

The Dimension of Trivariate C^1 Splines on Generic Bipyramid Cells

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Abstract

A bipyramid cell is a tetrahedral partition with one interior vertex v_0 , n boundary vertices that are coplanar with v_0 and surround it, and two further boundary vertices on opposite sides of this base plane. Colvin, DiMatteo and Sorokina determined the dimension of the space $S_d^1(\Delta)$ of C^1 splines of degree at most d on such a cell Δ in their collinear and coplanar cases. In the generic case they gave lower and upper bounds and conjectured that the dimension equals the upper bound whenever the two bounds differ. We prove an exact formula for every generic bipyramid cell. Let m be the number of distinct lines spanned by the n interior edges in the base plane. Then $\dim S_d^1(\Delta) = 1, 4, 11$ for $d = 0, 1, 2$, and for $d \geq 3$ it equals $2n\binom{d}{3} + 6d - 1 + (4 - m)_+$ if $m \geq 3$, and $8\binom{d}{3} + 8d - 3 - (4 - d)_+$ if $m = 2$. For $d \geq 3$ these are exactly the upper bounds as printed in the Oberwolfach abstract of Colvin, DiMatteo and Sorokina, and for $d = 2$ the upper bound is attained unless $m = 3$. So the conjecture holds in every case except $m = 3, d = 2$, where the dimension is 11 and the printed upper bound is 12; the smallest example has a triangular base. We could not access the journal version of their paper; if its bounds for $d \geq 3$ are the printed ones and it states this one case differently, the conjecture holds as stated there, and in any case the exact formula answers the question. An elementary cofactor argument shows directly that $\dim S_2^1(\Delta) = 11$ for every generic cell. The spline space splits into homogeneous components; the dimensions of those of degree at least 5 also follow from a general formula of Alfeld, Neamtu and Schumaker, so the new part of the formula is the determination of the components of degrees 3 and 4. The proof reduces the problem to bivariate splines on the planar fan in the base plane: each half of the cell projects onto this fan, and gluing the two halves forces the trace on the base plane to be C^2 across every ray. The same reduction gives a uniform proof of their formulas in the collinear and coplanar cases. The formula agrees with Macaulay2 values of DiPasquale and Villamizar for $n = m = 5$ and with exact rational computations. This is an unrefereed note.

1 Introduction

For a tetrahedral partition Δ of a region in $\mathbb{R}^3$ and integers $r, d \geq 0$, let $S_d^r(\Delta)$ be the space of C^r functions on $|\Delta|$ whose restriction to every tetrahedron is a polynomial of degree at most d . Computing $\dim S_d^r(\Delta)$ is a basic problem of multivariate spline theory. For trivariate splines it is difficult already for the star of a single interior vertex, a *cell*, in low degree; see [ANS96, Sha14, MV14, DV21].

Colvin, DiMatteo and Sorokina [CDS16, Sor15] studied *bipyramid cells*. Following [Sor15], such a cell Δ has

- exactly one interior vertex v_0 ;
- boundary vertices $v_1, \dots, v_n$ that are coplanar with v_0 and form a polygon surrounding v_0 in the *base plane* B ;

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- two boundary vertices v_{n+1} and v_{n+2} on opposite sides of B ;
- the $2n$ tetrahedra $T_i^+ = [v_0, v_i, v_{i+1}, v_{n+1}]$ and $T_i^- = [v_0, v_i, v_{i+1}, v_{n+2}]$, $1 \leq i \leq n$, with indices of $v_1, \dots, v_n$ taken modulo n .

The triangles $[v_0, v_i, v_{i+1}]$ triangulate the polygon, so the angle at v_0 from v_i to v_{i+1} lies in $(0, \pi)$. Let m be the number of distinct lines through v_0 that contain one of the edges $[v_0, v_i]$, $1 \leq i \leq n$ (the number of “slopes”). Then $n \geq 3$ and $2 \leq \lceil n/2 \rceil \leq m \leq n$ [Sor15]. In particular $m = 2$ forces $n = 4$: for $n = 3$ any two of the edges are consecutive, so no two of them are opposite and $m = 3$. Similarly, $m = 3$ forces $3 \leq n \leq 6$. When v_{n+1}, v_{n+2}, v_0 are not collinear, let P be the plane through them. Colvin, DiMatteo and Sorokina (CDS) distinguish four cases [Sor15]:

- *generic*: v_{n+1}, v_{n+2}, v_0 are not collinear, and P contains none of $v_1, \dots, v_n$;
- *coplanar I (II)*: they are not collinear, and P contains exactly one (two) of $v_1, \dots, v_n$;
- *collinear*: v_{n+1}, v_{n+2}, v_0 are collinear.

In the collinear and coplanar cases CDS determine $\dim S_d^1(\Delta)$ for $d \geq 2$ exactly [Sor15, Theorems 1–3]; we recall these formulas in Theorem 4.2. In the generic case they prove [Sor15, Theorem 4] that, for $m \geq 3$ and $d \geq 2$,

$$2n \binom{d}{3} + 6d - 2 \leq \dim S_d^1(\Delta) \leq 2n \binom{d}{3} + 6d - 1 + (4 - m)_+, \quad (1)$$

and that, for $m = 2$, $\dim S_2^1(\Delta) = 11$ and, for $d \geq 3$,

$$8 \binom{d}{3} + 8d - 5 \leq \dim S_d^1(\Delta) \leq 8 \binom{d}{3} + 8d - 3 - (4 - d)_+. \quad (2)$$

Here $t_+ = \max\{t, 0\}$ and $\binom{a}{b} = 0$ for $0 \leq a < b$. For the remaining cases, in which the two bounds differ, they write that they “conjecture that the dimension coincides with the upper bound” [Sor15]. This question is record OWR-13678-010 of the UnsolvedMath dataset [ula26]. We use the bounds exactly as printed in the Oberwolfach abstract [Sor15], the source of the record. We could not access the full text of the journal version [CDS16].

Our main result is an exact formula.

Theorem 1.1. *Let Δ be a generic bipyramid cell with n base vertices and m slopes. Then $\dim S_0^1(\Delta) = 1$, $\dim S_1^1(\Delta) = 4$, $\dim S_2^1(\Delta) = 11$, and for every $d \geq 3$*

$$\dim S_d^1(\Delta) = \begin{cases} 2n \binom{d}{3} + 6d - 1 + (4 - m)_+ & \text{if } m \geq 3, \\ 8 \binom{d}{3} + 8d - 3 - (4 - d)_+ & \text{if } m = 2 \text{ (so } n = 4\text{)}. \end{cases}$$

Explicitly, for $d \geq 3$ the dimension is $2n \binom{d}{3} + 6d - 1$ if $m \geq 4$, $2n \binom{d}{3} + 6d$ if $m = 3$, 28 if $m = 2$ and $d = 3$, and $8 \binom{d}{3} + 8d - 3$ if $m = 2$ and $d \geq 4$. In particular, it depends only on n , m and d . The homogeneous components are given in Proposition 4.1.

Corollary 1.2. *Let Δ be a generic bipyramid cell and let LB and UB be the bounds (1), (2) as printed in [Sor15]. In every case with $\text{LB} < \text{UB}$ we have $\dim S_d^1(\Delta) = \text{UB}$, except when $m = 3$ and $d = 2$; then $\text{LB} = 10 < \dim S_2^1(\Delta) = 11 < \text{UB} = 12$. The upper bound is always valid, and the lower bound is never attained when $\text{LB} < \text{UB}$.*

	$d = 2$	$d = 3$	$d \geq 4$
$m \geq 4$	11 = UB (LB = 10)	= UB	= UB
$m = 3$	11 < UB = 12 (LB = 10)	= UB	= UB
$m = 2$	11 (bounds coincide)	28 = UB (LB = 27)	= UB

Table 1: $\dim S_d^1(\Delta)$ on generic cells compared with the bounds (1), (2).

Thus the conjecture holds for all $d \geq 3$, and for $d = 2$ whenever $m \neq 3$. The exceptional cells are exactly the generic cells with $m = 3$, which have $3 \leq n \leq 6$, and the exception occurs only in degree 2. The term $(4 - m)_+$ in (1) is attained for $m = 3$ from degree 3 on, but not in degree 2. If the journal version [CDS16] has the bounds (1), (2) for $d \geq 3$ and states the case $m = 3$, $d = 2$ differently (for instance by requiring $d \geq 3$ there), then Theorem 1.1 shows that the conjecture holds as stated there. In any case Theorem 1.1 gives the dimension itself, so it answers the question for any version of the bounds. Sample values are listed in Table 2.

The degree-2 case has a short direct proof that does not use the rest of the paper.

Proposition 1.3. *Let Δ be a generic bipyramid cell, take $v_0 = 0$, and let ℓ_B be a linear form vanishing on B . Let $(\ell_B)_+^2$ be the function equal to ℓ_B^2 on the tetrahedra T_i^+ and to 0 on the T_i^- . Then $S_2^1(\Delta)$ consists of the functions $P + c(\ell_B)_+^2$, where P is a polynomial of degree at most 2 and $c \in \mathbb{R}$. In particular $\dim S_2^1(\Delta) = 11$ for all n and m .*

Example 1.4. Let $v_0 = (0, 0, 0)$, $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (-1, -1, 0)$, $v_4 = (0, 0, 1)$ and $v_5 = (1, -2, -1)$. Here $n = m = 3$. The plane through v_0, v_4, v_5 meets $B = \{z = 0\}$ in the line $\mathbb{R}(1, -2, 0)$, which contains none of v_1, v_2, v_3 , so the cell is generic. By Proposition 1.3, $S_2^1(\Delta)$ is spanned by the ten monomials of degree at most 2 and $(z_+)^2$, so $\dim S_2^1(\Delta) = 11$, whereas the conjectured value is UB = 12. By Theorem 1.1, $\dim S_d^1(\Delta) = 24$ and 48 for $d = 3, 4$, which are the upper bounds.

n	m	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
3	3	11 [10, 12]	24 [22, 24]	48 [46, 48]	90 [88, 90]	156 [154, 156]
4	2	11 [11, 11]	28 [27, 28]	61 [59, 61]	117 [115, 117]	205 [203, 205]
4	3	11 [10, 12]	26 [24, 26]	56 [54, 56]	110 [108, 110]	196 [194, 196]
4	4	11 [10, 11]	25 [24, 25]	55 [54, 55]	109 [108, 109]	195 [194, 195]
5	5	11 [10, 11]	27 [26, 27]	63 [62, 63]	129 [128, 129]	235 [234, 235]
6	3	11 [10, 12]	30 [28, 30]	72 [70, 72]	150 [148, 150]	276 [274, 276]

Table 2: $\dim S_d^1(\Delta)$ for generic cells (Theorem 1.1), with the bounds [LB, UB] of (1), (2).

Method and relation to earlier work. Put $v_0 = 0$. Every spline on a cell is a sum of homogeneous splines, a standard fact [ANS96], [DV21, (3)]. The upper half $\bigcup_i T_i^+$ of the cell is an “orange” around the edge $[v_0, v_{n+1}]$, and by the projection formula for oranges [SSVY24, Theorem 1.1] (see also [ASS92]) its splines reduce to bivariate splines on the planar fan F cut out on B by the edges $[v_0, v_i]$ (Lemma 2.3); the lower half is treated in the same way after a shear. The dimensions of bivariate splines on F are classical (Lemma 2.1; see [Sch79, LS07, GS98]). The step that we have not found in the literature is the gluing across B (Proposition 3.1): the two halves glue exactly when their traces on B agree and the jump of the normal derivatives equals the derivative of the trace in one fixed direction. With

Lemma 3.2, which shows that in the generic case the trace must be C^2 across every ray, this gives Theorem 1.1. The same two statements give a uniform proof of the CDS formulas in the collinear and coplanar cases (Theorem 4.2). The supersmoothness in Lemma 3.2 is of the kind studied in [Sor10], and CDS use additional smoothness of splines on bipyramid cells to prove their upper bound [Sor15]; see Remark 3.3.

For $k \geq 3r + 2$ the dimension of H_k^r is known on every cell: by a theorem of Alfeld, Neamtu and Schumaker [ANS96, Theorem 3], it is given by an explicit formula in which each interior edge enters through the number of distinct planes spanned by the interior faces around it; see also [DV21, Remark 4.6 and Section 6.2]. (We checked the statement in a preprint version of [ANS96] posted by its authors.) For bipyramid cells and $r = 1$ this formula reproduces Proposition 4.1 for all $k \geq 5$, and likewise the homogeneous dimensions used in the proof of Theorem 4.2; see Section 6. The spaces $H_0^1(\Delta)$ and $H_1^1(\Delta)$ consist of global polynomials, and $H_2^1(\Delta)$ is given by the elementary Proposition 1.3. So the new content of Theorem 1.1 and Proposition 4.1 is the determination of $H_3^1(\Delta)$ and $H_4^1(\Delta)$, and, given [ANS96], the CDS conjecture concerns only the homogeneous degrees 2, 3 and 4. Our proof does not use [ANS96]; it treats all degrees in the same way.

DiPasquale and Villamizar [DV21, Section 6.2] studied the generic case for one pentagonal base ($n = m = 5$). They call this configuration a non-generic bipyramid, because of the special position of its base; their Section 6.1 treats a bipyramid with generic vertex positions. They observed that the spline equal to z^{r+1} above B and to 0 below it is C^r . Using the upper bound of [MV14, Theorem 4.1], they showed that for $d \leq \lfloor (3r + 1)/2 \rfloor$ the dimension of the homogeneous splines of degree d equals $\binom{d+2}{2} + \binom{d+1-r}{2}$, the dimension of the span of the global polynomials and the multiples of this spline (for $r = 1$ this gives $\dim S_2^1(\Delta) = 11$ for their cell). For $d \geq 3r + 2$ they obtain $\dim H_d^r(\Delta)$ from [ANS96]. They also computed $\dim H_d^r(\Delta)$ with Macaulay2 for $r \leq 4$; so for their cell and $r = 1$ only the degrees 3 and 4 were left to the computation. Theorem 1.1 agrees with their values for $r = 1$ (Section 5). In the arXiv version that we read, they do not state a general formula or discuss the conjecture of CDS. Apart from the components of degree at least 5, which follow from [ANS96], and the component of degree 2 for their cell, we found no earlier statement of Theorem 1.1. We also found no earlier statement of Proposition 3.1 or of the exception at $(m, d) = (3, 2)$.

Organisation. Section 2 collects the standard ingredients. Section 3 contains the gluing step. Section 4 proves Theorem 1.1, Corollary 1.2, Proposition 1.3 and the formulas for the other cases. Section 5 compares the results with [DV21] and describes the exact computations. Section 6 relates the formulas to [ANS96] and contains further remarks.

2 Preliminaries

2.1 Splines on a cell

Let Δ be a bipyramid cell. For an interior face τ let ℓ_τ be an affine function vanishing on τ . A function on $|\Delta|$ that is a polynomial p_T on each tetrahedron T is C^r if and only if ℓ_τ^{r+1} divides $p_T - p_{T'}$ for every interior face $\tau = T \cap T'$ [Bil88], [BR91, Proposition 1.2]; see also [DV21, Proposition 2.4]. This criterion applies because Δ is hereditary: the tetrahedra containing a given vertex are connected through interior faces containing that vertex. We use it as the definition of $S_d^r(\Delta)$ from now on.

Translate so that $v_0 = 0$. Then every interior face contains 0, every ℓ_τ is a linear form, and the ideal (ℓ_τ^{r+1}) is homogeneous. Let $R = \mathbb{R}[x, y, z]$, let R_k be the space of forms of degree k , and let $H_k^r(\Delta)$ be the space of splines whose pieces all lie in R_k . A piecewise polynomial of degree at most d

is in $S_d^r(\Delta)$ if and only if each of its homogeneous components is, so

$$S_d^r(\Delta) = \bigoplus_{k=0}^d H_k^r(\Delta) \quad (3)$$

[ANS96], [DV21, (3)].

2.2 Normal form

An invertible linear map A of $\mathbb{R}^3$ maps Δ to a bipyramid cell $A\Delta$ of the same case with the same n and m , and $f \mapsto f \circ A^{-1}$ maps $H_k^r(\Delta)$ isomorphically onto $H_k^r(A\Delta)$. By the criterion above, $H_k^r(\Delta)$ depends only on which tetrahedra are adjacent and on the planes spanned by the interior faces. These do not change if a vertex v_j ($j \geq 1$) is replaced by a positive multiple.

Hence we may assume that $B = \{z = 0\}$, that $v_{n+1} = e_3 = (0, 0, 1)$ (apply the linear map that fixes B pointwise and sends v_{n+1} to e_3), and that $v_{n+2} = (w, -1)$ with $w = (c, e) \in \mathbb{R}^2$. After relabelling, $v_1, \dots, v_n$ are in counter-clockwise order. We identify B with $\mathbb{R}^2$ and regard $v_1, \dots, v_n$ as vectors in $\mathbb{R}^2$. If $w \neq 0$, the plane P through 0 , e_3 and $(w, -1)$ meets B in the line $\mathbb{R}w$. Therefore

- the cell is collinear if and only if $w = 0$;
- it is generic if and only if $w \neq 0$ and w is parallel to none of $v_1, \dots, v_n$;
- it is coplanar I (II) if and only if $w \neq 0$ and exactly one (two) of $v_1, \dots, v_n$ are parallel to w .

Let $\lambda_i \in \mathbb{R}[x, y]$ be a nonzero linear form with $\lambda_i(v_i) = 0$. The interior faces and their linear forms are

- $U_i = [0, v_i, e_3] = T_{i-1}^+ \cap T_i^+$, with form $\lambda_i(x, y)$;
- $L_i = [0, v_i, (w, -1)] = T_{i-1}^- \cap T_i^-$, with form $\lambda_i(x, y) + \lambda_i(w)z$;
- $B_i = [0, v_i, v_{i+1}] = T_i^+ \cap T_i^-$, with form z .

Finally let $D = c\partial_x + e\partial_y$ be the derivative in the direction w . For a linear form λ on $\mathbb{R}^2$ we have $D\lambda = \lambda(w)$.

2.3 Planar fans

The rays $\rho_i = \mathbb{R}_{\geq 0}v_i$ and the sectors $F_i = \text{cone}(v_i, v_{i+1})$ form a complete fan F in $\mathbb{R}^2$; the ray ρ_i separates F_{i-1} and F_i . For an integer j and orders $\mathbf{r} = (r_1, \dots, r_n)$ with $r_i \geq 0$, let $\mathcal{H}_j^{\mathbf{r}}(F)$ be the space of tuples $g = (g^{(1)}, \dots, g^{(n)})$ of binary forms of degree j such that $\lambda_i^{r_i+1}$ divides $g^{(i-1)} - g^{(i)}$ for every i (indices modulo n). These are the homogeneous splines of degree j on F that are C^{r_i} across ρ_i . If all r_i equal r we write $\mathcal{H}_j^r(F)$, and we set $\mathcal{H}_j^{\mathbf{r}}(F) = 0$ for $j < 0$. Let

$$\sigma_j = \dim \mathcal{H}_j^1(F), \quad \tau_j = \dim \mathcal{H}_j^2(F).$$

Lemma 2.1 (classical). *Let Λ be the set of the m lines through 0 that contain a ray of F ; we use the same letter λ for a line and for a linear form vanishing on it. Suppose that all rays on the same line $\lambda \in \Lambda$ have the same order r_λ . Then for every $j \geq 0$*

$$\dim \mathcal{H}_j^{\mathbf{r}}(F) = \sum_{i=1}^n (j - r_i)_+ + \left(j + 1 - \sum_{\lambda \in \Lambda} (j - r_\lambda)_+ \right)_+.$$

For constant $\mathbf{r}$, summing over $j \leq d$ gives the classical dimension formula for splines on a planar cell [Sch79, LS07]. The apolarity argument below is standard [GS98]; we include it for completeness.

Proof. For $g \in \mathcal{H}_j^{\mathbf{r}}(F)$ write $g^{(i-1)} - g^{(i)} = a_i \lambda_i^{r_i+1}$ with $a_i \in \mathbb{R}[x, y]_{j-r_i-1}$ ($a_i = 0$ if $j \leq r_i$). The sum of these differences over i telescopes, so $\sum_i a_i \lambda_i^{r_i+1} = 0$. Conversely, for any $g^{(n)} \in \mathbb{R}[x, y]_j$ and any (a_i) with $\sum_i a_i \lambda_i^{r_i+1} = 0$, the recursion $g^{(i)} = g^{(i-1)} - a_i \lambda_i^{r_i+1}$ ($i = 1, \dots, n$, starting from $g^{(0)} = g^{(n)}$) closes up and defines an element of $\mathcal{H}_j^{\mathbf{r}}(F)$. Since $\dim \mathbb{R}[x, y]_{j-r_i-1} = (j - r_i)_+$, this gives

$$\dim \mathcal{H}_j^{\mathbf{r}}(F) = (j+1) + \sum_i (j - r_i)_+ - \dim J_j,$$

where $J_j = \sum_i \lambda_i^{r_i+1} \mathbb{R}[x, y]_{j-r_i-1}$ is the degree- j part of the ideal $J = (\lambda^{r_\lambda+1} : \lambda \in \Lambda)$; here we use that rays on the same line have proportional forms λ_i and the same order.

On $\mathbb{R}[x, y]_j$ consider the pairing $\langle G, H \rangle = G(\partial_x, \partial_y)H$. It is nondegenerate, because distinct monomials are orthogonal and $\langle x^a y^b, x^a y^b \rangle = a!b!$. Let $\lambda = \alpha x + \beta y \neq 0$ and $\mu = \beta x - \alpha y$, so that μ vanishes on the vector (α, β) along which $\lambda(\partial)$ differentiates. Choose a linear form u with $u(\alpha, \beta) = 1$. In the coordinates (u, μ) the operator $\lambda(\partial)$ is $\partial/\partial u$, so for $H = \sum_q h_q u^q \mu^{j-q}$ we have $\lambda(\partial)^s H = 0$ if and only if $h_q = 0$ for all $q \geq s$, that is, if and only if $\mu^{(j-s+1)_+}$ divides H . Since the pairing is nondegenerate in degree $j - s$, H is orthogonal to $\lambda^s \mathbb{R}[x, y]_{j-s}$ if and only if $\lambda(\partial)^s H = 0$. Hence the orthogonal complement of J_j consists of the $H \in \mathbb{R}[x, y]_j$ that are divisible by $\mu_\lambda^{(j-r_\lambda)_+}$ for every $\lambda \in \Lambda$. The forms μ_λ are pairwise non-proportional, hence pairwise coprime, so this complement is $\prod_\lambda \mu_\lambda^{(j-r_\lambda)_+} \cdot \mathbb{R}[x, y]_{j-E}$ with $E = \sum_\lambda (j - r_\lambda)_+$, of dimension $(j+1-E)_+$. Therefore $\dim J_j = j+1 - (j+1-E)_+$, and the formula follows. $\square$

Corollary 2.2. *For the fan of a bipyramid cell ($m \geq 2$),*

$$\begin{aligned} \sigma_0 &= 1, & \sigma_1 &= 2, & \sigma_2 &= n + (3 - m)_+, & \sigma_j &= n(j-1) \ (j \geq 3), \\ \tau_0 &= 1, & \tau_1 &= 2, & \tau_2 &= 3, & \tau_3 &= n + (4 - m)_+, & \tau_4 &= 2n + (5 - 2m)_+, & \tau_j &= n(j-2) \ (j \geq 5). \end{aligned}$$

Proof. By Lemma 2.1, $\sigma_j = n(j-1)_+ + (j+1-m(j-1)_+)_+$ and $\tau_j = n(j-2)_+ + (j+1-m(j-2)_+)_+$. Since $m \geq 2$, we have $j+1-m(j-1) \leq 3-j \leq 0$ for $j \geq 3$ and $j+1-m(j-2) \leq 5-j \leq 0$ for $j \geq 5$. $\square$

2.4 Half-cells

Lemma 2.3 (half-cells). *Let $k \geq 0$.*

- (a) *Let $f_1^+, \dots, f_n^+ \in R_k$ and write $f_i^+ = \sum_{a=0}^k z^a g_a^{+,i}(x, y)$ with $g_a^{+,i} \in \mathbb{R}[x, y]_{k-a}$. Then λ_i^2 divides $f_{i-1}^+ - f_i^+$ for every i if and only if $g_a^+ = (g_a^{+,1}, \dots, g_a^{+,n})$ lies in $\mathcal{H}_{k-a}^1(F)$ for every a .*
- (b) *Let $f_1^-, \dots, f_n^- \in R_k$ and write $f_i^-(x, y, z) = \sum_{a=0}^k z^a g_a^{-,i}(x + cz, y + ez)$ with $g_a^{-,i} \in \mathbb{R}[x, y]_{k-a}$ (this representation exists and is unique, because $(x, y, z) \mapsto (x + cz, y + ez, z)$ is invertible). Then $(\lambda_i + \lambda_i(w)z)^2$ divides $f_{i-1}^- - f_i^-$ for every i if and only if $g_a^- \in \mathcal{H}_{k-a}^1(F)$ for every a .*

This is the homogeneous form of the projection formula for oranges [SSVY24, Theorem 1.1], in the case of $(3, 2)$ -oranges; for the history of the idea see [ASS92] and the introduction of [SSVY24].

Proof. (a) In $\mathbb{R}[x, y][z]$, a polynomial $\lambda \in \mathbb{R}[x, y]$ divides $\sum_a z^a h_a$ ($h_a \in \mathbb{R}[x, y]$) if and only if it divides every h_a ; apply this to λ_i^2 . (b) The substitution $\Phi(x, y, z) = (x + cz, y + ez, z)$ is a linear automorphism with $\lambda_i \circ \Phi = \lambda_i(x, y) + \lambda_i(w)z$. Composing with Φ^{-1} turns the condition into $\lambda_i^2 \mid G_{i-1} - G_i$ for $G_i = \sum_a z^a g_a^{-,i}(x, y)$, and (a) applies. $\square$

3 Gluing across the base plane

We keep the normal form of Section 2.2. The operator D acts on a tuple $g = (g^{(i)})$ piece by piece: $Dg = (Dg^{(i)})$.

Proposition 3.1 (gluing). *For every bipyramid cell in normal form and every $k \geq 0$, let*

$$W_k = \{g \in \mathcal{H}_k^1(F) : Dg \in \mathcal{H}_{k-1}^1(F)\}.$$

Then

$$\dim H_k^1(\Delta) = \dim W_k + \sigma_{k-1} + 2 \sum_{j=0}^{k-2} \sigma_j.$$

Proof. Let $f \in H_k^1(\Delta)$ have pieces f_i^+ on T_i^+ and f_i^- on T_i^- . Write them as in Lemma 2.3. By that lemma, the conditions across the faces U_i and L_i say exactly that $g_a^\pm \in \mathcal{H}_{k-a}^1(F)$ for $0 \leq a \leq k$. It remains to impose the conditions $z^2 \mid f_i^+ - f_i^-$ across the faces B_i . Modulo z^2 we have $f_i^+ \equiv g_0^{+,i} + z g_1^{+,i}$, and Taylor's formula $g(x + cz, y + ez) = \sum_{j \geq 0} \frac{z^j}{j!} (D^j g)(x, y)$ gives

$$f_i^- \equiv g_0^{-,i} + z (Dg_0^{-,i} + g_1^{-,i}) \pmod{z^2}.$$

A polynomial $h_0(x, y) + z h_1(x, y)$ is divisible by z^2 only if $h_0 = h_1 = 0$. So the conditions across $B_1, \dots, B_n$ are

$$g_0^+ = g_0^- \quad \text{and} \quad g_1^+ - g_1^- = Dg_0^-. \quad (4)$$

If (4) holds, then $Dg_0^- = g_1^+ - g_1^- \in \mathcal{H}_{k-1}^1(F)$, so $g_0^- \in W_k$. Conversely, for $g_0 \in W_k$, $g_1^- \in \mathcal{H}_{k-1}^1(F)$ and arbitrary $g_a^\pm \in \mathcal{H}_{k-a}^1(F)$ ($2 \leq a \leq k$), put $g_0^+ = g_0^- = g_0$ and $g_1^+ = g_1^- + Dg_0 \in \mathcal{H}_{k-1}^1(F)$; this defines an element of $H_k^1(\Delta)$. The resulting linear map

$$W_k \times \mathcal{H}_{k-1}^1(F) \times \prod_{a=2}^k \mathcal{H}_{k-a}^1(F)^2 \longrightarrow H_k^1(\Delta), \quad (g_0, g_1^-, (g_a^+, g_a^-)_{a \geq 2}) \longmapsto f,$$

is bijective, since f determines all $g_a^\pm$. Counting dimensions gives the claim. $\square$

Lemma 3.2 (trace smoothness). *W_k consists of the $g \in \mathcal{H}_k^1(F)$ such that λ_i^3 divides $g^{(i-1)} - g^{(i)}$ for every i with $\lambda_i(w) \neq 0$. Consequently:*

- (a) *in the generic case, $W_k = \mathcal{H}_k^2(F)$;*
- (b) *in the coplanar cases, $W_k = \mathcal{H}_k^r(F)$ with $r_i = 1$ for the one or two rays parallel to w and $r_i = 2$ for all other rays;*
- (c) *in the collinear case, $W_k = \mathcal{H}_k^1(F)$.*

Proof. Let $g \in \mathcal{H}_k^1(F)$ and write $g^{(i-1)} - g^{(i)} = \alpha_i \lambda_i^2$ with $\alpha_i \in \mathbb{R}[x, y]_{k-2}$. Since D is a derivation and $D\lambda_i = \lambda_i(w)$ is a constant,

$$Dg^{(i-1)} - Dg^{(i)} = \lambda_i^2 D\alpha_i + 2\lambda_i(w) \lambda_i \alpha_i.$$

This is divisible by λ_i^2 if and only if $\lambda_i(w) = 0$ or $\lambda_i \mid \alpha_i$, and $\lambda_i \mid \alpha_i$ means $\lambda_i^3 \mid g^{(i-1)} - g^{(i)}$. Parts (a)–(c) follow from the description of the cases in Section 2.2, since $\lambda_i(w) = 0$ if and only if $w = 0$ or w is parallel to v_i . $\square$

Remark 3.3. In the generic case, Lemma 3.2 says that the restriction to B of every homogeneous C^1 spline on Δ is C^2 across every ray. This is a supersmoothness phenomenon of the kind studied in [Sor10], and CDS use additional smoothness of splines on bipyramid cells to derive their upper bound [Sor15]. We could not see the precise statement in [CDS16]. Proposition 3.1 shows that this supersmoothness is the only obstruction: every trace in W_k extends, and the extensions form an affine space of the dimension given there.

4 Proofs

4.1 The generic case

Proposition 4.1. *Let Δ be a generic bipyramid cell. Then $\dim H_k^1(\Delta) = 1, 3, 7$ for $k = 0, 1, 2$, and for $k \geq 3$*

$$\dim H_k^1(\Delta) = 2n\binom{k-1}{2} + 6 + \varepsilon_k,$$

where $\varepsilon_3 = (4-m)_+ + (3-m)_+$, $\varepsilon_4 = (5-2m)_+ + 2(3-m)_+$ and $\varepsilon_k = 2(3-m)_+$ for $k \geq 5$. Thus $\varepsilon_k = 0$ for all $k \geq 3$ if $m \geq 4$; $(\varepsilon_3, \varepsilon_4, \varepsilon_5, \dots) = (1, 0, 0, \dots)$ if $m = 3$; and $(3, 3, 2, 2, \dots)$ if $m = 2$.

Proof. By Proposition 3.1 and Lemma 3.2(a), $h_k := \dim H_k^1(\Delta) = \tau_k + \sigma_{k-1} + 2\sum_{j=0}^{k-2} \sigma_j$. By Corollary 2.2, $h_0 = \tau_0 = 1$, $h_1 = \tau_1 + \sigma_0 = 3$ and $h_2 = \tau_2 + \sigma_1 + 2\sigma_0 = 7$. For $k \geq 3$ write $\sigma_j = n(j-1)_+ + s_j$ and $\tau_j = n(j-2)_+ + t_j$, where $(s_0, s_1, s_2) = (1, 2, (3-m)_+)$, $s_j = 0$ for $j \geq 3$, $(t_0, \dots, t_4) = (1, 2, 3, (4-m)_+, (5-2m)_+)$ and $t_j = 0$ for $j \geq 5$. Since $\sum_{j=0}^{k-2} (j-1)_+ = \binom{k-2}{2}$ and $(k-2) + \binom{k-2}{2} = \binom{k-1}{2}$,

$$h_k = 2n\binom{k-1}{2} + t_k + s_{k-1} + 2\sum_{j=0}^{k-2} s_j.$$

For $k = 3$ the last three terms are $(4-m)_+ + (3-m)_+ + 6$; for $k = 4$ they are $(5-2m)_+ + 0 + 2(3-m)_+$; for $k \geq 5$ they are $0 + 0 + 2(3-m)_+$. This is the claim. $\square$

Proof of Theorem 1.1. By (3) and Proposition 4.1, $\dim S_d^1(\Delta) = 1, 4, 11$ for $d = 0, 1, 2$, and for $d \geq 2$, using $\sum_{k=3}^d \binom{k-1}{2} = \binom{d}{3}$,

$$\dim S_d^1(\Delta) = 11 + \sum_{k=3}^d \left(2n\binom{k-1}{2} + 6 + \varepsilon_k \right) = 2n\binom{d}{3} + 6d - 1 + \sum_{k=3}^d \varepsilon_k.$$

For $d \geq 3$ the last sum is 0 if $m \geq 4$ and 1 if $m = 3$, which gives $2n\binom{d}{3} + 6d - 1 + (4-m)_+$. If $m = 2$, then $n = 4$ and the sum is 3 for $d = 3$ and $6 + 2(d-4) = 2d - 2$ for $d \geq 4$; this gives 28 for $d = 3$ and $8\binom{d}{3} + 8d - 3$ for $d \geq 4$, which is $8\binom{d}{3} + 8d - 3 - (4-d)_+$ in both cases. $\square$

4.2 Comparison with the bounds

Proof of Corollary 1.2. For $m \geq 3$ and $d \geq 2$, $\text{UB} - \text{LB} = 1 + (4-m)_+ \geq 1$, so the bounds differ; for $m = 2$ they differ exactly when $d \geq 3$, where $\text{UB} - \text{LB} = 2 - (4-d)_+ \geq 1$. For $d \geq 3$, Theorem 1.1 gives $\dim S_d^1(\Delta) = \text{UB}$. For $d = 2$ and $m \geq 3$, $\dim S_2^1(\Delta) = 11$ while $\text{LB} = 10$ and $\text{UB} = 11 + (4-m)_+$, which equals 11 exactly when $m \geq 4$. In all these cases $\dim S_d^1(\Delta) > \text{LB}$. $\square$

4.3 An elementary proof in degree two

Proof of Proposition 1.3. Let $f \in S_2^1(\Delta)$ with pieces p_T of degree at most 2. For every interior face $\tau = T \cap T'$ the difference $p_T - p_{T'}$ is divisible by ℓ_τ^2 , which has degree 2, so $p_T - p_{T'} = c_\tau \ell_\tau^2$ with a constant c_τ . Fix $i \in \{1, \dots, n\}$. The four tetrahedra containing the interior edge $[0, v_i]$ are $T_{i-1}^+, T_i^+, T_i^-, T_{i-1}^-$, and consecutive ones share the faces U_i, B_i, L_i, B_{i-1} . Going once around the edge, the differences telescope:

$$(p_{T_{i-1}^+} - p_{T_i^+}) + (p_{T_i^+} - p_{T_i^-}) + (p_{T_i^-} - p_{T_{i-1}^-}) + (p_{T_{i-1}^-} - p_{T_{i-1}^+}) = 0,$$

that is,

$$a \ell_{U_i}^2 + b \ell_B^2 + c \ell_{L_i}^2 = 0 \tag{5}$$

for some constants a, b, c , where $a = \pm c_{U_i}$, $c = \pm c_{L_i}$ and $b = c_{B_i} - c_{B_{i-1}}$ when c_{B_j} is defined by $p_{T_j^+} - p_{T_j^-} = c_{B_j} \ell_B^2$. The planes of U_i, B and L_i all contain the line $\mathbb{R}v_i$. They are pairwise distinct: $U_i \neq B$ and $L_i \neq B$ because $v_{n+1}, v_{n+2} \notin B$, and $U_i = L_i$ would put v_i in the plane through $0, v_{n+1}, v_{n+2}$, which the generic case excludes. So $\ell_{U_i}, \ell_B, \ell_{L_i}$ are pairwise non-proportional linear forms in the two-dimensional space of linear forms vanishing on $\mathbb{R}v_i$. After scaling, they are s, t and $s+t$ for a basis s, t of this space, and $s^2, t^2, (s+t)^2$ are linearly independent. Hence $a = b = c = 0$ in (5).

Thus $c_{U_i} = c_{L_i} = 0$ for all i , so f equals one polynomial P^+ on all T_i^+ and one polynomial P^- on all T_i^- , and $c_{B_i} = c_{B_{i-1}}$ for all i , so $P^+ - P^- = c \ell_B^2$ for a single constant c . Therefore $f = P^- + c(\ell_B)_+^2$. Conversely, every such function is a C^1 spline, since its only jumps are $c \ell_B^2$ across the faces B_i . The ten monomials of degree at most 2 and $(\ell_B)_+^2$ are linearly independent, because $(\ell_B)_+^2$ is not a polynomial. So $\dim S_2^1(\Delta) = 11$. $\square$

The argument uses genericity only through the pairwise distinctness of the three planes at each base edge. In the coplanar case II the plane P cuts Δ along interior faces; if ℓ_P is a linear form vanishing on P , the function equal to ℓ_P^2 on one side of P and to 0 on the other is a further quadratic spline, and accordingly $\dim S_2^1(\Delta) = 12$ in Theorem 4.2(c).

4.4 The collinear and coplanar cases

The same reduction reproves the exact formulas of CDS.

Theorem 4.2 ([Sor15, Theorems 1–3]). *Let Δ be a bipyramid cell and $d \geq 2$.*

- (a) *In the collinear case, $\dim S_d^1(\Delta) = 8\binom{d}{3} + 12(d-1) + (3-d)_+$ if $m = 2$, and $2n\binom{d}{3} + n(d-1) + 6d - 4$ if $m \geq 3$.*
- (b) *In the coplanar case I, $\dim S_d^1(\Delta) = 2n\binom{d}{3} + 7d - 3$.*
- (c) *In the coplanar case II, $\dim S_d^1(\Delta) = 8\binom{d}{3} + 10d - 8$ if $m = 2$, and $2n\binom{d}{3} + 8d - 4$ if $m \geq 3$.*

Proof. We use Proposition 3.1, Lemma 3.2, Lemma 2.1 and Corollary 2.2. For $k \geq 3$ we have $\sigma_{k-1} + 2 \sum_{j=0}^{k-2} \sigma_j = n(k-2) + 2n\binom{k-2}{2} + 6 + c_k$ with $c_3 = (3-m)_+$ and $c_k = 2(3-m)_+$ for $k \geq 4$, by the computation in the proof of Proposition 4.1.

Coplanar cases. Let $p \in \{1, 2\}$ be the number of base vertices on P ; they lie on the line $\mathbb{R}w$, which carries order 1, and the other $m-1$ lines carry order 2. By Lemma 2.1,

$$\dim W_k = p(k-1)_+ + (n-p)(k-2)_+ + (k+1 - (k-1)_+ - (m-1)(k-2)_+)_+,$$

which equals $1, 2, p + 2$ for $k = 0, 1, 2$, equals $n + p + (3 - m)_+$ for $k = 3$, and equals $n(k - 2) + p$ for $k \geq 4$ (as $(m - 1)(k - 2) \geq 2$). Hence $\dim H_k^1(\Delta) = 1, 3, p + 6$ for $k = 0, 1, 2$ and $\dim H_k^1(\Delta) = 2n\binom{k-1}{2} + 6 + p + 2(3 - m)_+$ for $k \geq 3$. Summing,

$$\dim S_d^1(\Delta) = (10 + p) + 2n\binom{d}{3} + (d - 2)(6 + p + 2(3 - m)_+).$$

In case I the line $\mathbb{R}w$ carries exactly one ray; this is impossible for $m = 2$, where both lines carry two rays, so $m \geq 3$ and the formula becomes $2n\binom{d}{3} + 7d - 3$. In case II it becomes $2n\binom{d}{3} + 8d - 4$ if $m \geq 3$ and $8\binom{d}{3} + 10d - 8$ if $m = 2$ (then $n = 4$).

Collinear case. Here $W_k = \mathcal{H}_k^1(F)$, so $\dim H_k^1(\Delta) = \sigma_k + \sigma_{k-1} + 2 \sum_{j \leq k-2} \sigma_j$. This is $1, 3, n + 4 + (3 - m)_+$ for $k = 0, 1, 2$, and for $k \geq 3$ it is $n(k - 1) + n(k - 2) + 2n\binom{k-2}{2} + 6 + c_k = 2n\binom{k-1}{2} + n + 6 + c_k$. Summing, for $m \geq 3$ we get $(n + 8) + 2n\binom{d}{3} + (n + 6)(d - 2) = 2n\binom{d}{3} + n(d - 1) + 6d - 4$. For $m = 2$ ($n = 4$) we get 13 for $d = 2$ and $13 + 8\binom{d}{3} + 10(d - 2) + (2d - 5) = 8\binom{d}{3} + 12(d - 1)$ for $d \geq 3$, in agreement with (a). $\square$

5 Comparison with DiPasquale–Villamizar and exact computations

External confirmation. In [DV21, Section 6.2], DiPasquale and Villamizar consider a bipyramid over a pentagon whose five base vertices lie in a plane with the interior vertex, such that no base vertex lies in the plane through the interior vertex and the two apices, and no two base vertices are collinear with the interior vertex. In the terminology of CDS this is the generic case with $n = m = 5$ (DiPasquale and Villamizar call it a non-generic bipyramid). Their Table 1 (column “syndim” of the arXiv version) lists $\dim H_d^r(\Delta)$ computed with the `AlgebraicSplines` package of Macaulay2. For $r = 1$ these values coincide with Proposition 4.1:

k	2	3	4	5	6	7	8	9
$\dim H_k^1$, [DV21, Table 1]	7	16	36	66	106	156	216	286
Proposition 4.1, $n = m = 5$	7	16	36	66	106	156	216	286

Of these, the values for $k \geq 5$ also follow from [ANS96], and the value for $k = 2$ is derived in [DV21] from [MV14, Theorem 4.1]; the values for $k = 3, 4$ were obtained there by computation. For $r = 2, 3, 4$, our exact computations on generic pentagonal cells also reproduce the whole “syndim” column (degrees 3–11, 4–12 and 5–13). These values are not covered by Theorem 1.1; their agreement is a check of the programs.

Exact computations. For a cell with integer vertices, $H_k^r(\Delta)$ and $S_d^r(\Delta)$ are kernels of matrices with rational entries (integer entries in (i) and (ii) below). The programs in the release build these matrices in three ways:

- (i) with cofactor unknowns, from the equations $p_T - p_{T'} - \ell_\tau^{r+1} q_\tau = 0$ (the q_τ are then forced to be unique, so the kernel is isomorphic to the spline space);
- (ii) without cofactors, by requiring that the jump $p_T - p_{T'}$ and its derivatives of order at most r in a normal direction of τ vanish identically on the plane of τ , after substituting a parametrisation $sa + tb$ of that plane;
- (iii) in Bernstein–Bézier form: the unknowns are the coefficients of a continuous spline of degree d at the domain points, and the classical C^1 conditions are imposed across each interior face.

Ranks are computed exactly over $\mathbb{Q}$ by fraction-free elimination. Ranks modulo a prime are used only as cross-checks; for an integer matrix they never exceed the rank over $\mathbb{Q}$. The computations cover cells of all four cases, including symmetric positions, and they agree with Theorems 1.1 and 4.2 in every instance; the details are in the Verification paragraph below.

6 Remarks

The formula of Alfeld, Neamtu and Schumaker. Let $r \geq 0$ and $k \geq 3r + 2$. For a cell with interior vertex 0, the cones over its tetrahedra form a total trihedral decomposition of $\mathbb{R}^3$ in the sense of [ANS96], with the same homogeneous splines. If the cell has V interior edges, then [ANS96, Theorem 3] states that

$$\dim H_k^r = (k - r)(k - 2r)V - 2k^2 + 6kr - 3r^2 + 3r + 2 + \sum_v \sum_{j=1}^{k-r} (r + j + 1 - je_v)_+, \quad (6)$$

where v runs over the interior edges and e_v is the number of distinct planes that contain the interior faces around v ; no genericity assumption is made. Take $r = 1$ and a bipyramid cell, so that $V = n + 2$. An edge with $e_v \geq 3$ contributes nothing to the double sum, and an edge with $e_v = 2$ contributes 1. The interior faces around $[v_0, v_i]$ are U_i , B_{i-1} , B_i and L_i , so $e_v = 3$ for these edges in the generic case (see the proof of Proposition 1.3), while $e_v = m$ for $[v_0, v_{n+1}]$ and $[v_0, v_{n+2}]$. Hence (6) gives, for $k \geq 5$,

$$\dim H_k^1(\Delta) = (k - 1)(k - 2)(n + 2) - 2k^2 + 6k + 2 + 2(3 - m)_+ = 2n \binom{k-1}{2} + 6 + 2(3 - m)_+,$$

which is Proposition 4.1. In the coplanar cases $e_v = 2$ for the p base edges in P , and in the collinear case for all n base edges; this gives the values $2n \binom{k-1}{2} + 6 + p + 2(3 - m)_+$ and $2n \binom{k-1}{2} + n + 6 + 2(3 - m)_+$ used in the proof of Theorem 4.2. For $k \leq 4$ the theorem does not apply. On generic cells with $m \geq 4$, the right-hand side of (6) still equals $\dim H_k^1(\Delta)$ for $k = 3, 4$; for $m = 3$ it is smaller by 1 when $k = 3$, and for $m = 2$ it is smaller by 1 when $k = 3$ and when $k = 4$.

The lower bound. In degree 2 the lower bound 10 counts only the global quadratics, while $S_2^1(\Delta)$ also contains $(\ell_B)_+^2$. By Theorem 1.1, $\dim S_d^1(\Delta) - \text{LB}$ equals 1 for $m \geq 4$ and all $d \geq 2$, equals 1 for $m = 3$ and $d = 2$ and 2 for $m = 3$ and $d \geq 3$, and equals 1 for $m = 2$ and $d = 3$ and 2 for $m = 2$ and $d \geq 4$. As a computational observation (not used in the proofs), on seven generic cells with $(n, m) \in \{(3, 3), (4, 2), (4, 3), (4, 4), (5, 5), (6, 3), (7, 4)\}$ the bound LB coincides, for $2 \leq d \leq 7$, with $\sum_{k \leq d} \chi(R/\mathcal{J}, k)$, the summed Euler characteristic of the Billera–Schenck–Stillman complex, which is a lower bound for $\dim H_k^1(\Delta)$ in each degree [MV14], [DV21, Proposition 2.6]. On these cells, $\dim H_k^1(\Delta) - \chi(R/\mathcal{J}, k)$ is 1 in degree 2 for $m \geq 3$ (the spline $(\ell_B)_+^2$), 1 in degree 3 for $m = 3$, 1 in degrees 3 and 4 for $m = 2$, and 0 otherwise.

Higher smoothness. For C^r splines, (3) and Lemma 2.3 extend verbatim with λ_i^{r+1} in place of λ_i^2 . The gluing condition becomes $f_i^+ \equiv f_i^- \pmod{z^{r+1}}$, that is, $g_s^+ = \sum_{j=0}^s \frac{1}{j!} D^j g_{s-j}^-$ for $0 \leq s \leq r$. For $r \geq 2$ these conditions couple $g_0^-, \dots, g_{r-1}^-$, and we have not analysed the resulting system. We make no claim for $r \geq 2$.

Verification. The claims were checked as follows.

1. The programs with which the formula was found and first tested (Python, exact arithmetic, about four minutes) use formulation (i). They check Lemma 2.1 in 1674 cases (random fans, constant and mixed orders, degrees at most 8). They compare the formulas of Proposition 3.1 and Lemma 3.2 with direct computation on 185 cells of all four cases, with $3 \leq n \leq 10$, every admissible m and $k \leq 7$ ($k \leq 6$ for $n > 8$), including symmetric configurations; a subset is recomputed with a second implementation, which requires the jump and its gradient to vanish at $k + 1$ points of each face. They also evaluate the closed formulas against [Sor15, Theorems 1–4] for $n \leq 40$, $d \leq 30$, compute $\dim S_d^1(\Delta)$ for $2 \leq d \leq 4$ without the splitting (3) on five cells, compute generic cells with $11 \leq n \leq 14$ up to degree 8, reproduce [DV21, Table 1] for $r = 1, 2, 3, 4$ on three pentagonal cells, and compute the Euler characteristics of the remark above. No mismatch occurred.
2. A separate standalone script, `check_bipyramid.py` (standard library only, one to two minutes), was written for this note and imports nothing from the programs of item 1. It uses formulation (ii) with exact ranks over $\mathbb{Q}$. It computes $\dim S_d^1(\Delta) = 1, 4, 11, 24, 48$ ($d \leq 4$) for the cell of Example 1.4 without the splitting (3), and certifies the basis of Proposition 1.3 for it. It checks Proposition 4.1 on 48 generic cells (all admissible (n, m) with $3 \leq n \leq 8$, degrees $k \leq 6$, or $k \leq 5$ for $n \geq 7$), checks Theorem 4.2 on 38 collinear and coplanar cells for $2 \leq d \leq 5$, reproduces [DV21, Table 1] for $r \leq 4$, and verifies Corollary 1.2 for $n \leq 60$ and $d \leq 40$. It also counts the numbers e_v of (6) on each of these cells and confirms that (6) agrees with the computed $\dim H_k^1(\Delta)$ for $k \geq 5$ (115 comparisons).
3. A first independent verification run, which was AI-assisted, re-derived all proofs and wrote its own code, which does not import the programs of item 1. It used formulation (ii) with ranks modulo two 31-bit primes, and an exact rank over $\mathbb{Q}$ for the cell of Example 1.4. It found $\dim S_d^1(\Delta) = 1, 4, 11, 24, 48$ for that cell ($d \leq 4$), and no mismatch in 303 comparisons on random cells of all four cases with $3 \leq n \leq 8$ and $d \leq 5$, or in 160 comparisons of $\dim H_k^1(\Delta)$ with $k \leq 8$ and $n \leq 10$. It also re-ran the programs of item 1 and reproduced their recorded outputs. The code of this run is not part of the release.
4. A second independent verification run, also AI-assisted, re-read the Oberwolfach report, checked every step of the proofs, and wrote its own programs before reading those of items 1 and 2; they are part of the release. They use formulation (iii) for $S_d^1(\Delta)$ itself, without the splitting (3), on cells moved by random integer affine maps, and, as a second implementation of formulation (i), homogeneous cofactor equations; ranks are exact over $\mathbb{Q}$. They found $\dim S_d^1(\Delta) = 1, 4, 11, 24, 48, 90, 156, 252$ ($d \leq 7$) for the cell of Example 1.4, and no mismatch in 639 comparisons of $\dim S_d^1(\Delta)$ ($d \leq 6$) and 1004 comparisons of $\dim H_k^1(\Delta)$ ($k \leq 10$) on 126 random generic cells with $3 \leq n \leq 12$, on 13 generic cells in special position, or in 384 comparisons on collinear and coplanar cells. They also reproduced [DV21, Table 1] for $r \leq 4$, the Euler characteristics of the remark above and Lemma 2.1 in 540 cases, and confirmed that (6) agrees with Proposition 4.1 for $k \geq 5$. The run also re-ran the programs of items 1 and 2 and reproduced their recorded outputs.

Scope and priority. Theorem 1.1 determines $\dim S_d^1(\Delta)$ for every generic bipyramid cell and every d , and Corollary 1.2 answers the question of record OWR-13678-010 [ula26]: in the remaining cases the dimension equals the upper bound, except for $m = 3$, $d = 2$. The bounds we compare with are those printed in [Sor15, Theorem 4]. The full text of [CDS16] was not accessible to us, so we cannot say how its Theorem 4 is worded. If it has the same bounds for $d \geq 3$ and treats the case $m = 3$, $d = 2$ differently, the conjecture as stated there is true by Theorem 1.1; for any wording, Theorem 1.1 decides the question, since it gives the dimension itself. The homogeneous components

of Theorem 1.1 of degree at least 5 were known: they follow from [ANS96, Theorem 3] (Section 6). The component of degree 2 is elementary (Proposition 1.3). What is new, as far as we know, is the determination of $H_3^1(\Delta)$ and $H_4^1(\Delta)$, the gluing step (Proposition 3.1), and the resulting formula with its exception at $(m, d) = (3, 2)$. We searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar, and ran three web searches. OpenAlex and Semantic Scholar list only [DV21] as citing [CDS16]. As described above, [DV21] treats one pentagonal cell and does not state a general formula. The projection of [SSVY24] is used in Lemma 2.3, but that paper does not treat bipyramid cells. Later work on trivariate splines that we found, such as [DV24, CDM⁺26], concerns other partitions. Apart from the components of degree at least 5, and the component of degree 2 for the cell of [DV21], we found no earlier statement of Theorem 1.1, and we found none of Proposition 3.1 or of the exception at $(m, d) = (3, 2)$. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the dimension formula and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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