

# Star Transforms Without Type 2 Singular Directions and Conflitti's Conjecture on Elementary Symmetric Polynomials

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## Abstract

The star transform integrates a function on the plane along  $m$  rays with a common vertex, with directions  $\gamma_1, \dots, \gamma_m$  and nonzero weights  $c_1, \dots, c_m$ . In the inversion formula of Ambartsoumian and Latifi, the directions  $\psi$  at which  $\sum_j c_j \prod_{i \neq j} \langle \psi, \gamma_i \rangle$  vanishes, called singular directions of Type 2, cause instability. They showed that every star with an even number of rays has such directions, and that for  $m = 3$  every choice of weights admits ray directions without them. For odd  $m \geq 5$  this was left as a conjecture, which the Oberwolfach Report 21/2023 records as unproven. We prove the conjecture for every odd  $m$  and all nonzero real weights. The proof reduces the problem, by nearly parallel rays, to a rational function of one variable without real zeros, which we build by nesting explicit three-pole clusters. The same report recalls a conjecture of Conflitti (2006): for even  $r$ , the real zero set of the elementary symmetric polynomial  $e_r$  in  $n$  variables contains no linear subspace of dimension  $r$ . Previously the cases  $r = 2$  (Conflitti) and  $n = r + 1$  (Ambartsoumian and Latifi; this case also follows from the irreducibility of  $e_{n-1}$ ) were known. We prove the conjecture for all even  $r$  and all  $n$ , using Descartes' rule of signs and a parity argument. Hence the largest subspace in the zero set has dimension  $\min(n, r - 1)$ . For even  $r \geq 4$  we also show that the only  $(r - 1)$ -dimensional subspaces in the zero set are coordinate subspaces. Exact computer certificates for explicit stars with at most 21 rays illustrate the construction; the proofs do not depend on them. This is an unrefereed note.

## 1 Introduction

**Star transforms.** For a unit vector  $\gamma \in \mathbb{S}^1$  and  $f \in C_c(\mathbb{R}^2)$ , the divergent beam transform is  $\mathcal{X}_\gamma f(x) = \int_0^\infty f(x + t\gamma) dt$ . Given unit vectors  $\gamma_1, \dots, \gamma_m$  and nonzero real weights  $c_1, \dots, c_m$ , the *star transform* is  $\mathcal{S}f = \sum_{i=1}^m c_i \mathcal{X}_{\gamma_i} f$ ; it arises in single-scattering tomography [ZSM14, AL21]. Ambartsoumian and Latifi [AL21, Section 3] proved the inversion formula  $\mathcal{R}f(\psi, s) = q(\psi) \frac{d}{ds} \mathcal{R}(\mathcal{S}f)(\psi, s)$ , where  $\mathcal{R}$  is the Radon transform,  $q = -1/w$  and  $w(\psi) = \sum_i c_i / \langle \psi, \gamma_i \rangle$ . Clearing denominators,  $w(\psi) = P_{c,\gamma}(\psi) / \prod_i \langle \psi, \gamma_i \rangle$  with

$$P_{c,\gamma}(\psi) = \sum_{j=1}^m c_j \prod_{i \neq j} \langle \psi, \gamma_i \rangle. \quad (1)$$

The zeros of  $P_{c,\gamma}$  on  $\mathbb{S}^1$  are the *singular directions of Type 2* [AL21, (13) and (23)].<sup>1</sup> At these directions the inversion formula divides by zero, and the numerical reconstructions in [AL21, Section 8] show

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<sup>1</sup>In [AL21, (13)] the factor is printed as  $\langle \psi, \gamma_j \rangle$ ; the definition (23) there and the proof of Theorem 6(b) show that  $\langle \psi, \gamma_i \rangle$  is meant. The report [Amb23b], the source of the record, has the same index misprint in its condition (6) and Definition 5 (p. 1130), where the condition on a direction  $\psi_i$  reads  $\sum_j c_j \prod_{i \neq j} \langle \psi_i, \gamma_j \rangle = 0$ .

strong artifacts; an equivalent condition governs the instability of the different inversion method of Zhao, Schotland and Markel [ZSM14], as observed in [AL21, Amb23b]. Since  $P_{c,\gamma}$  is a binary form of degree  $m - 1$ , a star has no singular direction of Type 2 if and only if  $P_{c,\gamma}$  is a definite form.

Ambartsoumian and Latifi [AL21, Theorems 3–6] showed the following. If  $m$  is even, every star has singular directions of Type 2, because  $P_{c,\gamma}$  is then an odd function. If  $m$  is odd, the regular star with equal weights has none; for every choice of weights some rays produce singular directions of Type 2; and for  $m = 3$  and every choice of weights some rays produce none. The statement for even  $m$  had been shown earlier by Zhao, Schotland and Markel, in the stability analysis of their inversion method [ZSM14, Section 8.2]: by a sign argument,  $w$  always has zeros when  $m$  is even, and also when  $m \geq 3$  is odd and the vectors  $c_1\gamma_1, \dots, c_m\gamma_m$  lie in a closed half-plane. Ambartsoumian and Latifi stated the extension of their result for  $m = 3$  to all odd  $m$  as Conjecture 1 [AL21, Section 5.3], and the Oberwolfach report [Amb23b, p. 1131] records that it had not been proven. This question is the first part of record OWR-13750332-001 of the UnsolvedMath dataset [Uns26]. We answer it.

**Theorem 1.1.** *Let  $m \geq 1$  be odd and let  $c_1, \dots, c_m$  be nonzero real numbers. Then there are unit vectors  $\gamma_1, \dots, \gamma_m \in \mathbb{S}^1$  such that  $P_{c,\gamma}(\psi) \neq 0$  for every  $\psi \in \mathbb{S}^1$ . Any such vectors are pairwise non-parallel, and in particular distinct. In other words, for any prescribed nonzero weights there is a star transform with these weights and no singular direction of Type 2.*

So these stars have distinct directions, as the definition of the star transform in [AL21, Definition 2] requires. For  $m = 3$  this gives a new proof of [AL21, Theorem 6(b)]. The proof in Section 2 is constructive, and an explicit example with the zero-sum weights of [AL21, (29)] is worked out in Example 2.8.

**Conflitti’s conjecture.** Let  $e_r$  denote the  $r$ -th elementary symmetric polynomial in  $n$  variables, with  $e_0 = 1$  and  $e_r = 0$  for  $r > n$ . Conflitti showed that the real zero set of  $e_2$  is a circular cone, which contains no plane [Con06, Theorem 6, Corollary 7], and made the following conjecture.

**Conjecture 1.2** (Conflitti [Con06, Conjecture 8]). *If  $r$  is even, then  $e_r^{-1}(0) \subseteq \mathbb{R}^n$  contains no real vector subspace of dimension  $r$ .*

He described the case of  $e_{n-1}$  with  $n$  odd as “a task quite hard to tackle”. Ambartsoumian and Latifi proved that case with the regular star [AL21, Theorem 7], and [Amb23b] recalls the conjecture. In version 1.6.0 of the UnsolvedMath dataset [Uns26], the statement of record OWR-13750332-001 quotes both questions; its title is taken from the first, and its literature note concerns the second. In the revision of 28 September 2026 the record has a new title and a cleaned statement of the second question only. For odd  $r$  the analogous statement fails once  $n \geq 2r$ :  $e_r$  vanishes on the  $\lfloor n/2 \rfloor$ -dimensional space of vectors  $(a_1, -a_1, a_2, -a_2, \dots)$  [AGZ00, Con06].

**Theorem 1.3.** *Let  $r \geq 2$  be even and  $n \geq 1$ . Then  $e_r^{-1}(0) \subseteq \mathbb{R}^n$  contains no linear subspace of dimension  $r$ .*

**Corollary 1.4.** *Let  $r \geq 2$  be even. The largest dimension of a linear subspace of  $\mathbb{R}^n$  on which  $e_r$  vanishes is  $\min(n, r - 1)$ .*

**Theorem 1.5.** *Let  $r \geq 4$  be even and  $n \geq r - 1$ . Every  $(r - 1)$ -dimensional linear subspace of  $\mathbb{R}^n$  on which  $e_r$  vanishes is a coordinate subspace, that is, it is spanned by  $r - 1$  vectors of the standard basis.*

*Remark 1.6* (What was known). The cases  $n \leq r$  of Theorem 1.3 are trivial: for  $n < r$  there is no  $r$ -dimensional subspace, and for  $n = r$  we have  $e_r = x_1 \cdots x_n \neq 0$ . The cases  $n = r - 1$  and  $n = r$  of Theorem 1.5 are trivial as well: for  $n = r - 1$  the only  $(r - 1)$ -dimensional subspace is  $\mathbb{R}^n$ , and for  $n = r$

an  $(r-1)$ -dimensional subspace on which  $e_r = x_1 \cdots x_r$  vanishes lies in, and hence equals, a coordinate hyperplane. The case  $n = r+1$ , that is,  $e_{n-1}$  with  $n$  odd, is the one proved in [AL21, Theorem 7]. It is also elementary:  $e_{n-1}$  is irreducible for  $n \geq 3$ , so its zero set contains no hyperplane (Lemma 3.2). More generally, Chirvasitu [Chi25, Theorem 0.2] described all linear subspaces contained in the zero locus of  $e_{n-1}$  over any infinite field; this includes the case  $n = r+1$  of Theorems 1.3 and 1.5. The case  $r = 2$  is due to Conflitti. So the new content of Theorem 1.3 is  $r \geq 4$  and  $n \geq r+2$ , and that of Theorem 1.5 is  $n \geq r+2$ . After Conjecture 8, Conflitti also suggested a stronger statement; it fails for every even  $r \geq 4$  (Remark 3.4).

**Methods.** For Theorem 1.1, reversing a ray absorbs the sign of its weight, and nearly parallel rays reduce the problem to one about a rational function  $\sum_j a_j/(t - b_j)$  of one variable: its poles  $b_j$  must be chosen so that it has no real zero. Three poles can be placed explicitly, and larger configurations are obtained by inserting a small copy of a solution in place of one pole. For Theorem 1.3, Descartes' rule of signs shows that  $e_{r-1}$  and  $e_r$  vanish simultaneously only at vectors with at most  $r-2$  nonzero coordinates. On an  $r$ -dimensional subspace inside  $e_r^{-1}(0)$ , the odd form  $e_{r-1}$  would then vanish only on a finite union of subspaces of codimension at least 2, which is impossible for a form of odd degree.

**Scope of the proofs.** All results are proved by hand, for all parameters, in Sections 2 and 3. Exact computer certificates (Section 4) confirm explicit stars for listed weight vectors only; no statement of the paper depends on them.

## 2 Stars without singular directions of Type 2

**Lemma 2.1** (Reversing rays; cf. [AL21, Remark 2]). *Let  $\varepsilon_1, \dots, \varepsilon_m \in \{1, -1\}$ . Then  $P_{(\varepsilon_i c_i), (\varepsilon_i \gamma_i)} = (\prod_i \varepsilon_i) P_{c, \gamma}$ . Moreover,  $P_{c, \gamma}$  does not change if the pairs  $(c_i, \gamma_i)$  are permuted.*

*Proof.* The  $j$ -th term becomes  $\varepsilon_j c_j \prod_{i \neq j} \varepsilon_i \langle \psi, \gamma_i \rangle = (\prod_i \varepsilon_i) c_j \prod_{i \neq j} \langle \psi, \gamma_i \rangle$ . The second statement is clear.  $\square$

**Definition 2.2.** A *list* is a finite sequence  $L = ((b_1, a_1), \dots, (b_n, a_n))$  of *positions*  $b_j \in \mathbb{R}$  and *weights*  $a_j \in \mathbb{R} \setminus \{0\}$ . Put

$$g_L(t) = \sum_{j=1}^n a_j \prod_{i \neq j} (t - b_i), \quad A_L = \sum_{j=1}^n a_j.$$

The list is *good* if  $A_L \neq 0$  and  $g_L$  has no real zero.

If the positions are distinct, then  $g_L(t)/\prod_i (t - b_i) = \sum_j a_j/(t - b_j)$  is the affine analogue of  $w$ . The polynomial  $g_L$  has degree  $n-1$  with leading coefficient  $A_L$  when  $A_L \neq 0$ . The positions of a good list are distinct: if  $b_i = b_j$  with  $i \neq j$ , every term of  $g_L$  contains  $t - b_i$  or  $t - b_j$ , so  $g_L(b_i) = 0$ .

**Lemma 2.3** (Nearly parallel rays). *Let  $L = ((b_1, a_1), \dots, (b_n, a_n))$  be a good list, and let  $\varphi_1(\theta), \dots, \varphi_n(\theta)$  be real functions with  $\varphi_j(\theta) = \theta b_j + o(\theta)$  as  $\theta \rightarrow 0^+$ . Put  $\ell_j(\theta) = (\cos \varphi_j(\theta), \sin \varphi_j(\theta))$ . Then for all sufficiently small  $\theta > 0$  the form  $P_{a, \ell(\theta)}$  has no zero on  $\mathbb{S}^1$ .*

*Proof.* For  $\theta > 0$  and  $(u, v) \in \mathbb{R}^2$  we have  $\langle (\theta u, -v), \ell_j(\theta) \rangle = \theta \lambda_{j, \theta}(u, v)$  with  $\lambda_{j, \theta}(u, v) = u \cos \varphi_j(\theta) - v \sin \varphi_j(\theta)/\theta$ . Hence

$$P_{a, \ell(\theta)}(\theta u, -v) = \theta^{n-1} G_\theta(u, v), \quad G_\theta = \sum_j a_j \prod_{i \neq j} \lambda_{i, \theta}.$$

As  $\theta \rightarrow 0^+$ ,  $\cos \varphi_j(\theta) \rightarrow 1$  and  $\sin \varphi_j(\theta)/\theta \rightarrow b_j$ , so the coefficients of the binary form  $G_\theta$  converge to those of  $G_0(u, v) = \sum_j a_j \prod_{i \neq j} (u - b_i v)$ . The form  $G_0$  has no zero on the unit circle: if  $v \neq 0$  then  $G_0(u, v) = v^{n-1} g_L(u/v) \neq 0$ , and  $G_0(\pm 1, 0) = (\pm 1)^{n-1} A_L \neq 0$ . Let  $\mu > 0$  be the minimum of  $|G_0|$  on the unit circle. Coefficientwise convergence of forms of fixed degree is uniform on the circle, so  $|G_\theta| \geq \mu/2$  there for all small  $\theta$ . For such  $\theta$  and  $\psi \neq 0$ , the vector  $(u, v) = (\psi_1/\theta, -\psi_2)$  is nonzero, and by homogeneity  $P_{a, \ell(\theta)}(\psi) = \theta^{n-1} G_\theta(u, v) \neq 0$ .  $\square$

**Lemma 2.4** (Three poles). *Let  $W_1, W_2, W_3 > 0$  with  $W_1 + W_3 > W_2$ . Put  $S = W_1 + W_3$ ,  $D = W_3 - W_1$ ,  $A = S - W_2$  and  $b^* = D(2W_2 - S)/S^2$ . For  $b \in \mathbb{R}$ , the list  $((-1, W_1), (b, -W_2), (1, W_3))$  is good if and only if  $S^2|b - b^*| < 4\sqrt{AW_1W_2W_3}$ . In particular the list*

$$T(W_1, W_2, W_3) = ((-1, W_1), (b^*, -W_2), (1, W_3))$$

*is good, and its weights sum to  $A$ .*

*Proof.* Expanding  $g(t) = W_1(t-b)(t-1) - W_2(t+1)(t-1) + W_3(t+1)(t-b)$  gives  $g(t) = At^2 + Bt + C$  with  $B = D - bS$  and  $C = W_2 - bD$ . Hence

$$B^2 - 4AC = S^2b^2 - 2D(S - 2A)b + D^2 - 4AW_2 = S^2(b - b^*)^2 + D^2 - 4AW_2 - S^2(b^*)^2,$$

because  $S - 2A = 2W_2 - S$ . Since  $S^2 - (2W_2 - S)^2 = 4W_2(S - W_2) = 4AW_2$  and  $S^2 - D^2 = 4W_1W_3$ , the constant term is  $4AW_2D^2/S^2 - 4AW_2 = -16AW_1W_2W_3/S^2$ . Thus

$$B^2 - 4AC = S^2(b - b^*)^2 - 16AW_1W_2W_3/S^2. \quad (2)$$

As  $A > 0$ , the quadratic  $g$  has no real zero exactly when (2) is negative, and the weights sum to  $A \neq 0$ .  $\square$

**Lemma 2.5** (Inserting a cluster). *Let  $L = ((b_1, a_1), \dots, (b_n, a_n))$  and  $K = ((\beta_1, \alpha_1), \dots, (\beta_p, \alpha_p))$  be good lists with  $\sum_l \alpha_l = a_1$ . For  $\delta > 0$  let  $L_\delta$  be the list obtained from  $L$  by replacing  $(b_1, a_1)$  with the  $p$  entries  $(b_1 + \delta\beta_l, \alpha_l)$ ,  $l = 1, \dots, p$ . Then  $L_\delta$  is good for all sufficiently small  $\delta > 0$ .*

*Proof.* The weights of  $L_\delta$  sum to  $A_L \neq 0$ , so  $g_{L_\delta}$  has degree  $n + p - 2$  and leading coefficient  $A_L$  for every  $\delta$ . Put

$$Q(t) = \prod_{j=2}^n (t - b_j), \quad \hat{g}(t) = \sum_{j=2}^n a_j \prod_{2 \leq i \leq n, i \neq j} (t - b_i), \quad H(u) = \prod_{l=1}^p (u - \beta_l).$$

Sorting the terms of  $g_L$  (respectively  $g_{L_\delta}$ ) according to whether the omitted factor comes from the first entry of  $L$  (respectively from the inserted cluster) or not, we get

$$g_L(t) = a_1 Q(t) + (t - b_1) \hat{g}(t), \quad (3)$$

$$g_{L_\delta}(t) = \delta^{p-1} [Q(t) g_K(u) + \delta H(u) \hat{g}(t)], \quad u = (t - b_1)/\delta. \quad (4)$$

Indeed,  $\sum_l \alpha_l \prod_{l' \neq l} (t - b_1 - \delta\beta_{l'}) = \delta^{p-1} g_K(u)$  and  $\prod_l (t - b_1 - \delta\beta_l) = \delta^p H(u)$ . The first of these tends to  $a_1(t - b_1)^{p-1}$  and the second to  $(t - b_1)^p$ , coefficientwise, as  $\delta \rightarrow 0$ . So by (4) and (3),  $g_{L_\delta}(t) \rightarrow (t - b_1)^{p-1} g_L(t)$  coefficientwise, hence uniformly on compact sets. The coefficients of  $g_{L_\delta}$  are polynomials in  $\delta$ , so by Cauchy's bound all zeros of  $g_{L_\delta}$ ,  $0 < \delta \leq 1$ , lie in a fixed disc  $|t| \leq M$ .

*Near  $b_1$ .* The positions of  $L$  are distinct, so  $Q(b_1) \neq 0$ . Choose  $\eta_0 \in (0, 1]$  and constants  $\kappa_1, \Lambda_1 > 0$  with  $|Q(t)| \geq \kappa_1$  and  $|\hat{g}(t)| \leq \Lambda_1$  for  $|t - b_1| \leq \eta_0$ . Since  $K$  is good,  $g_K$  has no real zero and exact

degree  $p - 1$ , so  $\kappa_2 = \inf_{u \in \mathbb{R}} |g_K(u)| / (1 + |u|)^{p-1}$  is positive. Put  $\Lambda_2 = \sup_{u \in \mathbb{R}} |H(u)| / (1 + |u|)^p < \infty$ . If  $|t - b_1| \leq \eta \leq \eta_0$  and  $u = (t - b_1)/\delta$ , then  $\delta(1 + |u|) \leq \delta + \eta$ , and (4) gives

$$|g_{L_\delta}(t)| \geq \delta^{p-1}(1 + |u|)^{p-1}[\kappa_1\kappa_2 - \Lambda_1\Lambda_2(\delta + \eta)].$$

With  $\eta = \min(\eta_0, \kappa_1\kappa_2/(3\Lambda_1\Lambda_2))$  and  $\delta < \kappa_1\kappa_2/(3\Lambda_1\Lambda_2)$  this is positive, so  $g_{L_\delta}$  has no zero in  $[b_1 - \eta, b_1 + \eta]$ .

*Away from  $b_1$ .* On the compact set  $E = \{t \in \mathbb{R} : |t - b_1| \geq \eta, |t| \leq M\}$  the limit satisfies  $|(t - b_1)^{p-1}g_L(t)| \geq \eta^{p-1} \min_E |g_L| > 0$ , so uniform convergence excludes zeros of  $g_{L_\delta}$  in  $E$  for small  $\delta$ . There are no zeros with  $|t| > M$ . Hence  $L_\delta$  is good for all small  $\delta > 0$ .  $\square$

**Lemma 2.6** (Nesting). *Let  $m = 2k + 1 \geq 3$  and  $u_1 \geq u_2 \geq \dots \geq u_m > 0$ . There is a good list with  $m$  entries whose weights are  $u_1, \dots, u_{k+1}, -u_{k+2}, \dots, -u_m$ , each used once.*

*Proof.* Put  $W_1 = u_1 + u_2 - u_m$  and  $W_{t+1} = W_t + u_{t+2} - u_{m-t}$  for  $1 \leq t \leq k - 1$ . For  $t \leq k - 1$  we have  $t + 2 \leq k + 1 < k + 2 \leq m - t$ , so  $u_{t+2} \geq u_{m-t}$  and  $W_{t+1} \geq W_t \geq W_1 \geq u_1 > 0$ . Let  $L_1 = T(u_1, u_m, u_2)$  from Lemma 2.4; it is good because  $u_1 + u_2 > u_m$ , its weights are  $u_1, -u_m, u_2$  and they sum to  $W_1$ . Given a good list  $L_t$  whose weights sum to  $W_t$ , the list  $T(W_t, u_{m-t}, u_{t+2})$  is good because  $W_t + u_{t+2} > W_t \geq u_1 \geq u_{m-t}$ . Insert  $L_t$  in place of its entry  $(-1, W_t)$  by Lemma 2.5, with a small  $\delta$ . The result  $L_{t+1}$  is good, its weights are those of  $L_t$  together with  $-u_{m-t}$  and  $u_{t+2}$ , and they sum to  $W_{t+1}$ . After  $k - 1$  steps,  $L_k$  has the required weights.  $\square$

*Proof of Theorem 1.1.* For  $m = 1$  we have  $P_{c,\gamma} = c_1 \neq 0$ . Let  $m = 2k + 1 \geq 3$ , and let  $u_1 \geq \dots \geq u_m$  be the numbers  $|c_1|, \dots, |c_m|$  in decreasing order. Take the good list  $((b_j, a_j))_{j \leq m}$  of Lemma 2.6 and a bijection  $\tau$  of  $\{1, \dots, m\}$  with  $|a_j| = |c_{\tau(j)}|$ . By Lemma 2.3 with  $\varphi_j(\theta) = \theta b_j$ , there is  $\theta > 0$  such that the unit vectors  $\ell_j = (\cos \theta b_j, \sin \theta b_j)$  give a form  $P_{a,\ell}$  without zeros on  $\mathbb{S}^1$ . Put  $\varepsilon_j = \text{sign}(a_j c_{\tau(j)})$  and  $\gamma_{\tau(j)} = \varepsilon_j \ell_j$ . Then  $c_{\tau(j)} = \varepsilon_j a_j$ , and by Lemma 2.1  $P_{c,\gamma} = \pm P_{a,\ell}$  has no zero on  $\mathbb{S}^1$ . The second statement of the theorem is shown in Remark 2.9.  $\square$

**Remark 2.7** (Signs in good lists). Let  $L$  be a good list with  $n$  entries, ordered so that  $b_1 < \dots < b_n$ . Since  $g_L$  has no real zero,  $n - 1 = \deg g_L$  is even and  $g_L$  has the sign of  $A_L$  everywhere. As  $g_L(b_j) = a_j \prod_{i \neq j} (b_j - b_i)$  has the sign of  $(-1)^{n-j} a_j$ , the weights alternate in sign, and the  $k + 1$  weights in odd places have the sign of  $A_L$ , where  $n = 2k + 1$ . Their absolute values therefore add up to more than those of the other  $k$  weights. This is why the proof pairs large and small magnitudes, and why rays must be reversed: for instance, if all  $c_i > 0$ , then  $k$  of the rays of the star constructed above point roughly opposite to the others. The alternation is also how the construction meets the necessary condition of Zhao, Schotland and Markel [ZSM14, Section 8.2] recalled in the introduction: for odd  $m \geq 3$ , the vectors  $c_i \gamma_i$  of a star without singular directions of Type 2 do not lie in a closed half-plane. In the star constructed above,  $c_{\tau(j)} \gamma_{\tau(j)} = a_j \ell_j$ , and for small  $\theta$  these vectors point alternately close to  $(1, 0)$  and close to  $(-1, 0)$ , in the order of the positions  $b_j$ .

**Example 2.8** (Zero-sum weights). Weights with  $\sum_i c_i = 0$  are important for the simultaneous recovery of attenuation and scattering, and [AL21, (29)] proposes  $c_1 = \dots = c_k = -1/k$ ,  $c_{k+1} = \dots = c_{2k+1} = 1/(k + 1)$ . For  $k = 2$  we use the proportional weights  $c = (-3, -3, 2, 2, 2)$ , so  $(u_1, \dots, u_5) = (3, 3, 2, 2, 2)$ . Here  $L_1 = T(3, 2, 3) = ((-1, 3), (0, -2), (1, 3))$  with  $g_{L_1} = 4t^2 + 2$ , and  $T(4, 2, 2) = ((-1, 4), (\frac{1}{3}, -2), (1, 2))$  with  $18g_{T(4,2,2)} = 72t^2 - 48t + 40$ . Inserting  $L_1$  with  $\delta = \frac{1}{2}$  gives the good list

$$L_2 = ((-\frac{3}{2}, 3), (-1, -2), (-\frac{1}{2}, 3), (\frac{1}{3}, -2), (1, 2)), \quad 18g_{L_2}(t) = 72t^4 + 96t^3 + 25t^2 + 14t + 33.$$

For the rays we use the exact rational unit vectors  $\ell(\tau) = ((1 - \tau^2)/(1 + \tau^2), 2\tau/(1 + \tau^2))$ , at the angle  $2 \arctan \tau$ , with  $\tau = b_j/2$ . These belong to the family of Lemma 2.3 with  $\varphi_j(\theta) = 2 \arctan(\theta b_j/2)$ ,

taken at  $\theta = 1$ ; definiteness is checked directly below. We match the entries  $(-\frac{3}{2}, 3)$  and  $(-\frac{1}{2}, 3)$  of  $L_2$  with  $c_1$  and  $c_2$ , the entries  $(-1, -2)$  and  $(\frac{1}{9}, -2)$  with  $c_3$  and  $c_4$ , and  $(1, 2)$  with  $c_5$ . Reversing the four rays whose list weight has the wrong sign gives

$$\gamma_1 = (-\frac{7}{25}, \frac{24}{25}), \gamma_2 = (-\frac{15}{17}, \frac{8}{17}), \gamma_3 = (-\frac{3}{5}, \frac{4}{5}), \gamma_4 = (-\frac{323}{325}, -\frac{36}{325}), \gamma_5 = (\frac{3}{5}, \frac{4}{5}).$$

A direct computation gives, with  $v = (x^2, xy, y^2)$ ,

$$\frac{5^6 \cdot 13 \cdot 17}{4} P_{c, \gamma}(x, y) = 923481x^4 - 1275450x^3y + 541472x^2y^2 + 120800xy^3 + 623616y^4 = v G v^\top,$$

$$G = \begin{pmatrix} 923481 & -637725 & -250000 \\ -637725 & 1041472 & 60400 \\ -250000 & 60400 & 623616 \end{pmatrix}.$$

The leading principal minors of  $G$  are 923481, 555086428407 and 296959066692499712, so  $G$  is positive definite, and this star has no singular direction of Type 2. By contrast, with the regular directions  $\gamma_j = (\cos \frac{2\pi j}{m}, \sin \frac{2\pi j}{m})$ ,  $m = 2k + 1$ , there are singular directions of Type 2 whenever the weights have both signs. Indeed, if  $\psi_j$  is  $\gamma_j$  rotated by  $\pi/2$ , then only the  $j$ -th term of (1) survives,  $P_{c, \gamma}(\psi_j) = c_j \prod_{i \neq j} \sin \frac{2\pi(i-j)}{m}$ , and the product has the sign  $(-1)^k$  for every  $j$ . So  $P_{c, \gamma}$  changes sign on  $\mathbb{S}^1$ .

*Remark 2.9* (The construction in practice). The proof gives no explicit bounds for  $\delta$  and  $\theta$ , but it is effective: one may try  $\delta, \theta = 2^{-1}, 2^{-2}, \dots$  and test goodness and definiteness exactly by Sturm's theorem. Goodness and definiteness need not be monotone in  $\delta$  and  $\theta$ ; in one test a value of  $\theta$  worked while  $\theta/3$  did not. Since the lemmas cover all sufficiently small values, the search still stops, and every value it returns is certified exactly. In this way we obtained and certified explicit stars for 82 weight vectors with  $m \leq 15$  and, in independent re-implementations, for further weight vectors with  $m \leq 21$  (Section 4). In any star without singular directions of Type 2, no two rays are parallel: if  $\gamma_i = \pm \gamma_j$  with  $i \neq j$ , every term of (1) vanishes at a unit vector orthogonal to  $\gamma_i$ . The construction aims at existence. For small  $\theta$  all rays lie close to one line, and  $|P_{c, \gamma}|$  can be very small in some directions: for  $m = 15$  and equal weights, the constructed star has  $\min |P_{c, \gamma}| / \max |P_{c, \gamma}| \approx 10^{-13}$  on  $\mathbb{S}^1$  (a numerical value). Better conditioned stars can be found numerically for small  $m$ , but we prove no quantitative bound.

### 3 Conflitti's conjecture

For  $y \in \mathbb{R}^n$  let  $s(y)$  be the number of nonzero coordinates of  $y$ , and  $E_y(t) = \prod_{i=1}^n (1 + y_i t) = \sum_{j \geq 0} e_j(y) t^j$ . Then  $E_y$  has degree  $s(y)$ ,  $e_0(y) = 1$  and  $e_{s(y)}(y) \neq 0$ . For  $J \subseteq [n] = \{1, \dots, n\}$  let  $\mathbb{R}^J = \{x \in \mathbb{R}^n : x_i = 0 \text{ for } i \notin J\}$ .

**Lemma 3.1.** *Let  $y \in \mathbb{R}^n$  and  $s = s(y)$ . Among  $e_0(y), \dots, e_s(y)$  no two consecutive numbers vanish. Consequently, if  $r \geq 2$  and  $e_{r-1}(y) = e_r(y) = 0$ , then  $s(y) \leq r - 2$ .*

This is classical: a real polynomial with a nonzero constant term and two consecutive vanishing coefficients below the leading one has non-real roots. See Todhunter [Tod85, Chapter V, Arts. 68–72] and Burnside and Panton [BP92, Art. 21 and Chapter IX, Example 18]. We include the short proof.

*Proof.* The polynomial  $E_y$  has only real roots, namely  $-1/y_i$  for  $y_i \neq 0$ , and none of them is 0. Let  $V$  and  $V'$  be the numbers of sign changes in the coefficient sequences of  $E_y(t)$  and  $E_y(-t)$ , ignoring zeros. By Descartes' rule of signs,  $V$  is at least the number of positive roots and  $V'$  at least the number of negative roots, so  $V + V' \geq s$ . Now let  $e_a(y)$  and  $e_b(y)$ ,  $a < b$ , be nonzero coefficients such that all coefficients between them vanish. The pair contributes a sign change to  $V$  if  $e_a e_b < 0$ , and to  $V'$  if

$(-1)^{a+b}e_a e_b < 0$ . So it contributes exactly 1 to  $V + V'$  if  $b - a$  is odd, and 0 or 2 if  $b - a$  is even. In all cases the contribution is at most  $b - a$ , and at most  $b - a - 1$  if  $b - a \geq 3$ . Since  $e_0(y) \neq 0 \neq e_s(y)$ , these gaps add up to  $s$ . If two consecutive coefficients among  $e_0, \dots, e_s$  vanished, some gap would be at least 3, and we would get  $V + V' \leq s - 1$ , a contradiction.

For the consequence, if  $s \geq r + 1$ , then  $e_{r-1}$  and  $e_r$  would be consecutive vanishing coefficients with  $1 \leq r - 1 < r \leq s$ . If  $s = r$  or  $s = r - 1$ , then  $e_r(y)$  or  $e_{r-1}(y)$  equals  $e_s(y) \neq 0$ . So  $s \leq r - 2$ .  $\square$

**Lemma 3.2.** *For  $n \geq 3$  the polynomial  $e_{n-1}(x_1, \dots, x_n)$  is irreducible, and its real zero set contains no hyperplane.*

*Proof.* Write  $e_{n-1} = x_1 F_1 + F_0$  with  $F_1 = e_{n-2}(x_2, \dots, x_n)$  and  $F_0 = x_2 \cdots x_n$ . In a factorization  $e_{n-1} = FG$  the degrees in  $x_1$  add up to 1, so one factor, say  $G$ , does not involve  $x_1$ . Then  $G$  divides  $F_1$  and  $F_0$ . The irreducible factors of  $F_0$  are the  $x_j$ ,  $j \geq 2$ , and none of them divides  $F_1$ , because setting  $x_j = 0$  leaves the monomial  $\prod_{i \neq 1, j} x_i$  (here  $n \geq 3$  is used). So  $G$  is constant. If  $e_{n-1}$  vanished on a hyperplane  $\{\lambda = 0\}$ , with  $\lambda$  a nonzero linear form, then  $\lambda$  would divide  $e_{n-1}$ : after a linear change of coordinates with  $\lambda$  as the first coordinate, the part of  $e_{n-1}$  free of that coordinate vanishes on  $\mathbb{R}^{n-1}$  and is therefore zero. This contradicts irreducibility, since  $\deg e_{n-1} \geq 2$ .  $\square$

**Lemma 3.3.** *Let  $r \geq 2$  be even, and let  $V \subseteq \mathbb{R}^n$  be a linear subspace with  $\dim V \geq 2$  on which  $e_r$  vanishes. Then there is  $J \subseteq [n]$  with  $|J| \leq r - 2$  and  $\dim(V \cap \mathbb{R}^J) \geq \dim V - 1$ .*

*Proof.* Let  $\mathcal{J}$  be the set of  $J \subseteq [n]$  with  $|J| \leq r - 2$ , put  $U_J = V \cap \mathbb{R}^J$  and  $U = \bigcup_{J \in \mathcal{J}} U_J$ , and suppose that  $\dim U_J \leq \dim V - 2$  for all  $J \in \mathcal{J}$ . If  $y \in V$  and  $e_{r-1}(y) = 0$ , then  $s(y) \leq r - 2$  by Lemma 3.1, since also  $e_r(y) = 0$ ; hence  $y \in U$ . A real vector space is not a finite union of proper subspaces. So there is  $y \in V \setminus U$ , and there is  $z \in V$  outside the subspaces  $\mathbb{R}y + U_J$ ,  $J \in \mathcal{J}$ , which have dimension at most  $\dim V - 1$ . If  $\alpha y + \beta z \in U_J$  with  $(\alpha, \beta) \neq (0, 0)$ , then  $\beta \neq 0$  gives  $z \in \mathbb{R}y + U_J$  and  $\beta = 0$  gives  $y \in U_J$ ; both are impossible. Hence  $e_{r-1}$  has no zero on  $\text{span}(y, z) \setminus \{0\}$ . But  $e_{r-1}$  is homogeneous of odd degree  $r - 1$ , so the continuous function  $\phi \mapsto e_{r-1}(y \cos \phi + z \sin \phi)$  takes the values  $e_{r-1}(y)$  and  $-e_{r-1}(y)$  at  $\phi = 0$  and  $\phi = \pi$ . It vanishes somewhere in between, at a nonzero vector of  $\text{span}(y, z)$ . This contradiction proves the lemma.  $\square$

*Proof of Theorem 1.3.* Let  $V \subseteq \mathbb{R}^n$  be an  $r$ -dimensional subspace on which  $e_r$  vanishes. By Lemma 3.3 some  $J$  with  $|J| \leq r - 2$  satisfies  $\dim(V \cap \mathbb{R}^J) \geq r - 1$ , which is impossible because  $\dim \mathbb{R}^J \leq r - 2$ .  $\square$

*Proof of Corollary 1.4.* If  $n \leq r - 1$ , then  $e_r = 0$  on  $\mathbb{R}^n$ . If  $n \geq r$ , then  $e_r$  vanishes on  $\mathbb{R}^{\{1, \dots, r-1\}}$ , since each monomial of  $e_r$  involves  $r$  coordinates. A subspace of dimension larger than  $r$  contains one of dimension  $r$ , so Theorem 1.3 gives the upper bound.  $\square$

*Proof of Theorem 1.5.* Let  $V$  be an  $(r - 1)$ -dimensional subspace on which  $e_r$  vanishes. By Lemma 3.3 there is  $J$  with  $|J| \leq r - 2$  and  $\dim(V \cap \mathbb{R}^J) \geq r - 2$ , so  $|J| = r - 2$  and  $\mathbb{R}^J \subseteq V$ . Choose  $w \in V$  with  $V = \mathbb{R}^J \oplus \mathbb{R}w$ ; subtracting from  $w$  its coordinates in  $J$ , we may assume that  $w \neq 0$  and  $w_i = 0$  for  $i \in J$ . For  $x \in \mathbb{R}^J$  and  $\sigma \in \mathbb{R}$  the vectors  $x$  and  $\sigma w$  have disjoint supports, so  $E_{x+\sigma w} = E_x E_{\sigma w}$  and

$$0 = e_r(x + \sigma w) = \sum_{i=0}^{r-2} e_i(x) \sigma^{r-i} e_{r-i}(w),$$

because  $e_i(x) = 0$  for  $i > |J|$ . The polynomials  $e_i(x) \sigma^{r-i}$ ,  $0 \leq i \leq r - 2$ , are nonzero on  $\mathbb{R}^J \times \mathbb{R}$  and have distinct degrees in  $\sigma$ , so they are linearly independent. Hence  $e_2(w) = \cdots = e_r(w) = 0$ . Since  $r \geq 4$ , in particular  $e_2(w) = e_3(w) = 0$ , and Lemma 3.1 gives  $s(w) \leq 1$ . So  $w$  is a multiple of a standard basis vector  $e_q$  with  $q \notin J$ , and  $V = \mathbb{R}^{J \cup \{q\}}$ .  $\square$

For  $r = 2$  the conclusion of Theorem 1.5 fails as soon as  $n \geq 3$ :  $e_2$  vanishes on the line spanned by  $(1, 1, -\frac{1}{2}, 0, \dots, 0)$ .

*Remark 3.4* (Conflitti’s heuristic). After Conjecture 8, Conflitti suggested that for even  $r$  every subspace of  $e_r^{-1}(0)$  containing a vector with at least  $r$  nonzero coordinates is a line. This is false for every even  $r \geq 4$ . The simplest counterexamples are the subspaces of type (b) in Chirvasitu’s description of the linear subspaces in the zero locus of  $e_{n-1}$  [Chi25, Theorem 0.2(1)(b)]: if  $[n] = \pi_1 \sqcup \dots \sqcup \pi_d$  and each  $v_i \in \mathbb{R}^{\pi_i}$  has all coordinates in  $\pi_i$  nonzero with  $\sum_{q \in \pi_i} 1/(v_i)_q = 0$  (so  $|\pi_i| \geq 2$ ), then  $e_{n-1}$  vanishes on  $\text{span}(v_1, \dots, v_d)$ . Indeed, if  $x = \sum_i t_i v_i$  with all  $t_i \neq 0$ , then  $e_{n-1}(x) = (\prod_q x_q) \sum_i t_i^{-1} \sum_{q \in \pi_i} 1/(v_i)_q = 0$ ; otherwise  $x$  has two zero coordinates, and every monomial of  $e_{n-1}$  vanishes at  $x$ . For  $n = r + 1$ , one block  $(1, 1, -\frac{1}{2})$  and  $r/2 - 1$  blocks  $(1, -1)$  give an  $(r/2)$ -dimensional subspace of  $e_r^{-1}(0) \subseteq \mathbb{R}^{r+1}$  whose generic vectors have all  $r + 1$  coordinates nonzero; for  $r = 4$  it is the plane spanned by  $(1, -1, 0, 0, 0)$  and  $(0, 0, 1, 1, -\frac{1}{2})$ . Padding with zeros gives such subspaces in  $\mathbb{R}^n$  for every  $n \geq r + 1$ . By Chirvasitu’s classification,  $r/2$  is the largest possible dimension for  $n = r + 1$ , and by Theorems 1.3 and 1.5 the dimension is at most  $r - 2$  for every  $n$ .

## 4 Computations and concluding remarks

Theorem 1.1 settles Conjecture 1 of [AL21] and Theorem 1.3 settles Conjecture 1.2. Two questions remain natural. First, for given weights, how stable can the inversion be, for instance how large can  $\min_\psi |w(\psi)| / \sum_i |c_i|$  be? Second, for even  $r \geq 6$  and  $n \geq r + 2$ , what is the largest dimension of a subspace of  $e_r^{-1}(0)$  that contains a vector with at least  $r$  nonzero coordinates? By Remark 3.4 it lies between  $r/2$  and  $r - 2$ ; for  $r = 4$  it equals 2.

**Verification.** The claims were checked as follows. None of the computations is used in the proofs.

1. Programs by the author (Python, exact rational arithmetic) do the following.

- `star_construction.py` carries out the construction of Section 2 for 82 weight vectors with  $m \leq 15$ : equal weights for  $m = 3, 5, \dots, 15$ ; the zero-sum weights of [AL21, (29)] for  $k = 1, \dots, 7$ ;  $c = (1, \dots, m)$  and  $c_i = (-1)^i i$  for  $m = 5, 9, 13$ ; powers of 10 with random signs for  $m = 5, 7, 9, 11$ ; one dominant weight, one tiny weight, or weights of two very different sizes for  $m = 5, 9$ ; and 50 random weight vectors. The parameters  $\delta$  and  $\theta$  are found by halving. Every star is certified: the rays are exact rational unit vectors,  $P(1, 0) \neq 0$ , and  $t \mapsto P(t, 1)$  has no real zero both by Sturm’s theorem and by a Descartes-rule bisection of its square-free part. A second program re-checks the 82 stars from the certificate file alone.
- Further programs check the identities of Lemma 2.4 symbolically, those of Lemmas 2.1, 2.3 and 2.5 on random instances, and Lemma 3.1 on all 137,256 integer vectors in  $\{-3, \dots, 3\}^n$ ,  $n \leq 6$ . Numerical least-squares searches over Grassmannians, which are evidence only, agree with Theorems 1.3 and 1.5.

2. Independent verification runs, which were AI-assisted, checked every proof line by line. Their re-check programs in the release were written separately, from the text of the paper; those that certify stars use only the standard library.

- All 82 stars were certified again, by Sturm’s theorem and by an exact Taylor-bisection bound that shows  $P(t, 1) \neq 0$  and  $P(1, t) \neq 0$  for  $t \in [-1, 1]$ .
- The release contains two re-implementations of the construction. The first uses random admissible middle positions as in Lemma 2.4, inserts each cluster at the other outer entry,

and lets  $\delta$  and  $\theta$  run through the powers  $3^{-j}$ . It certified 20 further weight vectors. Among them are equal weights for  $m = 17, 19$  and  $21$ , the zero-sum weights for  $k = 8, 9$  and  $10$ , the weights  $1, \dots, 17$ , and weights whose ratios reach  $10^{10}$ . The second lets  $\delta$  and  $\theta$  run through the powers  $5^{-j}$ , uses random admissible middle positions for half of the weight vectors and, for a third of them, rays perturbed by  $o(\theta)$  as allowed in Lemma 2.3. It certified 62 weight vectors with  $m \leq 17$ , nine of them with zero sum.

- The identity (2) and the value  $B = 2AD/S$  at  $b = b^*$  were proved by exact evaluation on a grid that is large enough for a polynomial identity test, and again symbolically. Example 2.8 was recomputed step by step. Lemma 3.1 was tested again exhaustively, the second time on 652,828 integer vectors with  $n \leq 8$ , and the subspaces of Remark 3.4 were shown to lie in  $e_r^{-1}(0)$  for  $r \leq 12$ .

**Scope and priority.** Theorem 1.1 answers the first question of record OWR-13750332-001, the extension of [AL21, Theorem 6(b)] to all odd  $m$ , for all nonzero weights. Theorem 1.3 proves Conjecture 1.2 for all even  $r$  and all  $n$ ; its new content is  $r \geq 4$  and  $n \geq r + 2$  (Remark 1.6). The case  $n = r + 1$  is due to Ambartsoumian and Latifi [AL21], and Chirvasitu’s classification [Chi25] covers it and the case  $n = r + 1$  of Theorem 1.5; his subspaces also refute Conflitti’s heuristic. We read [Amb23b], [Con06], [Chi25], [AAL25], and the preprint versions of [AL21] and [ZSM14]. We could not access the chapter “Star Transform” (pp. 143–163) of Ambartsoumian’s monograph [Amb23a], because the publisher’s page is protected by a bot check. The report [Amb23b], which cites the monograph, still describes the extension of Theorem 6(b) as unproven, and the 2025 preprint [AAL25] does not mention a solution. Searches of arXiv, OpenAlex, zbMATH and Crossref for both questions, two web searches for Conflitti’s conjecture and one for the star-transform conjecture, all in September 2026, found no proof of either conjecture: OpenAlex lists three works citing [Con06] and five citing [AL21], none of them on these questions. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the constructions and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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