms $M_0 = K(\overline{X})$ independently of the first step of the proof of Lemma 5.2: a sublattice of $K(\overline{X})$ with these graded pieces is all of $K(\overline{X})$, by descending induction on the degree (an additional check of the Euler form is therefore redundant). Then $M_k = \sum_{g \in G} g M_{k-w(g)}$ with $M_k = K(\overline{X})$ for $k \leq 0$. The membership $e_n \in M_k$ is decided as follows. (a) *Odd primes do not obstruct.* A monomial of weight d in G has lowest-degree term the corresponding product of e_1 and of the $2e_j$ in $\text{CH}^d(\overline{X})$, and after inverting 2 these products span $\text{CH}^d(\overline{X})$; by descending induction on d , $M_k \otimes \mathbb{Z}[\frac{1}{2}] = T^k K(\overline{X}) \otimes \mathbb{Z}[\frac{1}{2}]$. As $e_n \in T^n K(\overline{X})$, the question $e_n \in M_k$ for $k \leq n$ is 2-local. (b) *2-adic certificate.* The lattice $M_k + 2^a K(\overline{X})$ with $a = 25$ is computed exactly over $\mathbb{Z}/2^a$ in an integral basis of $K(\overline{X})$. If e_n is not in it, $e_n \notin M_k$. If e_n is in it and $2^{a-1} T^k K(\overline{X}) \subset M_k + 2^a K(\overline{X})$, which the program checks at every such level, then the exponent of T^k/M_k at 2 is less than 2^a (since T^k is saturated in $K(\overline{X})$), and $e_n \in M_k \otimes \mathbb{Z}_{(2)}$, so $e_n \in M_k$ by (a). Positive answers use only $M_k \subset \gamma^k$; negative answers use the full Lemma 5.2.

Results and checks. Table 1 lists $\lambda(n)$ for $n \leq 9$ together with the lower bound $n + 1 - j$ given by Proposition 3.2 from printed index results. The program reproduces $\chi(\mathcal{O}_{\overline{X}}) = 1$ and $\chi(e_n) = 1$, and it reproduces Totaro’s torsion index: by [Kar17, Theorem 4.3], $\gamma^D K(X'_{F(S)})$ (with $D = \dim \overline{X}$) is generated by the classes of closed points, so the g.c.d. of the Euler characteristics on $\gamma^D K(\overline{X})$ must be the index of the maximal orthogonal Grassmannian of a generic form, which is $2^{u(n)}$ [Kar17, Introduction], [Tot05]. The computed values are $2^{u(n)}$ for $n = 1, \dots, 8$, with $u(n) = 0, 0, 1, 1, 1, 2, 3, 4$. For $n = 9$ the computation took about 17 minutes.

n	$\lambda(n)$	bound	input for the bound of Proposition 3.2
1	1	1	$u(1) = 0$ [Tot05]
2	2	2	$u(2) = 0$ [Tot05]
3	2	2	$u(3) = 1$ [Tot05]
4	4	4	forms of dimension 10 in I^3 are isotropic [Kar25, Rem. 2.6]
5	4	4	$u(5) = 1$ [Tot05]
6	4	4	$u(6) = 2$ [Tot05]
7	5	4	$u(7) = 3$ [Tot05]
8	7	5	[Kar25, Corollary 4.4]
9	8	5; 6	$u(9) = 4$ [Tot05]; [Kar23, §8] (computer-assisted)

Table 1: The computed largest k with $e_n \in \gamma^k K(\text{OGr}(n, 2n+1))$, and the lower bound $n + 1 - j$ of Proposition 3.2. Conjecture 1.1 asserts $\lambda(n) < n + 1 - \lfloor (n+1)/2 \rfloor$ for $n \geq 8$, that is $\lambda(8), \lambda(9) \leq 4$. For $n = 9$ the computer-free bound is 5; the bound 6 uses the computer-assisted result of [Kar23, §8] (Remark 4.3).

Independent reproduction. One verification run wrote its own model from the definitions: the Borel presentation of $H^*(\overline{X}; \mathbb{Q})$, integration by Atiyah–Bott localisation at the 2^n fixed points, and $\text{ch}(e_n)$ determined by the projection formula and the Hirzebruch–Riemann–Roch theorem on $\overline{X}$ and on Σ_p , without the formula of (M3); its lattices were computed modulo 2^{64} with the analogous certificate. For $n = 1, \dots, 9$ it found the same values of $\lambda(n)$ and of the torsion exponents, and its $\text{ch}(e_n)$ agrees with (M3). Deliberately wrong models (replacing L by $L^{\otimes 2}$, or omitting $c_1^K(L)$) fail the check against

Totaro’s torsion index. The other verification run reproduced the levels and the torsion exponents for $n \leq 6$ with programs of its own and re-ran the program for $n = 8$. One of the runs also ran its program for $n = 10$ and found $e_{10} \in \gamma^8 \setminus \gamma^9$, which would contradict Conjecture 1.1 at $n = 10$ as well; this was computed by one program only and is not claimed here. A third verification run wrote a further implementation: integral coordinates $\sum_d d! \operatorname{ch}_d$, in which the product has binomial coefficients; $\lambda^i(T^\vee)$ from the Adams operations by Newton’s identities; and all lattices computed exactly over $\mathbb{Z}$ by Hermite normal forms, so that odd primes are treated by the computation itself. For $n = 1, \dots, 8$ it found the same values of $\lambda(n)$ and of the torsion exponents, and the index of M_k in $T^k K(\bar{X})$ a power of 2 for every k ; its values of $\chi(L^k)$, $\chi(\lambda^i T^\vee)$, $\chi(e_n L^k)$ and $\chi(e_n \lambda^i T^\vee)$ agree with those given by the Borel–Weil theorem and the Weyl dimension formula on $\bar{X}$ and on $\Sigma_p \simeq \operatorname{OGr}(n-1, 2n-1)$, which tests (M2) and (M3).

6 Reproducibility, verification, scope and priority

6.1 Reproducibility

No proof in Sections 3 and 4 depends on a computation. The reproducibility package has five parts. The program `check_note.py`, written with this note (standard library, a few seconds), checks the finite and exact parts of the proof: Totaro’s exponents $u(l)$ from the formula of [Tot05, Theorem 0.1] against his table, the list of $n \leq 400$ with $u(n) < [(n+1)/2]$ and the inequalities of Remark 4.5; the term numbers and the values of j in the proof of Theorem 1.2; the g.c.d. step of Proposition 3.2(i); Lemma 3.1 through Chern characters for $n \leq 12$, and its telescoped form against the formula printed in [Kar17]; and, with an implementation of the model of Section 5 independent of the earlier programs, the levels $\lambda(n)$ and the torsion exponents for $n \leq 6$ (for $n = 7$ with an option, about two minutes). The folder `original/` contains the programs with which the results of Section 5 were first obtained and their recorded outputs ($n = 8$: about three minutes; $n = 9$: about 17 minutes), and the folders `verification_run_A/` and `verification_run_B/` the programs and outputs of the first two verification runs described below, and the folder `independent_run_2/` those of the third run (standard library and the package `python-flint` for exact Hermite normal forms). Some recorded outputs were reproduced again when this note was written and by the third run; the details are in the package.

Verification. The results were first obtained with the proofs of Sections 3 and 4 and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each worked from its own copies of the sources and wrote its own programs. The first run compared the statement with [Kar16] and [Kar17]; re-derived Karpenko’s printed implication and Proposition 3.2 step by step, including the two implicit steps of Remark 3.3; checked the inputs against [Tot05] (Theorem 0.1, its table, Theorem 1.1 and the statement for forms in the introduction) and [Kar25] (the abstract, Corollary 4.4 and Section 1); re-derived the model (M1)–(M4), Lemma 5.2 and the 2-adic certificate; reproduced $\lambda(n)$ and the torsion exponents for $n \leq 6$ with programs of its own; and re-ran the original program for $n = 8$. The second run read the preprint of November 2015 and the final version of [Kar17] and found the cited parts unchanged; wrote the independent model described at the end of Section 5; re-ran all original programs, including the one for $n = 9$, with identical outputs; repeated the literature search; and checked the bibliographic data with Crossref. Neither run found a gap in the proof. The corrections they required were made: the cases $n = 9, 11$ now use Totaro’s statement for forms, and the surjectivity of the map from torsors to forms is used only in Remark 4.3; the computer-assisted input of Remark 4.3 is marked as such there and in Table 1; the independence of l_{j-1} from the subspace and the separability of L/K are stated in Proposition 3.2(i); the statements about [DKM21]

and [BD26] and the citation of the final version of [Kar17] in Remark 2.1 were corrected; and the novelty is described as in the introduction. When this note was written, the program `check_note.py` checked Lemma 3.1, Totaro’s exponents and the cases $n \leq 7$ of Section 5 once more.

A third verification run, also AI-assisted, examined the final text of this note as if nothing had been checked. It read again the cited parts of [Kar16], of the final version of [Kar17], of [Tot05], [Kar25], [Kar23], [KM22] and [DKM21], and compared every quoted statement with them; it re-derived Lemma 3.1, Proposition 3.2, the four steps of the proof of Theorem 1.2, Lemma 4.1, the model (M1)–(M4), Lemma 5.2 and the 2-adic certificate line by line. With programs of its own it recomputed Totaro’s exponents, the term numbers, the consequences of the row $d = 18$ of [Kar25, Corollary 4.4], the g.c.d. step, and Lemma 3.1 and (2) through Chern characters on the split quadrics of dimension $2n \leq 24$. It wrote a further implementation of the model of Section 5, described at the end of that section, and re-ran the programs of the package from the archive, with identical results. It checked all DOIs with Crossref and repeated the literature search. It found no gap in the proofs. The corrections it required were made: the statements about $n \geq 13$ in the abstract and in Remark 4.5 were made precise (for $n = 13$ one has $u(13) = [(13 + 1)/2]$), the quotation of Conjecture 3.3 was aligned with [Kar17], the use of characteristic 0 is described completely in Remark 4.2, and the documentation of the package was completed.

Scope and priority. The note shows that Conjecture 1.1 (Conjecture 4.7 of [Kar17], Conjecture 3 of [Kar16]) is false, by proving that it fails for $n = 8, 9$ and 11 for every admissible form in characteristic 0. It does not decide the statement of the conjecture for any other n ; in particular $n = 10$ and $n = 12$ are not decided. The question, the printed implication whose argument is the core of the proof, [Kar17, Lemma 4.1 (after Panin [Pan94]), Proposition 4.2, Theorem 4.3, Remark 4.8], the existence of admissible forms and the input for $n = 8$ [Kar25] (with the computer-assisted alternative [Kar23]) are due to Karpenko; the torsion index of the spin groups, the values $u(9) = 4$, $u(11) = 5$ and their consequence for forms are due to Totaro [Tot05]; the identity of Lemma 3.1 is quoted in [Kar17] from [Kar90, §3.2]. We also used Swan’s computation [Swa85], the book [EKM08] (Propositions 85.2, 86.13 and 86.17, Theorem 86.12, Corollary 86.10), the Clifford algebra formulas of [Lam05, Ch. V], the description of the torsion index as the l.c.m. of the indexes of torsors [KM22, §1] and the specialisation theorem [KM06, Theorem 6.4], the K-theoretic form of [Kar17, (1.2)] via [BR12], the surjectivity $R(P) \rightarrow K(G/P)$ [Pit72, Ste75, Pan94], and [DKM21, Theorem 7.2]. Every ingredient was in print by 2025, and for $n = 9$ and 11 the disproof was available as soon as [Kar17] appeared. Our contribution is only to state the consequence, to fill in the two implicit steps, to give the general form Proposition 3.2, and the computer-assisted supplement of Section 5, whose exact levels we did not find in print. No priority is claimed beyond that.

What was read: Karpenko’s abstract in [Kar16] in the publisher’s file; [Kar17] in the author’s final preprint (revised 4 June 2016), to which our section and statement numbers refer, the journal version not being compared ([DKM21] cites Conjecture 3.3 of the journal version under the same number); of [Tot05], in the author’s file, the introduction, Theorem 0.1 with its table and Theorem 1.1; of [Kar25], in the author’s preprint of 28 August 2023, the abstract, Sections 1 and 2 and Section 4 with Corollary 4.4; of [Kar23], in the author’s preprint revised 23 November 2021, Sections 1 and 5–8; of [KM22] Section 1; of [DKM21] the introduction and the remark after Theorem 7.2; of [BD26] the abstract. The proofs of the cited theorems were not checked. We did not read [Kar90], [Swa85], [Pit72], [Ste75], [Pan94], [BR12], [KM06], nor the cited parts of [EKM08, Lam05]; what we use from them is taken from [Kar17, Kar25] or is a standard fact, and Lemma 3.1 is proved here.

The searches were made on 20 October 2026: the list of the nine works citing [Kar17] in Semantic Scholar (the citing list of OpenAlex was refused with HTTP status 429 and not retried); Karpenko’s publication list and his later papers, searched as text; arXiv listings of Karpenko, Totaro and Yagita;

zbMATH for “gamma filtration” from 2016 on; and two web searches. The third verification run repeated them on the same day (arXiv queries on the gamma and topological filtrations, Karpenko and the spin groups; Crossref; the same nine citing works in Semantic Scholar; OpenAlex, again refused with HTTP status 429; zbMATH; Karpenko’s publication list of 14 September 2026; one further web search). No printed answer to the question was found in these sources, and no printed statement that Conjecture 4.7 or Conjecture 3.3 of [Kar17] is false. The papers of Baek and Devyatov [BD26] and the works cited there concern a different conjecture of Karpenko, on the Chow ring and the graded Grothendieck ring of generic flag varieties, and [DKM21] discusses Conjecture 3.3 but not Conjecture 4.7 or the gamma filtration. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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