

A Counterexample to a Homogeneous Fractional Hardy Conjecture on the Heisenberg Group

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26 October 2026

Abstract

A conjecture in the 2023 Oberwolfach report on hypoelliptic operators proposes replacing a homogeneous weight in a fractional Hardy inequality by the power of the standard Heisenberg homogeneous norm, while retaining the displayed constant. We give a negative answer. On the first Heisenberg group, one explicit compactly supported real function violates the proposed inequality for every fractional order $99/100 \leq s < 1$. Its fractional Rayleigh quotient is less than $273/284$, whereas the proposed constant is greater than $98/101$. The proof combines an exact radial energy calculation with two elementary Gamma-ratio inequalities and spectral concavity. A single smooth compactly supported witness also exists for the entire interval. The result concerns the proposed constant; it does not rule out a Hardy inequality with a smaller constant.

1 The conjecture and normalization

Use coordinates $(x, y, t) \in \mathbb{H}^1 = \mathbb{R}^3$, Lebesgue measure $dp = dx dy dt$, and the normalization

$$X = \partial_x + \frac{y}{2}\partial_t, \quad Y = \partial_y - \frac{x}{2}\partial_t, \quad \mathcal{L} = -(X^2 + Y^2), \quad \rho = ((x^2 + y^2)^2 + 16t^2)^{1/4}.$$

These are the sublaplacian and homogeneous norm in Roncal–Thangavelu [2, §§2.3, 3.1]. Let $\mathcal{L}_s$ be their conformally invariant fractional power. In the joint spectral decomposition indexed by $k \in \{0, 1, \dots\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, the multipliers of $\mathcal{L}$ and $\mathcal{L}_s$ are, respectively,

$$a_{k,\lambda} = (2k+1)|\lambda|, \quad m_s(k, \lambda) = (2|\lambda|)^s \frac{\Gamma(k+1+s/2)}{\Gamma(k+1-s/2)}. \quad (1)$$

In particular $\mathcal{L}_1 = \mathcal{L}$. This operator is distinguished from the ordinary spectral power $\mathcal{L}^s$.

In [1, p. 1373], Thangavelu reports joint work with Roncal. The conjecture following formula (9) would imply, for $n = 1$,

$$\langle \mathcal{L}_s f, f \rangle \geq C(s) \int_{\mathbb{H}^1} \frac{|f|^2}{\rho^{2s}} dp, \quad C(s) = 4^s \left(\frac{\Gamma(1+s/2)}{\Gamma(1-s/2)} \right)^2, \quad 0 < s < 1. \quad (2)$$

The left side may be read as a quadratic form. Our explicit witness belongs even to the operator domain $D(\mathcal{L})$.

Classical Heisenberg Hardy inequalities distinguish the horizontal gradient from differentiation along homogeneous dilations. See the classical work [3], the distance-specific analysis [4], and the recent quantitative unweighted inequalities of d’Arca, Fanelli, Franceschi and Prandi [5]. We do not claim this distinction as new. The result below is a concrete fractional counterexample to the particular constant in (2), with a uniform explicit interval of orders.

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Theorem 1. Set $R = e^{3\pi}$ and define the real function

$$f(p) = \begin{cases} \rho(p)^{-1} \sin^2(\frac{1}{3} \log \rho(p)), & 1 < \rho(p) < R, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

For every $99/100 \leq s < 1$,

$$\frac{\langle \mathcal{L}_s f, f \rangle}{\int_{\mathbb{H}^1} |f|^2 \rho^{-2s} dp} < \frac{273}{284} < \frac{98}{101} < C(s). \quad (4)$$

Consequently the proposed replacement in (2) is false. There also exists one nonzero real $h \in C_c^\infty(\mathbb{H}^1 \setminus \{0\})$ that violates (2) for every $99/100 \leq s < 1$.

2 Two Gamma bounds and spectral comparison

Lemma 2. For $a > 0$ and $0 < s < 1$,

$$\frac{a}{(a+s)^{1-s}} \leq \frac{\Gamma(a+s)}{\Gamma(a)} \leq a^s. \quad (5)$$

On $\mathbb{H}^1$, the associated form estimate is

$$\langle \mathcal{L}_s v, v \rangle \leq (2-s)^s \langle \mathcal{L}^s v, v \rangle \leq (2-s)^s \langle \mathcal{L} v, v \rangle^s \|v\|_2^{2(1-s)} \quad (6)$$

for v in the form domain of $\mathcal{L}$.

Proof. If Z has Gamma distribution with shape a and unit scale, then $\mathbb{E}Z = a$ and $\mathbb{E}Z^s = \Gamma(a+s)/\Gamma(a)$. Concavity of r^s proves the upper bound. Applying the same argument with shape $a+s$ and exponent $1-s$ gives $\Gamma(a+1)/\Gamma(a+s) \leq (a+s)^{1-s}$, proving the lower bound.

Substitute $a = k + 1 - s/2$ in the upper bound and use (1):

$$m_s(k, \lambda) \leq ((2k+2-s)|\lambda|)^s \leq (2-s)^s a_{k,\lambda}^s.$$

Integrating this pointwise multiplier inequality against the nonnegative Plancherel spectral measure proves the first part of (6). Jensen's inequality for the normalized spectral measure of $\mathcal{L}$ associated with v proves the second. The assertion for $v = 0$ is immediate. $\square$

3 Exact energy of the witness

Write $f(p) = F(\rho(p))$. The values of F and F' are zero at both endpoints of its support. Thus f is C^1 , is piecewise smooth, and has bounded weak second derivatives on a compact annulus. Euclidean mollification converges in $W^{2,2}$ on a fixed compact annulus, hence in the graph norm of $\mathcal{L}$. This proves $f \in D(\mathcal{L})$ and justifies the integration by parts below.

Introduce $\theta \in (-\pi/2, \pi/2)$ and the polar angle $\phi \in [0, 2\pi)$ by

$$x^2 + y^2 = r^2 \cos \theta, \quad 4t = r^2 \sin \theta, \quad r = \rho.$$

The change of variables and a direct horizontal differentiation give

$$dp = \frac{1}{4} r^3 dr d\theta d\phi, \quad |\nabla_{\mathbb{H}} \rho|^2 = \cos \theta.$$

Indeed, the numerators of $X\rho$ and $Y\rho$, with common denominator ρ^3 , are $(x^2 + y^2)x + 4ty$ and $(x^2 + y^2)y - 4tx$. Their squared sum is $\rho^4(x^2 + y^2)$.

Set

$$D = \int_{\mathbb{H}^1} |f|^2 \rho^{-2} dp, \quad E = \langle \mathcal{L}f, f \rangle, \quad M = \|f\|_2^2.$$

Integration in the two angular variables yields

$$D = \frac{\pi^2}{2} \int_1^R r F(r)^2 dr, \quad E = \pi \int_1^R r^3 F'(r)^2 dr.$$

For $u = \log r$ put $g(u) = \sin^2(u/3)$ on $[0, 3\pi]$. Since $F'(r) = r^{-2}(g'(u) - g(u))$,

$$\int_0^{3\pi} g^2 du = \frac{9\pi}{8}, \quad \int_0^{3\pi} (g')^2 du = \frac{\pi}{6}, \quad \int_0^{3\pi} gg' du = 0.$$

Consequently

$$D = \frac{9\pi^3}{16}, \quad E = \frac{31\pi^2}{24}, \quad q := \frac{E}{D} = \frac{62}{27\pi}. \quad (7)$$

Let $D_s = \int |f|^2 \rho^{-2s} dp$. The support condition $1 \leq \rho \leq R$ gives $D_s \geq D$ and $M \leq R^2 D$. Lemma 2 therefore implies

$$\frac{\langle \mathcal{L}_s f, f \rangle}{D_s} \leq [(2-s)q]^s e^{6\pi(1-s)}. \quad (8)$$

4 Rational bounds and smooth approximation

We now prove all the strict comparisons in (4) without numerical Gamma evaluations. Assume $99/100 \leq s < 1$. Using $3 < \pi < 22/7$ gives

$$(2-s)q < \frac{101}{100} \frac{62}{81} = \frac{3131}{4050} < 1.$$

The needed power bound is $(3131/4050)^{99/100} < 39/50$. The ratio of the base to $39/50$ is $3131/3159$. Bernoulli's inequality yields

$$\left(\frac{3131}{3159}\right)^{99} \leq \frac{3131}{3131 + 99 \cdot 28} = \frac{3131}{5903} < \frac{39}{50}.$$

Taking the appropriate hundredth root proves the claim. Since the base is below one, we have $[(2-s)q]^s < 39/50$. Also, by the exponential series,

$$e^{6\pi(1-s)} \leq e^{3\pi/50} < e^{33/175} < \frac{1}{1 - 33/175} = \frac{175}{142}.$$

The upper bound (8) is thus strictly less than $(39/50)(175/142) = 273/284$.

For the proposed constant, the lower bound in (5) gives

$$C(s) \geq \frac{4^s(1-s/2)^2}{(1+s/2)^{2(1-s)}}.$$

Here $(1-s/2)^2 \geq 1/4$. From $\log 4 < 2$ and $e^{-x} > 1-x$ we obtain $4^s \geq 4^{99/100} > 98/25$. Finally, concavity gives

$$(1+s/2)^{2(1-s)} \leq (3/2)^{1/50} \leq 1 + \frac{1}{50} \frac{1}{2} = \frac{101}{100}.$$

Therefore $C(s) > 98/101$, completing (4), since

$$\frac{98}{101} - \frac{273}{284} = \frac{259}{28684} > 0.$$

It remains to justify the smooth version. Mollify f inside a fixed compact annulus bounded away from zero. The real smooth approximants f_j converge to f in the form norm of $\mathcal{L}$. For $99/100 \leq s \leq 1$, the multiplier proof above also gives $\mathcal{L}_s \leq 2(I + \mathcal{L})$ as forms. By Cauchy–Schwarz in each form norm, $\langle \mathcal{L}_s f_j, f_j \rangle$ converges uniformly in s to $\langle \mathcal{L}_s f, f \rangle$. The weights ρ^{-2s} are uniformly bounded on the fixed annulus, so $D_s(f_j)$ converges uniformly as well. The function $C(s)$ is continuous and bounded on this closed interval. The gap for f is at least $(259/28684)D > 0$, uniformly in s . Thus every sufficiently large j gives one smooth witness for all $99/100 \leq s < 1$. This completes the proof of Theorem 1.

5 Scope and reproducibility

The result refutes the proposed homogeneous replacement with the constant printed in [1, p. 1373]. It does not determine the optimal homogeneous Hardy constant, and it makes no assertion of failure for every $0 < s < 99/100$. The known nonhomogeneous and differently weighted inequalities are not contradicted.

The accompanying standard-library Python checker verifies the horizontal gradient polynomial identity, the exact Fourier-polynomial integrals and all rational comparisons. These checks supplement the spectral and Sobolev arguments; they are not a formal verification of the theorem. The work was developed with AI assistance and checked by the originating researcher. It has not undergone independent peer review. The bounded literature comparison found no identical fractional counterexample and makes no absolute priority claim.

References

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