

Nonconvexity of the First Artin Cone in Six Variables

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Abstract

The level-one Artin cone consists of homogeneous forms that become sums of squares after multiplication by some nonzero quadratic sum of squares. We prove that this cone is not convex for sextics in six variables. The two witnesses are classical Motzkin forms in disjoint triples of variables. Each has a quadratic sum-of-squares multiplier, but their sum has none, even if inhomogeneous multipliers of degree at most two are allowed. The obstruction follows by restricting to zeros in one variable block and inspecting the lowest homogeneous part in the other. The minimum sum-of-squares multiplier degree of the sum is exactly four. Closedness then gives positive-definite witnesses by small perturbation. This supplies a negative answer to the universal convexity question for Artin cones, including its formulation in the 2015 Oberwolfach problem session.

1 The convexity question

Write $H_{n,2d}$ for the real homogeneous forms of degree $2d$ in n variables, including the zero form, and $\Sigma_{n,2d}$ for the cone of sums of squares of degree- d forms. Following Ahmadi and Hall [1, Definition 3.1], define

$$\mathcal{A}_{n,2d}^r = \{p \in H_{n,2d} : pq \in \Sigma_{n,2d+2r} \text{ for some } 0 \neq q \in \Sigma_{n,2r}\}. \quad (1)$$

Here *cone* means closure under nonnegative scalar multiplication, without assuming convexity.

Ahmadi asked whether sets defined by bounded-degree SOS multipliers are convex in the 2015 Oberwolfach problem session [2, p. 2708, item 9]. Ahmadi and Hall formulated the homogeneous question explicitly and noted that even level $r = 1$ was unresolved [1, Open problem 2]. It is also stated in Kunisky's notes [3, Open Problem 2.1]. We give a counterexample at $(n, 2d, r) = (6, 6, 1)$.

The requirement $q \neq 0$ in (1) is essential. It is explicit in [1]; if the zero multiplier were allowed in the shorthand of the Oberwolfach question, the condition would impose no restriction. Our argument also covers nonhomogeneous SOS multipliers of total degree at most two.

Set

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2. \quad (2)$$

This is the classical Motzkin form. For disjoint triples $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$, put

$$F(X, Y) = M(X), \quad G(X, Y) = M(Y).$$

Theorem 1. *The forms F and G belong to $\mathcal{A}_{6,6}^1$, but $F + G$ does not. More strongly, there is no nonzero SOS polynomial $q(X, Y)$ of total degree at most two for which $q(F + G)$ is SOS. Consequently $\mathcal{A}_{6,6}^1$ is not convex. The least total degree of a nonzero SOS multiplier for $F + G$ is four.*

The Motzkin form and its quadratic multiplier identity are classical. The proof combines them with a restriction argument at a spanning set of zeros. Section 4 shows that the nonconvexity also has positive-definite witnesses. No assertion about convexity for every other parameter triple is needed or made.

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2 Classical identities and lowest homogeneous parts

For a polynomial h , its lowest homogeneous part means its nonzero homogeneous component of smallest total degree.

Lemma 2. *The lowest homogeneous part of a nonzero SOS polynomial is SOS. A homogeneous degree- $2d$ SOS polynomial is a sum of squares of homogeneous degree- d forms.*

Proof. Write $h = \sum_i h_i^2$, omitting identically zero summands, and let k be the least degree of a nonzero homogeneous component among the h_i . The degree- $2k$ part of h is the sum of the squares of their degree- k parts. It is nonzero: a sum of squares of real polynomials can vanish identically only if every summand does. This proves the first assertion. The analogous argument with the largest degree shows that if h is homogeneous of degree $2d$, all nonzero components of the h_i have degree d . $\square$

Lemma 3. *The form M is nonnegative but not SOS. It vanishes at*

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad a = (1, 1, 1),$$

which form a basis of $\mathbb{R}^3$. Moreover,

$$(x^2 + y^2)M(x, y, z) = [xy(x^2 + y^2 - 2z^2)]^2 + [xz(z^2 - y^2)]^2 + [yz(z^2 - x^2)]^2. \quad (3)$$

Proof. The arithmetic–geometric mean inequality applied to x^4y^2 , x^2y^4 and z^6 gives nonnegativity. The zeros and their linear independence are immediate. Identity (3) follows by expansion; it is the homogenization of the classical identity in [4, equation (3.19), p. 84].

For completeness, suppose $M = \sum_i h_i^2$. By Lemma 2 the h_i are homogeneous cubics. The zero coefficients of x^6 and y^6 first exclude x^3 and y^3 from every h_i . The coefficients of x^4z^2 and y^4z^2 now exclude x^2z and y^2z : the only other possible products producing these monomials use the already excluded pure cubes. Next the zero coefficients of x^2z^4 and y^2z^4 exclude xz^2 and yz^2 . The remaining cubic monomials are

$$x^2y, \quad xy^2, \quad z^3, \quad xyz.$$

Among products of two of these monomials, only $(xyz)^2$ contributes to $x^2y^2z^2$. Its coefficient in $\sum_i h_i^2$ must therefore be nonnegative, whereas it equals -3 in M . $\square$

We will use the following local consequence of Lemma 2. If P is a non-SOS homogeneous form and $b(X)P(X)$ is SOS, then $b(0)$ cannot be positive. Otherwise the lowest homogeneous part is $b(0)P$, an SOS form, a contradiction. This conclusion also holds when the product is identically zero, since then $b(0) > 0$ is impossible.

3 The obstruction to a quadratic multiplier

Proof of Theorem 1. Let

$$q_X = x_1^2 + x_2^2, \quad q_Y = y_1^2 + y_2^2.$$

Identity (3) proves that $q_X F$ and $q_Y G$ are SOS. Both multipliers are nonzero quadratic SOS forms, so $F, G \in \mathcal{A}_{6,6}^1$.

Suppose now that q is an SOS polynomial of degree at most two and $q(F + G)$ is SOS. Fix a real zero b of M and restrict to $Y = b$. Restriction preserves SOS and gives

$$q(X, b)M(X) \text{ SOS.}$$

The consequence of Lemma 2 just noted implies $q(0, b) \leq 0$. Since q is SOS, $q(0, b) \geq 0$, and hence

$$q(0, b) = 0, \quad q(b, 0) = 0 \quad \text{whenever } M(b) = 0. \quad (4)$$

The second equality follows by interchanging the blocks. Taking $b = 0$ also gives $q(0, 0) = 0$.

Every SOS polynomial of degree at most two is a sum of squares of affine-linear polynomials. Indeed, higher-degree leading square parts cannot cancel, by the same argument as in Lemma 2. Vanishing at the origin eliminates the constants in those affine-linear polynomials. Thus $q = \sum_i \ell_i^2$ with each ℓ_i linear.

Using the three zeros from Lemma 3, (4) says that q vanishes at

$$(e_1, 0), (e_2, 0), (a, 0), (0, e_1), (0, e_2), (0, a). \quad (5)$$

At a zero of a sum of real squares each square vanishes. Every ℓ_i therefore vanishes on the six vectors (5), which form a basis of $\mathbb{R}^6$. Hence every ℓ_i is zero, and $q = 0$.

This proves the asserted impossibility for nonzero q . If $(F + G)/2$ belonged to $\mathcal{A}_{6,6}^1$, multiplying its SOS certificate by 2 would give one for $F + G$. Thus the midpoint fails to belong to the cone, proving nonconvexity.

Finally $q_X q_Y$ is a nonzero quartic SOS form, and

$$q_X q_Y (F + G) = q_Y (q_X F) + q_X (q_Y G) \quad (6)$$

is SOS, since sums and products of SOS polynomials are SOS. The preceding obstruction excludes multiplier degrees zero and two. A nonzero SOS polynomial cannot have odd total degree, again because its highest square terms cannot cancel. Therefore the minimum multiplier degree is exactly four. $\square$

Remark 4. Formula (6) is an explicit twelve-square certificate: multiply each of the three squares for $q_X F$ by y_1^2 and y_2^2 , and each of the three squares for $q_Y G$ by x_1^2 and x_2^2 . No numerical infeasibility calculation is used in the obstruction.

4 Positive-definite witnesses

The preceding endpoints vanish on large sets. That degeneracy is useful for the proof but is not necessary for nonconvexity.

Lemma 5. *The set $\mathcal{A}_{6,6}^1$ is closed in the coefficient topology.*

Proof. Let $p_j \rightarrow p$ with $p_j \in \mathcal{A}_{6,6}^1$. Write a corresponding quadratic multiplier as $q_j(u) = u^T Q_j u$, where Q_j is positive semidefinite. Rescaling does not affect the SOS certificate, so we can impose $\text{tr } Q_j = 1$. The positive semidefinite trace-one matrices form a compact set. Passing to a subsequence gives $Q_j \rightarrow Q$ with Q positive semidefinite and $\text{tr } Q = 1$. Thus $q(u) = u^T Q u$ is nonzero and $p_j q_j \rightarrow p q$.

For clarity, the fixed-degree SOS cone used in this last limit is closed for the following elementary reason. Let $v(u)$ be the vector of all degree-four monomials, so any SOS octic s_j has a Gram representation $s_j = v^T B_j v$ with B_j positive semidefinite. On the unit sphere in $\mathbb{R}^6$, the moment matrix

$$H = \int_{S^5} v v^T d\sigma$$

is positive definite: a nonzero homogeneous quartic cannot vanish on the whole sphere. If s_j converges in coefficients, its integrals are bounded, and

$$\int_{S^5} s_j d\sigma = \text{tr}(B_j H) \geq \lambda_{\min}(H) \text{tr } B_j$$

bounds the PSD matrices B_j . A convergent subsequence of Gram matrices gives an SOS Gram matrix for the limit. Applying this to $s_j = p_j q_j$ yields $p q \in \Sigma_{6,8}$, proving the lemma. $\square$

Corollary 6. *There exist positive-definite sextics $F_\varepsilon, G_\varepsilon$ in $\mathcal{A}_{6,6}^1$ whose midpoint is not in $\mathcal{A}_{6,6}^1$.*

Proof. Put

$$R(X, Y) = (\|X\|^2 + \|Y\|^2)^3, \quad F_\varepsilon = F + \varepsilon R, \quad G_\varepsilon = G + \varepsilon R.$$

For every $\varepsilon > 0$ the two forms are positive definite. They still belong to $\mathcal{A}_{6,6}^1$: use q_X and q_Y , respectively, and note that R and the products $q_X R, q_Y R$ are SOS. Their midpoint converges to $(F + G)/2$ as $\varepsilon \downarrow 0$. Since that limit is outside the closed set $\mathcal{A}_{6,6}^1$, all sufficiently small positive ε give the assertion. $\square$

No explicit value of the perturbation threshold is asserted. The exact unperturbed pair already suffices to answer the universal convexity question.

5 Scope and reproducibility

The conclusion resolves the convexity question in item 9 of [2], under its nonzero-multiplier convention, and gives a negative instance of the general question in [1, 3]. The dataset record OWR-13941-002 combines this question with several unrelated optimization questions. Those other questions are not resolved by this paper.

The source archive includes a standard-library Python checker for the two quadratic identities, the twelve-square quartic identity, the coefficient elimination in Lemma 3, and the ranks in (5). All these computations use exact integer or rational arithmetic and also run under Python’s optimization flag. Finite nonnegativity samples are included only as smoke tests; the analytic proof is independent of them.

AI assistance was used in developing the argument, checking the algebra, searching sources, and preparing the manuscript. This is a self-audited, unrefereed preprint; independent review and formal verification are not claimed. The bounded literature comparison does not establish absolute priority.

References

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