

Collapsed Boundary Data Do Not Force Sobolev Labeling

Alper Ferudun*

24 October 2026

Abstract

We construct a smooth polynomial, uniformly gradient-convex energy on two-valued maps of the unit disk whose minimizers have completely collapsed zero boundary data but admit no global labeling by two Sobolev functions. The potential is $\lambda|y^2 - (1 - |z|^2)^2 z|^2$, with complex multiplication. An unordered square-root competitor has energy $16\pi/5$, independently of λ , whereas the minimum over single-valued maps grows on the order of $\sqrt{\lambda}$. An explicit comparison at $\lambda = 2^{20}$ excludes every globally labeled minimizer. The density is quasiconvex in the multiple-valued sense. Thus collapsed boundary data do not imply collapse in the general value-dependent framework of an Oberwolfach question of Spadaro. The conclusion does not address the narrower class of autonomous energies depending only on the gradient.

1 The question and the energy

Write $D = \{z \in \mathbb{C} : |z| < 1\}$, identifying $\mathbb{C}$ with $\mathbb{R}^2$. The space $\mathcal{A}_Q(\mathbb{C})$ consists of unordered Q -tuples $\sum_{i=1}^Q \llbracket y_i \rrbracket$, with matching distance

$$\mathcal{G} \left(\sum_i \llbracket y_i \rrbracket, \sum_i \llbracket w_i \rrbracket \right)^2 = \min_{\sigma \in S_Q} \sum_i |y_i - w_{\sigma(i)}|^2.$$

We use the metric Sobolev spaces and approximate differentials of De Lellis and Spadaro [3]. Expressions involving Du_i below refer to the almost everywhere approximate differential; they do not presuppose a global Sobolev selection.

Open Question 3 in Spadaro's contribution to Oberwolfach Report 53/2015 asks whether a quasiconvex-energy minimizer with boundary $Q\llbracket \phi \rrbracket$ must be $Q\llbracket \Phi \rrbracket$ [5, p. 3043]. The preceding framework permits both spatial and value dependence. We give a counterexample in precisely that generality, and prove the stronger failure of every global Sobolev labeling.

The Sobolev and lower-semicontinuity foundations are established in [3, 2]; value-dependent material energies are studied in [4]. Ball and Zarnescu compare orientable and nonorientable minimizers in liquid-crystal models [1]. Those energy-comparison and lifting ideas are important antecedents. Their simply connected $W^{1,2}$ lifting result assumes a fixed nonzero scalar order parameter. Our unordered amplitude can vanish, at the center and on the boundary, so that hypothesis does not apply. The contribution here is an explicit smooth energy on a disk, a finite parameter value, and quantitative bounds excluding every Sobolev labeling.

Set

$$h(z) = (1 - |z|^2)^2 z, \quad f_\lambda(z, y, A) = |A|^2 + \lambda|y^2 - h(z)|^2, \quad \lambda > 0. \quad (1)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: AlperTheKing.

Here y^2 is the complex square and $|A|$ is the real Frobenius norm. For $Q = 1, 2$, let

$$\mathcal{X}_Q = \{u \in W^{1,2}(D, \mathcal{A}_Q(\mathbb{C})) : \text{Tr } u = Q\llbracket 0 \rrbracket\}, \quad (2)$$

$$E_{Q,\lambda}(u) = \int_D \sum_{i=1}^Q (|Du_i|^2 + \lambda |u_i^2 - h(z)|^2) dz, \quad (3)$$

$$m_Q(\lambda) = \inf_{u \in \mathcal{X}_Q} E_{Q,\lambda}(u). \quad (4)$$

For $Q = 1$ we identify $\mathcal{X}_1$ with $W_0^{1,2}(D, \mathbb{C})$.

Theorem 1. *The symmetric Q -integrand in (3) is smooth, nonnegative, and Q -quasiconvex, with gradient Hessian $2I$. The minima $m_Q(\lambda)$ are attained. For every $\lambda \geq 64$,*

$$\left(\frac{99}{8192} \sqrt{\lambda} - \pi \right) \log 2 \leq m_1(\lambda) \leq \frac{8\pi}{5} + \frac{16}{15\sqrt{15}} \sqrt{\lambda}, \quad (5)$$

$$m_2(\lambda) \leq \frac{16\pi}{5}. \quad (6)$$

In particular,

$$m_2(2^{20}) \leq \frac{16\pi}{5} < \frac{517}{42} \leq 2m_1(2^{20}). \quad (7)$$

Every minimizer in $\mathcal{X}_2$ at $\lambda = 2^{20}$ therefore has no representation $\llbracket v_1 \rrbracket + \llbracket v_2 \rrbracket$ with $v_1, v_2 \in W^{1,2}(D, \mathbb{C})$. In particular it is not $2\llbracket \Phi \rrbracket$.

2 Quasiconvexity and attainment

In the quasiconvexity definition of [2, Definition 0.1], the spatial point and the Q values are frozen while the gradients are varied. The potential in (1) is therefore constant in that test. For an affine multiplicity block of q sheets, a competitor w on a unit-volume cube C has boundary $q\llbracket a + Lx \rrbracket$. Its barycenter $\eta(w) = q^{-1} \sum_i w_i$ has the same single-valued boundary. Consequently

$$\int_C \sum_{i=1}^q |Dw_i|^2 \geq q \int_C |D\eta(w)|^2 \geq q|L|^2.$$

The first inequality is Cauchy–Schwarz. For the second, write $\eta(w) = a + Lx + \psi$, where ψ has zero trace; the cross term vanishes because $\int_C D\psi = 0$. Summing over the affine blocks proves the required Q -quasiconvexity. No global selection of w was used. Alternatively the density is polyconvex in the sense of [2, Definition 0.4 and Theorem 0.5].

For fixed λ the integrand has growth

$$0 \leq e_\lambda(z, a, A) \leq C_\lambda \left(1 + \sum_i |a_i|^4 + \sum_i |A_i|^2 \right).$$

This meets [2, Theorem 0.2]: the Sobolev exponent and domain dimension both equal two, so any finite value-growth exponent is allowed. Thus the functional is weakly lower semicontinuous.

For completeness, the direct method also gives attainment. A minimizing sequence has bounded Dirichlet energy. The scalar function $\mathcal{G}(u, Q\llbracket 0 \rrbracket)$ has zero trace and gradient bounded by $|Du|$, so Poincaré’s inequality bounds its L^2 norm. The compactness and trace results in [3, Propositions 2.10–2.11] provide a strongly L^2 -convergent subsequence with bounded Dirichlet energy and

the prescribed trace. Lower semicontinuity gives a minimizer. One may instead combine lower semicontinuity of the Dirichlet term with Fatou's lemma for the continuous nonnegative potential along an almost everywhere convergent subsequence. The zero map has finite energy, so all the infima under discussion are finite.

3 A bounded-energy unordered competitor

For $0 \leq r \leq 1$, put $a(r) = (1 - r^2)\sqrt{r}$. Define

$$U(re^{i\theta}) = \llbracket a(r)e^{i\theta/2} \rrbracket + \llbracket -a(r)e^{i\theta/2} \rrbracket, \quad U(0) = 2\llbracket 0 \rrbracket. \quad (8)$$

Changing θ by 2π interchanges the sheets and leaves the unordered value unchanged. Each local sheet squares to $h(z)$; the potential vanishes, and the outer trace is $2\llbracket 0 \rrbracket$.

To verify Sobolev membership directly, retain (8) on $r \geq \varepsilon$, and replace $a(r)$ by $a(\varepsilon)r/\varepsilon$ in the inner disk. These unordered maps are Lipschitz and converge in L^2 to U . Their inner-disk Dirichlet energy is $(5\pi/2)a(\varepsilon)^2 = O(\varepsilon)$. The energy outside the core converges to the finite integral below. Metric Sobolev compactness gives $U \in W^{1,2}$, and its almost everywhere differential determines the energy.

A calculation gives

$$ra'(r)^2 + \frac{a(r)^2}{4r} = \frac{1}{2} - 3r^2 + \frac{13}{2}r^4, \quad \int_0^1 \left(ra'^2 + \frac{a^2}{4r} \right) dr = \frac{4}{5}.$$

Counting both sheets,

$$E_{2,\lambda}(U) = 4\pi \cdot \frac{4}{5} = \frac{16\pi}{5}. \quad (9)$$

This proves (6). We do not assert that U itself minimizes the perturbed functional.

4 The scalar transition cost

Lemma 2. *Let $\rho, \beta > 0$ and let $w \in W^{1,2}([0, 2\pi], \mathbb{C})$ satisfy $w(2\pi) = -w(0)$. Then*

$$\int_0^{2\pi} (|w'|^2 + \beta|w^2 - \rho|^2) d\theta \geq \min \left\{ \frac{9\pi}{8}\beta\rho^2, \frac{11}{6}\sqrt{\beta}\rho^{3/2} \right\}. \quad (10)$$

Proof. Write $w = x + iy$. The identity

$$|w^2 - \rho|^2 = (x^2 - \rho)^2 + 2(x^2 + \rho)y^2 + y^4 \geq (x^2 - \rho)^2$$

rules out an inexpensive complex detour. If $|x| \leq \sqrt{\rho}/2$ everywhere, the potential is at least $9\rho^2/16$, giving the first bound.

Otherwise extend w antiperiodically and choose a starting point θ_0 with $|x(\theta_0)| \geq \sqrt{\rho}/2$. The other endpoint of the interval of length 2π has the opposite sign. The energy density is periodic, so shifting the interval preserves its integral. Moreover,

$$|w'|^2 + \beta|w^2 - \rho|^2 \geq 2\sqrt{\beta}|x'| |x^2 - \rho|.$$

Apply the chain rule to the Lipschitz primitive of $(\rho - s^2)\mathbf{1}_{\{|s| \leq \sqrt{\rho}/2\}}$. Its total change along this interval forces a cost of at least

$$2\sqrt{\beta} \int_{-\sqrt{\rho}/2}^{\sqrt{\rho}/2} (\rho - s^2) ds = \frac{11}{6}\sqrt{\beta}\rho^{3/2}.$$

This argument uses absolute continuity, without assuming monotonicity of x . □

Let $v \in W^{1,2}(D, \mathbb{C})$, without imposing any boundary condition. For almost every $r \in [1/4, 1/2]$, the angular slice of v is a periodic $W^{1,2}$ function. Set

$$\rho = r(1 - r^2)^2, \quad w(\theta) = e^{-i\theta/2}v(re^{i\theta}), \quad V = |w^2 - \rho|^2.$$

Then w is antiperiodic and $9/64 \leq \rho \leq 1$. We have

$$|\partial_\theta v|^2 = |w' + iw/2|^2 \geq \frac{1}{2}|w'|^2 - \frac{1}{4}|w|^2, \quad |w|^2 \leq 2\rho + \rho^{-1}V. \quad (11)$$

For the second inequality put $s = |w|^2/\rho$ and note that $V/\rho^2 \geq (s-1)^2$ and $2 + (s-1)^2 - s = (s-3/2)^2 + 3/4 \geq 0$.

Define $J_r = \int_0^{2\pi} (|\partial_\theta v|^2 + \lambda r^2 V) d\theta$. Equation (11) yields

$$J_r \geq \frac{1}{2} \int_0^{2\pi} (|w'|^2 + \beta V) d\theta - \pi\rho, \quad \beta = 2\lambda r^2 - \frac{1}{2\rho}.$$

For $\lambda \geq 64$, we have $\beta \geq \lambda/8 - 32/9 \geq \lambda/16 > 0$. Lemma 2 now gives

$$\begin{aligned} J_r &\geq \min \left\{ \frac{729\pi\lambda}{1048576}, \frac{99\sqrt{\lambda}}{8192} \right\} - \pi \\ &= \frac{99\sqrt{\lambda}}{8192} - \pi. \end{aligned} \quad (12)$$

Indeed, the ratio of the first term to the second is $81\pi\sqrt{\lambda}/1408 > 1$ for $\lambda \geq 64$. Dropping the nonnegative radial energy and the region outside the annulus, then integrating (12) with dr/r , proves the lower bound in (5). It holds for every scalar Sobolev map, even without the zero boundary restriction.

At $\lambda = 2^{20}$ the lower bound is $(99/8 - \pi) \log 2$. Using $\pi < 22/7$ and $\log 2 > 2/3$, we obtain

$$m_1(2^{20}) > \left(\frac{99}{8} - \frac{22}{7} \right) \frac{2}{3} = \frac{517}{84} > \frac{176}{35} > \frac{8\pi}{5}. \quad (13)$$

The logarithm estimate follows by integrating the strict convexity tangent bound for $1/t$ at $t = 3/2$ on $[1, 2]$. For an elementary certificate of the other bound, $\int_0^1 x^4(1-x)^4/(1+x^2) dx = 22/7 - \pi > 0$. Thus (7) is an explicit finite-parameter inequality.

5 Matching order and exclusion of labeling

For $0 < \delta < \pi$, define on $[-\pi, \pi]$

$$t_\delta(\theta) = \begin{cases} -1, & \theta \leq -\delta, \\ \theta/\delta, & |\theta| < \delta, \\ 1, & \theta \geq \delta. \end{cases}$$

The function $v_\delta(re^{i\theta}) = a(r)t_\delta(\theta)e^{i\theta/2}$ is single-valued: its endpoint values at $-\pi$ and π agree. It has zero outer trace and finite Sobolev energy at the origin. Its angular integrals are

$$\int_{-\pi}^{\pi} t_\delta^2 = 2\pi - \frac{4\delta}{3}, \quad \int_{-\pi}^{\pi} (t'_\delta)^2 = \frac{2}{\delta}, \quad \int_{-\pi}^{\pi} (t_\delta^2 - 1)^2 = \frac{16\delta}{15}.$$

Together with $\int_0^1 a^2/r = 8/15$ and $\int_0^1 ra^4 = 1/60$, these give

$$E_{1,\lambda}(v_\delta) = \frac{8\pi}{5} - \frac{16\delta}{15} + \frac{16}{15\delta} + \frac{4\lambda\delta}{225}.$$

Choosing $\delta = \sqrt{60/\lambda} < \pi$ and dropping the negative term proves the upper bound in (5). Thus the scalar minimum has order $\sqrt{\lambda}$, whereas the unordered minimum stays bounded.

Finally suppose a minimizer at $\lambda = 2^{20}$ has a global Sobolev labeling $u = \llbracket v_1 \rrbracket + \llbracket v_2 \rrbracket$. The identity $\mathcal{G}(u, 2\llbracket 0 \rrbracket)^2 = |v_1|^2 + |v_2|^2$ and compatibility of traces with Lipschitz composition force both scalar traces to vanish. The derivative and potential energies split into their scalar sums; on coincidence sets, the derivatives of the coinciding Sobolev maps agree almost everywhere. Therefore

$$E_{2,2^{20}}(u) = E_{1,2^{20}}(v_1) + E_{1,2^{20}}(v_2) \geq 2m_1(2^{20}),$$

contradicting (7). If $u = 2\llbracket \Phi \rrbracket$, the inverse diagonal map is Lipschitz with scale $1/\sqrt{2}$, so Φ is a Sobolev map and would give such a labeling. This completes the proof of Theorem 1.

Remark 3. The conclusion is nonexistence of a global Sobolev labeling, not a description of the singular set or a formula for an actual minimizer. The potential depends on the spatial point and on the map value. Removing those terms changes the problem; no counterexample for autonomous gradient-only energies is asserted.

The accompanying standard-library Python checker verifies the polynomial identities and rational constants, with additional finite checks at 1,025 rational annular radii. These diagnostics do not replace the Sobolev or variational proof. This unrefereed manuscript was developed with assistance from OpenAI Codex for mathematical exploration, literature retrieval, drafting and computational checks. It has undergone the originating researcher's detailed self-audit; independent review, proof-assistant certification and absolute priority are not claimed.

References

- [1] J. M. Ball and A. Zarnescu, *Orientability and energy minimization in liquid crystal models*, Arch. Ration. Mech. Anal. **202** (2011), 493–535. doi:10.1007/s00205-011-0421-3.
- [2] C. De Lellis, M. Focardi and E. Spadaro, *Lower semicontinuous functionals for Almgren's multiple valued functions*, Ann. Acad. Sci. Fenn. Math. **36** (2011), 393–410. doi:10.5186/aasfm.2011.3626.
- [3] C. De Lellis and E. Spadaro, *Q-valued functions revisited*, Mem. Amer. Math. Soc. **211** (2011), no. 991. arXiv:0803.0060.
- [4] M. Focardi, P. M. Mariano and E. Spadaro, *Multi-Value Microstructural Descriptors for Complex Materials: Analysis of Ground States*, Arch. Ration. Mech. Anal. **217** (2015), 899–933. doi:10.1007/s00205-015-0846-1.
- [5] E. Spadaro, joint work with M. Focardi and P. M. Mariano, *Semicontinuous energies defined in the space of multiple valued maps*, in *Mini-Workshop: Scales in Plasticity*, Oberwolfach Report 53/2015, 3040–3043. doi:10.4171/OWR/2015/53.