

Purely Singular Copulas Are Typical for the Markov-Kernel Metric D_1

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Abstract

Durante, Fernández-Sánchez and Trutschnig (Oberwolfach Report 20/2015) asked whether the class of purely singular d -dimensional copulas is of second category in $(\mathcal{C}_d, D_1)$, where D_1 is the metric of Trutschnig defined through the conditional distributions (Markov kernels) of the copulas. The answer is yes: for every $d \geq 2$ the purely singular copulas form a dense G_δ set in $(\mathcal{C}_d, D_1)$, hence a co-meager set. This follows in a few lines from known results, and we claim no priority for it: the purely singular copulas form a G_δ set already for the coarser uniform metric, $(\mathcal{C}_d, D_1)$ is complete, and the check-min approximation $B_N(A)$ of a copula A , which carries the mass of each cell of the N -grid uniformly on the main diagonal of the cell, is purely singular and tends to A in D_1 . For $d = 2$ this density is classical; Durante, Fernández-Sánchez and Trutschnig use it in “A typical copula is singular” (2015), citing Mikusiński and Taylor (Metrika 2010), and with Theorem 4.1 of that paper it settles the case $d = 2$, which the paper nevertheless lists as open. For $d \geq 3$ the density may also be in Mikusiński and Taylor (2010), a paper on approximations of n -copulas that we could not read; we prove it with the bound $D_1(A, B_N(A)) \leq \|H_A - P_N H_A\|_1 + (d-1)/N$, where H_A is the conditional distribution function and P_N averages over the columns of the grid. New, to our knowledge, is that the copulas whose conditional distributions are, for almost every value of the conditioning coordinate, singular and without atoms also form a dense G_δ set in $(\mathcal{C}_d, D_1)$: in this metric a typical copula has diffuse singular conditional distributions, whereas for $d = 2$ and the uniform metric they are typically point masses. If D_1 is replaced by the L^1 distance of the copula functions, as in the restatement of the question in version 1.7.0 of the UnsolvedMath collection, the question reduces to the known result for the uniform metric. This is an unrefereed note.

1 Introduction

1.1 The question

Let $d \geq 2$ and $I = [0, 1]$. A d -copula is the distribution function of a probability measure μ_A on I^d whose one-dimensional marginals are the Lebesgue measure λ on I (see, e.g., [DS15]); thus $A(u) = \mu_A([0, u])$ for $u \in I^d$, and $\mathcal{C}_d$ denotes the set of d -copulas. Let $\mu_A = \mu_A^{\text{ac}} + \mu_A^{\text{s}}$ be the Lebesgue decomposition of μ_A with respect to the Lebesgue measure λ^d . The copula A is *purely singular* if $\mu_A^{\text{ac}} = 0$, that is, if μ_A is concentrated on a λ^d -null Borel set. We write $\mathcal{S}_d$ for the set of purely singular d -copulas.

Write a point of I^d as $u = (x, y)$ with $x \in I$ and $y \in I^{d-1}$, and let $K_A: I \times \mathcal{B}(I^{d-1}) \rightarrow [0, 1]$ be the Markov kernel of A , that is, a regular conditional distribution of the last $d - 1$ coordinates given the

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first one, so that $\mu_A(G) = \int_I K_A(x, G_x) d\lambda(x)$ with $G_x = \{y : (x, y) \in G\}$ for every Borel set $G \subseteq I^d$. Put $H_A(x, y) = K_A(x, [0, y])$. The metric

$$D_1(A, B) = \int_{I^{d-1}} \int_I |H_A(x, y) - H_B(x, y)| d\lambda(x) d\lambda^{d-1}(y) \quad (1)$$

was introduced by Trutschnig [Tru11] for $d = 2$ and extended to $d \geq 3$ by Fernández-Sánchez and Trutschnig [FT15]; the space $(\mathcal{C}_d, D_1)$ is complete and separable but not compact [Tru11, MFT21]. The uniform metric $d_\infty(A, B) = \max_u |A(u) - B(u)|$ makes $\mathcal{C}_d$ compact, and the topology of D_1 is strictly finer.

In the extended abstract [Dur15] of Oberwolfach Report 20/2015, F. Durante, in joint work with J. Fernández-Sánchez and W. Trutschnig, uses the Markov kernel to study singular copulas and recalls Baire category results proved in [DFST15]: the absolutely continuous copulas are of first category both in $(\mathcal{C}_d, d_\infty)$ and in $(\mathcal{C}_d, D_1)$, and the purely singular copulas are co-meager in $(\mathcal{C}_d, d_\infty)$ [Dur15, Theorem 3]. It remains open, the abstract says, “whether the class of purely singular copulas is of second category” in $(\mathcal{C}_d, D_1)$ [Dur15, p. 1126]. For $d = 2$ the same question closes the paper [DFST15, Section 5], whose results are written for $d = 2$ and are said to extend to $d \geq 3$ with obvious modifications [DFST15, Section 1]. The question is record OWR-13677-003 of the UnsolvedMath collection [ula26], where it is listed as open.

Three remarks on the statement; they are taken up in Section 5. (i) The abstract writes D_1 as “the D_1 metric in” [Tru11], which is (1) for $d = 2$; it does not write the formula for $d \geq 3$. We use (1), which conditions on the first coordinate as the kernel representation of [Dur15, p. 1125] does; [MFT21] restates the multivariate metric of [FT15] with conditioning on the last coordinate. All our results hold for every conditioning coordinate, for the sum and the maximum of the d resulting metrics, and for the companion metrics D_p and D_∞ (Remark 3.7). (ii) “Purely singular” means $\mu_A^{\text{ac}} = 0$: Theorem 3 of [Dur15] writes “(purely) singular” for the class that [DFST15] shows to be co-meager for d_∞ , which is $\mathcal{S}_d$. On p. 1124 the abstract also speaks loosely of “singular copulas (or, copulas with a singular component)”; under that reading the question is trivial (Section 5). (iii) In version 1.7.0 of [ula26] the record restates the question with D_1 defined as the L^1 distance $\int |C_1 - C_2|$ of the copula functions; in version 1.6.0 its statement was the sentence of the source. That metric is equivalent to d_∞ , and under it the question is the known statement for d_∞ (Proposition 5.1).

1.2 Results

Theorem 1.1. *For every $d \geq 2$ the set $\mathcal{S}_d$ of purely singular d -copulas is a dense G_δ set in $(\mathcal{C}_d, D_1)$. Consequently $\mathcal{S}_d$ is co-meager, and in particular of second category, in $(\mathcal{C}_d, D_1)$.*

So the answer to the question of [Dur15, DFST15] is yes; the question asks for the second category, and $\mathcal{S}_d$ is even co-meager. As explained in Section 1.3, Theorem 1.1 follows in a few lines from known results. The source decomposes the kernel of $C \in \mathcal{C}_2$ into an absolutely continuous part, a singular part without point masses and a discrete part [Dur15, p. 1125]. For a typical copula in $(\mathcal{C}_d, D_1)$ only the middle part survives:

Theorem 1.2. *For every $d \geq 2$ the set of copulas $A \in \mathcal{C}_d$ such that, for λ -almost every $x \in I$, the conditional distribution $K_A(x, \cdot)$ is singular with respect to λ^{d-1} and has no atoms, is a dense G_δ set in $(\mathcal{C}_d, D_1)$.*

Corollary 1.3. *Let $d \geq 2$. The copulas $A \in \mathcal{C}_d$ that are purely singular, whose measure μ_A has full support I^d , and whose conditional distributions $K_A(x, \cdot)$ are singular and atomless for almost every x , form a dense G_δ set in $(\mathcal{C}_d, D_1)$. In particular the completely dependent copulas and the copulas whose support is a λ^d -null set form meager sets in $(\mathcal{C}_d, D_1)$.*

This is in contrast with the uniform metric: in $(\mathcal{C}_2, d_\infty)$ a typical copula is mutually completely dependent [Kim68, Theorem 1.1 and the remark after its proof], [DT25, Theorem 3.8, Corollary 3.10], so its conditional distributions are point masses. In $(\mathcal{C}_2, D_1)$ the completely dependent copulas form a closed set [Dur15, Theorem 2], which is even nowhere dense [FT26, Section 4.1]; Corollary 1.3 shows that a typical copula in $(\mathcal{C}_d, D_1)$ has diffuse but singular conditional distributions. In the terminology of [DT25], where a bivariate copula is called singular if its mass is carried by the atomless singular parts of its conditional distributions, Theorem 1.2 for $d = 2$ says that the singular copulas are typical in $(\mathcal{C}_2, D_1)$, while they are meager in $(\mathcal{C}_2, d_\infty)$. Copulas of this kind were constructed by Trutschnig and Fernández-Sánchez [TF14]: according to [DFST15, Section 5], there are singular bivariate copulas, possibly with full support, whose conditional distribution functions are continuous, with derivative zero almost everywhere. Theorem 1.2 and Corollary 1.3 say that in $(\mathcal{C}_d, D_1)$ singular atomless conditional distributions are typical.

The proof of Theorem 1.1 has three steps (Section 3). The set $\mathcal{S}_d$ is a countable intersection of sets that are open already for d_∞ (Lemma 3.2); these sets are D_1 -open because D_1 dominates a power of d_∞ (Lemma 3.1). For each N , the *diagonal checkerboard* $B_N(A)$, which carries the mass $\mu_A(Q)$ of every cell Q of the N -grid uniformly on the main diagonal of Q (the check-min approximation of order N in the terminology of [MT10, DFST15]), is purely singular and satisfies $D_1(A, B_N(A)) \leq \|H_A - P_N H_A\|_1 + (d-1)/N \rightarrow 0$, where P_N averages H_A over the columns of the grid (Lemma 3.3). Finally $(\mathcal{C}_d, D_1)$ is complete (Appendix A), and the Baire category theorem applies. For Theorem 1.2 the diagonals are replaced by cells carrying rotated Cantor measures, and the absence of atoms is shown to be a G_δ property (Section 4).

1.3 What is new and what is not

We found no text that answers the question (the searches are described in the paragraph “Scope and priority” at the end of Section 6), but the answer follows in a few lines from known results, and we claim no priority for Theorem 1.1. Its three ingredients are these.

(i) $\mathcal{S}_d$ is a G_δ set already for d_∞ (Lemma 3.2, a standard argument), hence for the finer metric D_1 ; for $d = 2$, [DFST15, Theorem 4.1] gives a dense G_δ set for d_∞ contained in $\mathcal{C}_{\text{sing}} = \mathcal{S}_2$.

(ii) The check-min approximations $B_N(A)$ are purely singular and tend to A in D_1 . For $d = 2$ this is classical. In the language of doubly stochastic operators, convex combinations of operators induced by invertible measure-preserving maps are dense in the strong operator topology [Bro66, Theorem 2], [Kim68, Theorem 2.2 and Lemma 2.2], [Iwa80, Corollary 1], strong operator convergence implies convergence in D_1 , and for $N = 2^n$ the approximants of [Kim68], with translations as the measure-preserving maps, are exactly the $B_N(A)$ (Remark 3.6). In the language of copulas, the dyadic checkerboard approximations converge weakly conditionally [KFT21b, Theorem 3.2], and the proof applies verbatim to the $B_{2^n}(A)$ (Remark 3.6). [DFST15] uses this density: before its Theorem 4.1 it states that the rational check-min copulas are D_1 -dense, and in the proof of its Theorem 3.2 it takes check-min copulas D_1 -close to Π from [MT10, Theorem 6]. For $d \geq 3$ the density may also be in [MT10], which treats check-min, checkerboard, Bernstein and shuffle-of-Min approximations of n -copulas in arbitrary dimension and the ways in which they converge: [MT09, Theorem 11] quotes from [MT10] that the checkerboard and Bernstein approximations of n -copulas ∂ -converge, and, for bivariate copulas, [AW26, AF26] describe the check-min approximations as converging in D_1 , citing [MT10]. For d -copulas, ∂ -convergence is D_1 -convergence for every conditioning coordinate (Remark 3.8). We could not read [MT10]; Lemma 3.3 proves the density for every d , with an explicit bound.

(iii) $(\mathcal{C}_d, D_1)$ is complete [Tru11, MFT21] (Appendix A).

With (i)–(iii), Theorem 1.1 follows from the Baire category theorem. In particular the case $d = 2$ follows from [DFST15, Theorem 4.1] together with the density stated in [DFST15] itself

(Remark 3.6(d)); nevertheless [DFST15, Section 5] lists the question as open: “It remains an open question whether Csing is of second category in $(C, D1)$.” New, to our knowledge, are Theorem 1.2 and Corollary 1.3; copulas with singular atomless conditional distributions were constructed in [TF14], but we found no statement that they are typical. Smaller additions are the explicit bound of Lemma 3.3(c) and the identity of Remark 3.5 for $d = 2$, which [MT10] might also contain. The arguments are short and use standard tools.

2 Preliminaries

Throughout, $d \geq 2$, $A, B \in \mathcal{C}_d$, and $u = (x, y)$ with $x \in I$, $y \in I^{d-1}$. For $y, y' \in I^{d-1}$ we write $[0, y] = \prod_j [0, y_j]$ and $y \leq y'$ coordinatewise. A *box* is a product $R = \prod_{j=1}^d \langle a_j, b_j \rangle \subseteq I^d$ of intervals with $a_j < b_j$, each end open or closed. For $N \geq 1$ let $J_r = [\frac{r-1}{N}, \frac{r}{N})$ for $r < N$ and $J_N = [\frac{N-1}{N}, 1]$; for $i = (i_1, \dots, i_d) \in \{1, \dots, N\}^d$ let $Q_i = J_{i_1} \times \dots \times J_{i_d}$ and $Q'_i = J_{i_2} \times \dots \times J_{i_d}$. The cells Q_i partition I^d , and for fixed i_1 the cells Q'_i partition I^{d-1} .

Lemma 2.1 (standard facts). *(a) Every hyperplane $\{u : u_j = t\}$ is μ_A -null. Hence for every box R ,*

$$\mu_A(R) = V_A(a, b) := \sum_{\varepsilon \in \{0,1\}^d} (-1)^{d-|\varepsilon|} A(c^\varepsilon), \quad c_j^\varepsilon = \begin{cases} b_j, & \varepsilon_j = 1, \\ a_j, & \varepsilon_j = 0, \end{cases}$$

and $\sum_{i: i_j=r} \mu_A(Q_i) = \mu_A(\{u : u_j \in J_r\}) = \frac{1}{N}$ for all j and r .

(b) $|A(u) - A(v)| \leq \sum_j |u_j - v_j|$ for $u, v \in I^d$.

(c) H_A is Borel on I^d . Two versions of K_A agree, as measures, for λ -almost every x ; hence H_A , and D_1 , are well defined. Moreover $A(x, y) = \int_0^x H_A(t, y) dt$ for all $(x, y) \in I^d$.

(d) For a convex combination $C = \sum_k c_k A_k$ of copulas, $\sum_k c_k K_{A_k}$ is a version of K_C .

Proof. (a) The j -th marginal of μ_A is λ , which has no atoms. The formula holds for the half-open box $(a, b]$ by inclusion–exclusion, and the boundary of R is contained in finitely many hyperplanes. The last statement is the case of the slab $\{u_j \in J_r\}$. (b) $|A(u) - A(v)|$ is at most the μ_A -mass of $[0, u] \triangle [0, v]$, which lies in the union of the slabs between u_j and v_j , of masses $|u_j - v_j|$. (c) Let $y_j^{(m)} = \lceil 2^m y_j \rceil / 2^m$. The function $(x, y) \mapsto K_A(x, [0, y^{(m)}])$ is a finite sum of products of a Borel function of x and the indicator of a Borel set of y 's, and $[0, y^{(m)}]$ decreases to $[0, y]$, so H_A is a pointwise limit of Borel functions. Two versions agree for almost every x on a countable π -system generating $\mathcal{B}(I^{d-1})$, hence as measures [Kal02, Chapter 6]. The last formula is the disintegration identity for $G = [0, x] \times [0, y]$. (d) The disintegration identity is linear in the measure. $\square$

The metric D_1 . By Lemma 2.1(c), (1) defines $D_1(A, B) = \|H_A - H_B\|_{L^1(\lambda^d)}$. It is a metric: if $D_1(A, B) = 0$, then $A(x, y) = B(x, y)$ for all x and almost all y , hence for all y by continuity. Completeness of $(\mathcal{C}_d, D_1)$ is proved in [Tru11] for $d = 2$ and stated for all d in [MFT21]; for completeness we give a short proof in Appendix A. For $d = 2$, $D_1(A, \Pi) \leq \frac{1}{3}$ with equality if and only if A is completely dependent [Tru11], as restated in [KFT21b, Section 2]. The averaging operator in the first variable is

$$(P_N f)(x, y) = N \int_{J_r} f(t, y) dt \quad (x \in J_r),$$

the conditional expectation with respect to $\sigma(J_1, \dots, J_N) \otimes \mathcal{B}(I^{d-1})$; in particular $\|P_N f\|_1 \leq \|f\|_1$.

Baire category. As in [Dur15, p. 1126], a subset of a metric space is *nowhere dense* if its closure has empty interior, *meager* (of first category) if it is a countable union of nowhere dense sets, *of second category* if it is not meager, and *co-meager* if its complement is meager. In a nonempty complete metric space a dense G_δ set is co-meager (its complement is a countable union of closed sets with empty interior), and by the Baire category theorem (see, e.g., [Oxt80]) a co-meager set is of second category.

3 Purely singular copulas form a dense G_δ set

Lemma 3.1 (D_1 dominates d_∞). *For all $A, B \in \mathcal{C}_d$,*

$$d_\infty(A, B)^d \leq 2(4(d-1))^{d-1} D_1(A, B).$$

Hence every d_∞ -open subset of $\mathcal{C}_d$ is D_1 -open.

Proof. Let $\delta = d_\infty(A, B) = |A(x_0, y_0) - B(x_0, y_0)|$, and assume $\delta > 0$. Put $s = \delta/(4(d-1))$, so $s \leq \frac{1}{4}$, and $R = \{y \in I^{d-1} : |y_j - y_{0,j}| \leq s \text{ for all } j\}$. For $y \in R$ we have $\sum_j |y_j - y_{0,j}| \leq \delta/4$, so by Lemma 2.1(b), applied to A and to B , $|A(x_0, y) - B(x_0, y)| \geq \delta/2$. By Lemma 2.1(c), $|A(x_0, y) - B(x_0, y)| = |\int_0^{x_0} (H_A - H_B)(t, y) dt| \leq \int_I |H_A - H_B|(t, y) dt$. Each side of R has length at least s , so $\lambda^{d-1}(R) \geq s^{d-1}$ and $D_1(A, B) \geq \frac{\delta}{2} s^{d-1} = \delta^d / (2(4(d-1))^{d-1})$. $\square$

A *rational box* is a set $(a_1, b_1) \times \cdots \times (a_d, b_d) \cap I^d$ with rational $a_j < b_j$ that meets I^d ; it is relatively open in I^d and a box in the sense of Section 2. For $k \geq 1$ let $\mathcal{V}_k$ be the countable family of finite unions V of rational boxes with $\lambda^d(V) < \frac{1}{k}$, and

$$\mathcal{G}_k = \{A \in \mathcal{C}_d : \mu_A(V) > 1 - \frac{1}{k} \text{ for some } V \in \mathcal{V}_k\}.$$

Lemma 3.2 ($\mathcal{S}_d$ is a G_δ set). *Every $\mathcal{G}_k$ is d_∞ -open, hence D_1 -open, and $\mathcal{S}_d = \bigcap_{k \geq 1} \mathcal{G}_k$.*

Proof. Let $V = R_1 \cup \cdots \cup R_m \in \mathcal{V}_k$. By inclusion-exclusion, $\mu_A(V) = \sum_{\emptyset \neq J \subseteq \{1, \dots, m\}} (-1)^{|J|+1} \mu_A(R_J)$ with $R_J = \bigcap_{j \in J} R_j$, a box or empty; by Lemma 2.1(a) each $\mu_A(R_J)$ is a signed sum of 2^d values of A . Hence $|\mu_A(V) - \mu_B(V)| \leq (2^m - 1)2^d d_\infty(A, B)$, the set $\{A : \mu_A(V) > 1 - \frac{1}{k}\}$ is d_∞ -open, and so is its union $\mathcal{G}_k$ over $V \in \mathcal{V}_k$. By Lemma 3.1, $\mathcal{G}_k$ is D_1 -open.

Let $A \in \mathcal{S}_d$, so $\mu_A(Z) = 1$ for a Borel set Z with $\lambda^d(Z) = 0$. By outer regularity there is an open set $O \subseteq \mathbb{R}^d$ with $Z \subseteq O$ and $\lambda^d(O) < \frac{1}{k}$. Every point of $O \cap I^d$ lies in a rational box contained in $O \cap I^d$; enumerate these boxes as $R_1, R_2, \dots$ and put $V_m = R_1 \cup \cdots \cup R_m$. Then $\mu_A(V_m) \uparrow \mu_A(O \cap I^d) = 1$ and $\lambda^d(V_m) \leq \lambda^d(O) < \frac{1}{k}$, so $\mu_A(V_m) > 1 - \frac{1}{k}$ for some m , and $A \in \mathcal{G}_k$.

Conversely let $A \in \bigcap_k \mathcal{G}_k$. Write $\mu_A = h \lambda^d + \mu_A^s$ and $c = \int h d\lambda^d = \mu_A^{ac}(I^d)$. For each k choose $V_k \in \mathcal{V}_k$ with $\mu_A(V_k) > 1 - \frac{1}{k}$. Then $\int_{V_k} h d\lambda^d = \mu_A(V_k) - \mu_A^s(V_k) > 1 - \frac{1}{k} - (1 - c) = c - \frac{1}{k}$. Since $\lambda^d(V_k) \rightarrow 0$ and h is integrable, $\int_{V_k} h \rightarrow 0$. Hence $c = 0$ and $A \in \mathcal{S}_d$. $\square$

The diagonal checkerboard. Fix $A \in \mathcal{C}_d$ and $N \geq 1$. For each cell Q_i let σ_i be the uniform distribution on its main diagonal, that is, the law of $(\frac{i-1+T}{N}, \dots, \frac{i-1+T}{N})$ with T uniform on $[0, 1)$. Define the probability measure

$$\mu_{B_N} = \sum_i \mu_A(Q_i) \sigma_i, \quad B_N(u) = B_N(A)(u) = \mu_{B_N}([0, u]).$$

Lemma 3.3 (density). (a) $B_N(A) \in \mathcal{C}_d$.

(b) $B_N(A) \in \mathcal{S}_d$.

$$(c) \ D_1(A, B_N(A)) \leq \|H_A - P_N H_A\|_1 + \frac{d-1}{N}.$$

$$(d) \ D_1(A, B_N(A)) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Proof. (a) The j -th marginal of σ_i is the uniform distribution $N\lambda|_{J_{ij}}$. By Lemma 2.1(a) the j -th marginal of μ_{B_N} is $\sum_r (\sum_{i: i_j=r} \mu_A(Q_i)) N\lambda|_{J_r} = \sum_r \lambda|_{J_r} = \lambda$.

(b) μ_{B_N} is concentrated on the union of N^d segments, a compact λ^d -null set, since $d \geq 2$.

(c) *The kernel of B_N .* For x in the interior of J_r put $t = Nx - (r-1) \in (0, 1)$ and $p_i(x) = (\frac{i_2-1+t}{N}, \dots, \frac{i_d-1+t}{N})$, a point of the closure $\overline{Q'_i}$. Let

$$K(x, \cdot) = \sum_{i: i_1=r} w_i \delta_{p_i(x)}, \quad w_i = N\mu_A(Q_i),$$

and define $K(x, \cdot)$ arbitrarily at the points $x = \frac{r}{N}$. By Lemma 2.1(a), $\sum_{i: i_1=r} w_i = 1$. For a Borel set $G \subseteq I^d$ the substitution $x = \frac{r-1+t}{N}$ gives

$$\int_I K(x, G_x) dx = \sum_i \mu_A(Q_i) \int_0^1 \mathbf{1}_G\left(\frac{i_1-1+t}{N}, \dots, \frac{i_d-1+t}{N}\right) dt = \sum_i \mu_A(Q_i) \sigma_i(G) = \mu_{B_N}(G),$$

so K is a version of K_{B_N} , and $H_{B_N}(x, y) = \sum_{i: i_1=r} w_i \mathbf{1}_{[p_i(x) \leq y]}$.

Comparison with $P_N H_A$. For x in the interior of J_r , the disintegration identity with $G = J_r \times [0, y]$ gives

$$P_N H_A(x, y) = N \int_{J_r} K_A(t, [0, y]) dt = N\mu_A(J_r \times [0, y]) = \sum_{i: i_1=r} \rho_i([0, y]),$$

where $\rho_i(E) = N\mu_A(J_r \times (E \cap Q'_i))$ is a measure of mass w_i carried by $\overline{Q'_i}$. Write $\overline{Q'_i} = \prod_{j=2}^d [\alpha_j, \alpha_j + \frac{1}{N}]$ and $E_i = \bigcup_{j=2}^d \{y : \alpha_j \leq y_j < \alpha_j + \frac{1}{N}\}$. We claim

$$|\rho_i([0, y]) - w_i \delta_{p_i(x)}([0, y])| \leq w_i \mathbf{1}_{E_i}(y).$$

On E_i both terms lie in $[0, w_i]$. If $y \notin E_i$, then for every j either $y_j < \alpha_j$ or $y_j \geq \alpha_j + \frac{1}{N}$. If $y_j < \alpha_j$ for some j , then $[0, y]$ misses $\overline{Q'_i}$ and both terms vanish; otherwise $\overline{Q'_i} \subseteq [0, y]$ and both terms equal w_i . Since $\lambda^{d-1}(E_i) \leq \frac{d-1}{N}$ and $\sum_{i: i_1=r} w_i = 1$,

$$\int_{I^{d-1}} |P_N H_A(x, y) - H_{B_N}(x, y)| dy \leq \frac{d-1}{N} \quad (2)$$

for almost every x , hence $\|P_N H_A - H_{B_N}\|_1 \leq \frac{d-1}{N}$, and (c) follows from the triangle inequality.

(d) P_N is a contraction of $L^1(\lambda^d)$. For continuous g on I^d , $|P_N g - g| \leq \omega_g(\frac{1}{N})$, where ω_g is the modulus of continuity, so $\|f - P_N f\|_1 \leq 2\|f - g\|_1 + \omega_g(\frac{1}{N})$ for $f \in L^1(\lambda^d)$. Since continuous functions are dense in L^1 , $\|H_A - P_N H_A\|_1 \rightarrow 0$, and (d) follows from (c). $\square$

Proof of Theorem 1.1. By Lemma 3.2, $\mathcal{S}_d = \bigcap_k \mathcal{G}_k$ with D_1 -open sets $\mathcal{G}_k$. By Lemma 3.3, $\mathcal{S}_d$ is D_1 -dense, hence so is every $\mathcal{G}_k \supseteq \mathcal{S}_d$. Thus $\mathcal{S}_d$ is a dense G_δ set. Each $\mathcal{C}_d \setminus \mathcal{G}_k$ is closed with empty interior, so $\mathcal{C}_d \setminus \mathcal{S}_d = \bigcup_k (\mathcal{C}_d \setminus \mathcal{G}_k)$ is meager. Since $(\mathcal{C}_d, D_1)$ is a nonempty complete metric space (Appendix A), $\mathcal{C}_d$ is not meager by the Baire category theorem, so $\mathcal{S}_d$ is not meager either. $\square$

Remark 3.4 (the uniform metric). The same lemmas give the result for d_∞ : the sets $\mathcal{G}_k$ are d_∞ -open, $d_\infty(A, B_N(A)) \rightarrow 0$ by Lemma 3.1 and Lemma 3.3(d), and $(\mathcal{C}_d, d_\infty)$ is compact, hence complete. So $\mathcal{S}_d$ is a dense G_δ set in $(\mathcal{C}_d, d_\infty)$ for every $d \geq 2$. This is [Dur15, Theorem 3, second item], proved for $d = 2$ in [DFST15, Theorem 4.1, Corollary 4.1] by a different G_δ set.

Remark 3.5 ($d = 2$: an exact identity). For $d = 2$ the cell estimate (2) can be computed exactly: for every $A \in \mathcal{C}_2$ and $N \geq 1$,

$$\|P_N H_A - H_{B_N}\|_1 = \frac{1}{3N}, \quad \text{hence} \quad \left| D_1(A, B_N(A)) - \frac{1}{3N} \right| \leq \|H_A - P_N H_A\|_1. \quad (3)$$

Indeed, let $x \in J_r$, $t = Nx - (r - 1)$, and $y = \frac{s-1+v}{N} \in J_s$ with $v \in [0, 1)$. The cells (r, s') with $s' \neq s$ contribute equally to $P_N H_A(x, y)$ and to $H_{B_N}(x, y)$, so the difference is $w_{rs}(G_{rs}(v) - \mathbf{1}[t \leq v])$, where G_{rs} is the distribution function of $\rho_{(r,s)}/w_{rs}$, rescaled to $[0, 1)$. With $U_{rs} \sim G_{rs}$, integration over v and then over t gives the contribution

$$\frac{w_{rs}}{N^2} \int_0^1 \left(\int_0^t G_{rs} + \int_t^1 (1 - G_{rs}) \right) dt = \frac{w_{rs}}{N^2} \left(\frac{1}{2} - \mathbb{E}[U_{rs}(1 - U_{rs})] \right).$$

Summing over r and s and using $\sum_{r,s} w_{rs} = N$ and $\sum_r \rho_{(r,s)} = N\lambda|_{J_s}$ (uniform second marginal), so that $\sum_r w_{rs} \mathbb{E}[U_{rs}(1 - U_{rs})] = \int_0^1 v(1 - v) dv = \frac{1}{6}$, we obtain $\frac{1}{N^2}(\frac{N}{2} - \frac{N}{6}) = \frac{1}{3N}$. In particular $D_1(\Pi, B_N(\Pi)) = \frac{1}{3N}$, since $H_\Pi(x, y) = y$ does not depend on x . The copula $B_N(\Pi)$ is the uniform mixture of the N copulas of the circular shifts $x \mapsto x + \frac{c}{N} \bmod 1$, $c = 0, \dots, N - 1$; its kernel has N atoms. For every A , $B_1(A) = M$, the upper Fréchet bound, and $D_1(\Pi, M) = \frac{1}{3}$ is the largest value of $D_1(\cdot, \Pi)$ [Tru11]. For $N \geq 2$, $B_N(\Pi)$ is not completely dependent, in accordance with the closedness of the completely dependent copulas in $(\mathcal{C}_2, D_1)$ [Dur15, Theorem 2]. These statements are checked in exact arithmetic in Section 6.

Remark 3.6 (the case $d = 2$ in the literature). (a) *Operators*. A copula $A \in \mathcal{C}_2$ defines the doubly stochastic operator $T_A f(x) = \int f(y) K_A(x, dy)$, and $T_A \mathbf{1}_{[0,y]} = H_A(\cdot, y)$. If $T_{A_n} \rightarrow T_A$ in the strong operator topology of $L^1(\lambda)$ or $L^2(\lambda)$, then $\int_I |H_{A_n} - H_A|(x, y) dx \rightarrow 0$ for every y , and $D_1(A_n, A) \rightarrow 0$ by dominated convergence. Operators induced by invertible measure-preserving maps correspond to mutually completely dependent copulas, whose measures are carried by graphs and are therefore singular, and so are finite mixtures of them. Hence the strong density of such mixtures [Bro66, Theorem 2], [Kim68, Theorem 2.2], [Iwa80, Corollary 1] gives the D_1 -density of $\mathcal{S}_2$.

(b) *Kim's approximants are diagonal checkerboards*. In [Kim68, Lemma 2.2] the doubly stochastic matrix $(N\mu_A(Q_{rs}))_{r,s}$, $N = 2^n$, is written by Birkhoff's theorem as a convex combination $\sum_k c_k P_{\pi_k}$ of permutation matrices, and each P_{π_k} is realised by an invertible measure-preserving map that sends J_r onto $J_{\pi_k(r)}$; the resulting operator has the same cell masses as T_A . If these maps are chosen as translations, the copula of the map for π_k is uniform on the diagonals of the cells $(r, \pi_k(r))$, each of mass $\frac{1}{N}$. The mixture is then a copula that is uniform on the diagonal of every cell and has the cell masses of A , that is, it is $B_N(A)$.

(c) *Copulas*. [KFT21b, Theorem 3.2] shows that the dyadic checkerboard approximations of A converge weakly conditionally to A , and weak conditional convergence implies D_1 -convergence (see [KFT21b]). The proof uses only the conditional distribution functions at the dyadic points y of level at most n , which are the same for the checkerboard approximation and for $B_{2^n}(A)$, because the atom of $K_{B_{2^n}(A)}(x, \cdot)$ in a cell lies in the interior of the cell. [DT25] defines checkerboard approximations with an arbitrary copula B in every cell and, in the proof of its Theorem 3.3, applies the theorem to them for B in a certain class $\mathcal{C}_\mathbb{Q}$; the argument just given covers $B = M$.

(d) *The G_δ set of [DFST15]*. Let $C_1, C_2, \dots$ be the rational check-min copulas of [DFST15], that is, the copulas that are uniform on the diagonal of every cell of some N -grid and have rational cell masses, and let $S_{t,n}$ be the d_∞ -balls of [DFST15, Theorem 4.1], centred at C_t . The second part of that theorem shows that $S = \bigcap_n \bigcup_t S_{t,n}$ contains only purely singular copulas; this inclusion does not depend on the metric under consideration. Each $S_{t,n}$ is D_1 -open by Lemma 3.1. The rational check-min copulas are D_1 -dense: by Lemma 3.3, $B_N(A)$ is D_1 -close to A , and replacing its cell masses q by rational ones q' with the same uniform margins changes D_1 by at most $\sum_i |q_i - q'_i|$. Such q' exist:

(Nq_{rs}) is a doubly stochastic matrix, by Birkhoff's theorem a convex combination of permutation matrices, and the coefficients can be replaced by nearby rational ones. Hence S is a dense G_δ set in $(\mathcal{C}_2, D_1)$, which gives Theorem 1.1 for $d = 2$ once more. The D_1 -density of check-min copulas also follows from facts quoted in [DFST15]: the D_1 -density of rational checkerboard copulas [Tru11], quoted before [DFST15, Theorem 4.1], and, in the proof of [DFST15, Theorem 3.2], check-min copulas D_1 -close to Π [MT10, Theorem 6] and the inequality $D_1(V_\tau(\Pi), V_\tau(A)) \leq D_1(\Pi, A)$ [Tru11] for the checkerboard operators V_τ . Indeed, right after quoting the D_1 -density of the rational checkerboard copulas, [DFST15] states that the same holds for the rational check-min copulas.

Remark 3.7 (other conditioning coordinates, D_p and D_∞). (a) Let $D_1^{(j)}$ be defined like D_1 with the j -th coordinate in the role of x ; [MFT21] uses $j = d$. Lemma 3.1 holds for $D_1^{(j)}$ with the same proof, and Lemma 3.2 does not involve the metric. The construction $A \mapsto B_N(A)$ commutes with permutations of the coordinates (the diagonal of a cell is mapped onto the diagonal of the permuted cell), and $D_1^{(j)}(A, B) = D_1(A^\pi, B^\pi)$, where $A^\pi(u) = A(\pi u)$ and π is the transposition of the coordinates 1 and j ; so Lemma 3.3 holds for $D_1^{(j)}$, as does Appendix A. Hence Theorem 1.1 holds for every $D_1^{(j)}$. It also holds for $\sum_j D_1^{(j)}$ and $\max_j D_1^{(j)}$: a Cauchy sequence for these metrics is Cauchy for each $D_1^{(j)}$, its $D_1^{(j)}$ -limits all coincide with its d_∞ -limit by Lemma 3.1, the sets $\mathcal{G}_k$ are open, and $D_1^{(j)}(A, B_N(A)) \rightarrow 0$ for each j .

(b) For $1 \leq p < \infty$ let $D_p(A, B) = \|H_A - H_B\|_{L^p(\lambda^d)}$. Since $0 \leq H \leq 1$, $D_1 \leq D_p$ and $D_p^p \leq D_1$; so D_p and D_1 have the same convergent and the same Cauchy sequences, and Theorem 1.1 holds for D_p .

(c) Let $D_\infty(A, B) = \sup_{y \in I^{d-1}} \int_I |H_A - H_B|(x, y) dx$, as in [MFT21]; the integral is well defined for every y by Lemma 2.1(c). Since $D_1 \leq D_\infty$, the sets $\mathcal{G}_k$ are D_∞ -open. For fixed y , the estimate in the proof of Lemma 3.3(c) gives $\int_I |P_N H_A - H_{B_N}|(x, y) dx \leq \sum_i \mu_A(Q_i) \mathbf{1}_{E_i}(y) \leq \sum_{j=2}^d \mu_A(\{u_j \in J_{r_j(y)}\}) = \frac{d-1}{N}$, where $J_{r_j(y)}$ is the interval containing y_j (for $y_j = 1$ the indicator vanishes). The map $y \mapsto f_y = H_A(\cdot, y) \in L^1(I)$ is 1-Lipschitz for the ℓ^1 -norm, because $|f_y - f_{y'}|(x) \leq K_A(x, [0, y \vee y'] \setminus [0, y \wedge y'])$ and the margins are uniform. So $\{f_y\}$ is compact in $L^1(I)$, the averaging operators converge to the identity uniformly on it, and $D_\infty(A, B_N(A)) \rightarrow 0$. Completeness: a D_∞ -Cauchy sequence (A_n) is D_1 -Cauchy and D_1 -converges to some A ; if $\sup_y \|f_y^n - f_y^m\|_1 < \varepsilon$ for $n, m \geq n_0$, then along a subsequence $f_y^m \rightarrow f_y$ in $L^1(I)$ for almost every y , so $\|f_y^n - f_y\|_1 \leq \varepsilon$ for almost every y , and for every y by the Lipschitz continuity in y . Hence Theorem 1.1 holds for D_∞ .

Remark 3.8 (∂ -convergence). Mikusiński and Taylor [MT09, Definition 3] call a sequence (C_k) of d -copulas ∂ -convergent to C if for every coordinate j and every value of the other coordinates, $\int_I |\partial_j C_k - \partial_j C| dt \rightarrow 0$, where t is the j -th coordinate. For every value y of the other coordinates, $\partial_j C(\cdot, y)$ is a version of the conditional distribution function $H_C^{(j)}(\cdot, y)$ with respect to the j -th coordinate [Dur15, p. 1125]. So ∂ -convergence means that $g_k^{(j)}(y) = \int_I |H_{C_k}^{(j)} - H_C^{(j)}|(t, y) dt \rightarrow 0$ for all j and y . The functions $g_k^{(j)}$ are 2-Lipschitz for the ℓ^1 -norm (Remark 3.7(c)), so pointwise convergence to 0 is the same as uniform convergence and, by the argument of Lemma 3.1, the same as $\int g_k^{(j)} dy = D_1^{(j)}(C_k, C) \rightarrow 0$. Hence ∂ -convergence is convergence in $D_1^{(j)}$ for every j , that is, in $\sum_j D_1^{(j)}$; for $d = 2$ this is the description in [KFT21b, Section 2]. [MT09, Theorem 11] quotes from [MT10] that the checkerboard and the Bernstein approximations of a d -copula ∂ -converge to it, and, for bivariate copulas, [AW26, Section 2.3] and [AF26, Remark 3.7], citing [MT10], state that the check-min approximations converge in D_1 and in its analogue for the second coordinate. If [MT10] proves the ∂ -convergence of the $B_N(A)$ in every dimension, then Lemma 3.3(d), for every $D_1^{(j)}$, is in [MT10]. We could not check this.

4 Conditional distributions of a typical copula

Let $\mathcal{T}_d$ be the set of $A \in \mathcal{C}_d$ such that $K_A(x, \cdot)$ has no atoms for λ -almost every x . Theorem 1.2 states that $\mathcal{S}_d \cap \mathcal{T}_d$ is a dense G_δ set; by the next lemma, $\mathcal{S}_d \cap \mathcal{T}_d$ is exactly the set described in Theorem 1.2.

Lemma 4.1 (Lebesgue decomposition and disintegration). *Let $\mu_A = h \lambda^d + \mu_A^s$ with μ_A^s concentrated on a λ^d -null Borel set Z . For λ -almost every x , the absolutely continuous part of $K_A(x, \cdot)$ with respect to λ^{d-1} is $h(x, \cdot) \lambda^{d-1}$, and its singular part is $K_A(x, \cdot \cap Z_x)$. In particular $A \in \mathcal{S}_d$ if and only if $K_A(x, \cdot)$ is singular for almost every x .*

Proof. Let $F \subseteq I^{d-1}$ and $E \subseteq I$ be Borel and $G = (E \times F) \setminus Z$. Computing $\mu_A(G)$ with the density and by disintegration gives $\int_E \int_{F \setminus Z_x} h(x, y) dy dx = \int_E K_A(x, F \setminus Z_x) dx$. As E is arbitrary, $K_A(x, F \setminus Z_x) = \int_{F \setminus Z_x} h(x, y) dy$ for almost every x . Taking F in a countable π -system that generates $\mathcal{B}(I^{d-1})$ and contains I^{d-1} , we get, for almost every x , equality of the two finite measures $F \mapsto K_A(x, F \setminus Z_x)$ and $F \mapsto \int_{F \setminus Z_x} h(x, \cdot)$. By Fubini's theorem, Z_x is λ^{d-1} -null for almost every x . So $K_A(x, \cdot) = h(x, \cdot) \lambda^{d-1} + K_A(x, \cdot \cap Z_x)$ with an absolutely continuous and a singular term, and uniqueness of the Lebesgue decomposition gives the claim. Integrating the absolutely continuous parts over x gives $\int h d\lambda^d = \mu_A^{\text{ac}}(I^d)$, which proves the last statement. $\square$

Lemma 4.2 (subsequences and kernels). *If $D_1(A_n, A) \rightarrow 0$, there are a subsequence (n_k) and a λ -null set E such that $K_{A_{n_k}}(x, \cdot) \rightarrow K_A(x, \cdot)$ weakly for every $x \notin E$.*

Proof. Let $g_n(x) = \int_{I^{d-1}} |H_{A_n}(x, y) - H_A(x, y)| dy$. Since $\int g_n = D_1(A_n, A) \rightarrow 0$, a subsequence satisfies $g_{n_k} \rightarrow 0$ outside a null set E_1 . Let $Z = \bigcup_j \{y : y_j = 1\}$, the union of the upper faces of I^{d-1} , and $E_2 = \{x : K_A(x, Z) > 0\}$; since $\int K_A(x, \{y_j = 1\}) dx = \mu_A(\{u_{j+1} = 1\}) = 0$, E_2 is null. Fix $x \notin E_1 \cup E_2$ and write $\nu_k = K_{A_{n_k}}(x, \cdot)$, $\nu = K_A(x, \cdot)$, with distribution functions F_k, F ; thus $F_k \rightarrow F$ in $L^1(\lambda^{d-1})$. As I^{d-1} is compact, it suffices to show that every weak limit ν' of a subsequence of (ν_k) equals ν . Let W_j be the countable set of $t \in (0, 1)$ with $\nu'(\{y_j = t\}) > 0$ or $\nu(\{y_j = t\}) > 0$, and $\Gamma = \prod_j ((0, 1) \setminus W_j)$. For $y \in \Gamma$ the relative boundary of $[0, y]$ lies in $\bigcup_j \{y' : y'_j = y_j\}$, which is ν - and ν' -null. Hence $F_k(y) \rightarrow F'(y)$ along the subsequence, by the portmanteau theorem, and F and F' are continuous at y . Along a further subsequence $F_k \rightarrow F$ almost everywhere, so $F' = F$ almost everywhere; by continuity at the points of Γ and density of $\{F = F'\}$, we get $F' = F$ on Γ . By inclusion-exclusion, ν and ν' agree on the boxes $\prod_j L_j$ with $L_j = [0, b_j]$ or $L_j = (a_j, b_j]$, where $a_j < b_j$ lie in $(0, 1) \setminus W_j$. These boxes form a π -system generating the Borel sets of $[0, 1)^{d-1}$. Moreover $\nu([0, 1)^{d-1}) = \lim F(b, \dots, b) = \lim F'(b, \dots, b) = \nu'([0, 1)^{d-1})$ as $b \uparrow 1$ through $\bigcap_j ((0, 1) \setminus W_j)$. By Dynkin's theorem $\nu = \nu'$ on $[0, 1)^{d-1}$. Since $x \notin E_2$, $\nu([0, 1)^{d-1}) = 1$; hence $\nu'([0, 1)^{d-1}) = 1$, $\nu'(Z) = 0$ and $\nu' = \nu$. $\square$

The face condition is needed when $d - 1 \geq 2$: the point masses at $(1, 0)$ and at $(1, \frac{1}{2})$ have distribution functions that agree on $[0, 1)^2$. Lemma 4.2 claims convergence only along a subsequence; D_1 -convergence does not imply weak convergence of the kernels along the whole sequence [KFT21b, Example 3.3], [MFT21, Example 2.1].

Lemma 4.3 (the largest atom). *For a probability measure ν on I^{d-1} let $a(\nu) = \sup_z \nu(\{z\})$.*

- (a) *Let $\mathcal{D}_m$ be the partition of I^{d-1} into dyadic cubes of side 2^{-m} , half-open as the intervals J_r . Then $a(\nu) = \lim_m \max_{c \in \mathcal{D}_m} \nu(c)$; in particular $x \mapsto a(K_A(x, \cdot))$ is Borel.*
- (b) *If $\nu_k \rightarrow \nu$ weakly, then $\limsup_k a(\nu_k) \leq a(\nu)$.*

Proof. (a) $M_m = \max_{c \in \mathcal{D}_m} \nu(c)$ is nonincreasing in m and at least $a(\nu)$. If $\lim M_m = c > a(\nu)$, choose $c_m \in \mathcal{D}_m$ with $\nu(c_m) \geq c$ and $z_m \in c_m$, and let z be a limit point of (z_m) . For every $r > 0$ the closed ball $\bar{B}(z, r)$ contains c_m for infinitely many m , so $\nu(\bar{B}(z, r)) \geq c$, and $\nu(\{z\}) \geq c > a(\nu)$, a contradiction. (b) Take a subsequence with $a(\nu_k) \rightarrow L = \limsup_k a(\nu_k)$ and points z_k with $\nu_k(\{z_k\}) \geq a(\nu_k) - \frac{1}{k}$ converging to some z . For $r > 0$ the portmanteau theorem gives $\nu(\bar{B}(z, r)) \geq \limsup_k \nu_k(\bar{B}(z, r)) \geq L$; letting $r \rightarrow 0$, $a(\nu) \geq \nu(\{z\}) \geq L$. $\square$

Lemma 4.4. *The functional $\Psi(A) = \int_I a(K_A(x, \cdot)) dx$ is upper semicontinuous on $(\mathcal{C}_d, D_1)$. Hence $\mathcal{T}_d = \{\Psi = 0\} = \bigcap_k \{\Psi < \frac{1}{k}\}$ is a G_δ set in $(\mathcal{C}_d, D_1)$.*

Proof. Ψ is well defined by Lemma 2.1(c) and Lemma 4.3(a). Let $D_1(A_n, A) \rightarrow 0$ and suppose that $\Psi(A_{n_k}) \rightarrow L > \Psi(A)$ along a subsequence. By Lemma 4.2, along a further subsequence the kernels converge weakly for almost every x , and by Lemma 4.3(b) and the reverse Fatou lemma (the integrands lie in $[0, 1]$), $L \leq \int_I \limsup_k a(K_{A_{n_k}}(x, \cdot)) dx \leq \Psi(A)$, a contradiction. $\square$

Checkerboards with Cantor noise. Let κ be the Cantor distribution, the law of $\sum_m 2\epsilon_m 3^{-m}$ with independent fair bits ϵ_m ; it has no atoms and is carried by the Cantor set K , $\lambda(K) = 0$. Let $\{s\}$ denote the fractional part of s and $\alpha_i = (\frac{i_1-1}{N}, \dots, \frac{i_d-1}{N})$ the lower corner of Q_i . Let τ_i be the law of $\alpha_i + \frac{1}{N}(T, \{T + W_2\}, \dots, \{T + W_d\})$, where T is uniform on $[0, 1]$, $W_2, \dots, W_d$ have law κ , and all are independent. Put $\mu_{B'_N} = \sum_i \mu_A(Q_i) \tau_i$ and let $B'_N = B'_N(A)$ be its distribution function.

Lemma 4.5. *$B'_N(A) \in \mathcal{S}_d \cap \mathcal{T}_d$, and $D_1(A, B'_N(A)) \leq \|H_A - P_N H_A\|_1 + \frac{d-1}{N} \rightarrow 0$.*

Proof. For fixed w , $\{T + w\}$ is uniform on $[0, 1]$, so every coordinate of τ_i is uniform on the corresponding interval, and $B'_N \in \mathcal{C}_d$ as in Lemma 3.3(a). For x in the interior of J_r , with $t = Nx - (r - 1)$, let

$$K'(x, \cdot) = \sum_{i: i_1=r} w_i \nu_{i,t}, \quad w_i = N\mu_A(Q_i),$$

where $\nu_{i,t}$ is the law of $\alpha'_i + \frac{1}{N}(\{t + W_2\}, \dots, \{t + W_d\})$ and α'_i is the lower corner of Q'_i . The disintegration identity for K' follows as in the proof of Lemma 3.3(c), with Fubini's theorem over (T, W) . Each $\nu_{i,t}$ is a product of $d - 1$ laws of the form $\{t + W_j\}$ (rescaled), each atomless and carried by the null set $(t + K) \bmod 1$. A product of atomless measures is atomless, a product of measures carried by null sets is carried by a λ^{d-1} -null set, and finite mixtures preserve both properties. So $K'(x, \cdot)$ is atomless and singular for every x off the grid; hence $B'_N \in \mathcal{T}_d$, and $B'_N \in \mathcal{S}_d$ by Lemma 4.1. Every $\nu_{i,t}$ is carried by $\bar{Q}'_i$, so the cell estimate in the proof of Lemma 3.3(c) applies verbatim, and the bound tends to 0 by Lemma 3.3(d). $\square$

Proof of Theorem 1.2 and Corollary 1.3. By Lemmas 3.2 and 4.4, $\mathcal{S}_d \cap \mathcal{T}_d$ is a G_δ set in $(\mathcal{C}_d, D_1)$, and by Lemma 4.5 it is dense; by Lemma 4.1 it is the set described in Theorem 1.2. Let $\mathcal{F}_d$ be the set of copulas with $\text{supp } \mu_A = I^d$. Then $\mathcal{F}_d = \bigcap_R \{A : \mu_A(R) > 0\}$ over the countably many rational boxes R , and each of these sets is d_∞ -open by Lemma 2.1(a), hence D_1 -open. $\mathcal{F}_d$ is D_1 -dense, since $(1 - \varepsilon)A + \varepsilon\Pi_d \in \mathcal{F}_d$ and, by Lemma 2.1(d), $D_1(A, (1 - \varepsilon)A + \varepsilon\Pi_d) = \varepsilon D_1(A, \Pi_d) \leq \varepsilon$, where Π_d is the independence copula. Hence $\mathcal{S}_d \cap \mathcal{T}_d \cap \mathcal{F}_d$ is a dense G_δ set, being a countable intersection of dense open sets in a complete metric space. Completely dependent copulas have point masses as conditional distributions and lie outside $\mathcal{T}_d$; copulas with a null support lie outside $\mathcal{F}_d$. Both classes are therefore contained in meager sets. $\square$

For d_∞ the analogue of $\mathcal{F}_d$ is co-meager by the same argument [DT25, Lemma 3.9], while $\mathcal{T}_2$ is meager in $(\mathcal{C}_2, d_\infty)$, because it misses the co-meager set of completely dependent copulas [Kim68, Theorem 1.1], [DT25, Theorem 3.8].

5 Other readings of the question

The source fixes the conventions: copulas on I^d with the Lebesgue decomposition of μ_C [Dur15, p. 1124], the Markov kernel with respect to the first coordinate [Dur15, p. 1125], “the D_1 metric in” [Tru11], and Baire category notions as in Section 2 [Dur15, p. 1126]. Under these conventions Theorem 1.1 answers the question. We list the other readings we found and what happens under each.

(R1) *The L^1 distance of the copula functions.* In version 1.7.0 of the UnsolvedMath collection [ula26] the record of the question restates it with the metric $L(C_1, C_2) = \int_{I^d} |C_1(u) - C_2(u)| du$ and asks whether the purely singular copulas are nonmeager. This is not the metric of the source, and the question becomes the known one for d_∞ :

Proposition 5.1. *For all $C_1, C_2 \in \mathcal{C}_d$,*

$$L(C_1, C_2) \leq d_\infty(C_1, C_2) \quad \text{and} \quad d_\infty(C_1, C_2)^{d+1} \leq 2(4d)^d L(C_1, C_2).$$

Hence L and d_∞ have the same convergent and the same Cauchy sequences, $(\mathcal{C}_d, L)$ is complete, the Baire category notions for L and for d_∞ coincide, and $\mathcal{S}_d$ is a dense G_δ set in $(\mathcal{C}_d, L)$ by Remark 3.4.

Proof. The first inequality is clear. Let $\delta = |C_1 - C_2|(u_0) = d_\infty(C_1, C_2) > 0$. By Lemma 2.1(b), $|C_1 - C_2| \geq \delta/2$ on the set of $u \in I^d$ with $|u_j - u_{0,j}| \leq \delta/(4d)$ for all j , whose volume is at least $(\delta/(4d))^d$. So $L(C_1, C_2) \geq \frac{\delta}{2}(\delta/(4d))^d$. $\square$

(R2) *Copulas with a singular component.* On p. 1124 the source speaks of “singular copulas (or, copulas with a singular component)”. If the question is read for the class of copulas with $\mu_A^s \neq 0$, it is trivial: this class is the complement of the absolutely continuous copulas, which are of first category in $(\mathcal{C}_d, D_1)$ by [Dur15, Theorem 3, first item] (for $d = 2$, [DFST15, Corollary 3.1]), and it also contains $\mathcal{S}_d$. The word “purely” in the question excludes this reading.

(R3) *Copulas with a null support.* If “purely singular” were read as “ $\text{supp } \mu_A$ is a λ^d -null set”, the answer would be no: by Corollary 1.3 these copulas form a meager set in $(\mathcal{C}_d, D_1)$, and by the same argument also in $(\mathcal{C}_d, d_\infty)$. This reading contradicts [Dur15, Theorem 3, second item] and is therefore not the source’s.

(R4) *Singular conditional distributions.* The source decomposes the kernel of $C \in \mathcal{C}_2$ as $K_C^a + K_C^s + K_C^d$ (absolutely continuous, singular without atoms, discrete) [Dur15, p. 1125], and [DT25] calls a bivariate copula singular if its mass is carried by the parts K_C^s . If “purely singular” is read as “ $K_C^a(x, \cdot) = K_C^d(x, \cdot) = 0$ for almost every x ”, the answer is yes by Theorem 1.2. This reading is not the source’s either: under it the second item of [Dur15, Theorem 3] would be false for $d = 2$, since this class is meager in $(\mathcal{C}_2, d_\infty)$ (end of Section 4).

(R5) *Other versions of D_1 .* The answer does not change if the kernel is taken with respect to another coordinate, as in [MFT21], if the d resulting metrics are added or maximised, or if D_1 is replaced by D_p or D_∞ (Remark 3.7). Under every reading consistent with [Dur15, Theorem 3], the answer is yes.

6 Computations and reproducibility

All theorems, lemmas and remarks of this note are proved without computation. The computations check the construction of Lemma 3.3 and the identities of Remark 3.5. The reproducibility package has four parts.

The program of this note (`check_note.py`, standard library, exact rational arithmetic, $d = 2$, about two seconds). For a checkerboard copula A and a fixed column of the N -grid, the integral over

y of $|H_A - H_{B_N}|$ is a piecewise quadratic function of the position of the atom of $K_{B_N}(x, \cdot)$; the program locates its breakpoints and integrates exactly by Simpson's rule. It checks: (A) $D_1(\Pi, B_N(\Pi)) = \frac{1}{3N}$ for $N \leq 12$; (B) for 58 random rational checkerboard copulas A on M -grids, $2 \leq M \leq 8$, with $1 \leq N \leq 10$ (including N not dividing M and $N > M$): the margins of $B_N(A)$ are uniform, $\|P_N H_A - H_{B_N}\|_1 = \frac{1}{3N}$, and $D_1(A, B_N(A))$ lies between $\frac{1}{3N} - \|H_A - P_N H_A\|_1$ and $\|H_A - P_N H_A\|_1 + \frac{1}{3N}$, hence below the bound of Lemma 3.3(c); the largest ratio of $D_1(A, B_N(A))$ to that bound was 0.547; (C) $d_\infty^2 \leq 8D_1$ (Lemma 3.1, $d = 2$) for 90 random pairs of checkerboard copulas (88 of them with $A \neq A'$), with largest ratio $d_\infty^2/D_1 = 0.5$; (D) the kernel of B_N , through $B_N(x, y) = \int_0^x H_{B_N}(t, y) dt$ at 125 rational points; (E) $B_N(\Pi)$ is the uniform mixture of the N circular shifts, $N \leq 8$; (F) two deliberately wrong inputs are rejected.

The programs with which the results were first obtained: an exact computation of $D_1(\Pi, B_N(\Pi))$ for $N \leq 12$, and midpoint-rule estimates of $D_1(A, B_N(A))$ and of the bound of Lemma 3.3(c) for six copulas, FGM($\frac{1}{2}$), $(M + \Pi)/2$ and Clayton(2) with $N \leq 64$ for $d = 2$, and FGM₃($\frac{1}{2}$), $(M_3 + \Pi_3)/2$ and Π_3 with $N \leq 32$ for $d = 3$. In all 39 cases the estimate lies below the bound, and $N D_1(A, B_N(A))$ stays between about 0.15 and 0.32. The values for $d = 3$ are midpoint estimates on a 256^3 grid and are 0.3–1% too high; for example, for $A = \Pi_3$ and $N = 1$ the estimate is 0.31765, while the exact value is $D_1(\Pi_3, M_3) = \frac{19}{60} = 0.31667$.

The programs of three independent verification runs: runs A and B, which examined the first written version (folders `verification_run_A/` and `verification_run_B/`), and a further run on the text of this note (folder `independent_run_2/`), each with its own programs and their recorded outputs; see the next paragraph.

Verification. The results were first obtained, with proofs and with the two programs just described. Two rounds of independent verification, all AI-assisted, followed; each run re-derived the proofs and wrote its own programs. In the first round two runs examined the first written version. Run A re-derived all proofs line by line, including the boundary cases of the cell estimate (grid points, the faces $y_j = 1$, the closed last interval) and the face condition in Lemma 4.2. It found and proved the identity of Remark 3.5. In exact arithmetic it checked $D_1(\Pi, B_N(\Pi)) = \frac{1}{3N}$ for $N \leq 12$, the bound of Lemma 3.3(c) on 36 random rational checkerboard copulas, and Lemma 3.1 for $d = 2$ on 195 pairs; numerically it tested completely dependent copulas (for instance $D_1(W, B_N(W)) = \frac{1}{2N}$), the case $d = 3$, and the approximants of Lemma 4.5 for $d = 2$. It also found the readings (R2) and (R3) of Section 5 and required corrections in the description of the source. Run B wrote an exact engine for D_1 in dimension 2, validated on known values such as $D_1(\Pi, M) = \frac{1}{3}$ and $D_1(M, W) = \frac{1}{2}$, and checked the bound of Lemma 3.3(c) exactly on 84 cases (largest ratio 0.573), together with deliberately wrong inputs. It computed $D_1(A, B_N(A))$ with high accuracy for smooth copulas in dimensions 2 and 3, which showed that the midpoint estimates for $d = 3$ are slightly too high. It found the classical density results of Remark 3.6(a), and it checked in exact arithmetic, on 18 cases, that Kim's approximants built with translations coincide with $B_N(A)$ for $N = 2, 4, 8$. Both runs re-ran the first programs and reproduced their recorded outputs. All corrections required by the two runs were made: the credit for $d = 2$ (Section 1.3 and Remark 3.6), the description of how the source defines D_1 for $d \geq 3$, the readings of Section 5, the wording of Remark 3.7(a), and the range of the constants above. When the note was written, the closest prior source [DFST15] was read in full, and Remark 3.6(d) and Corollary 1.3 were added.

In the second round a further run examined the text of the note line by line, the late additions included. It read the source and [DFST15] again, compared every cited statement with the version of the cited text that it could read ([Bro66, Kim68, Iwa80, KFT21b, DT25, MFT21, FT26, MT09]), re-derived all proofs, and found no gap. It wrote two programs: an exact one for $d = 2$, which refines to a common grid and integrates every cell in closed form, and confirmed $D_1(\Pi, B_N(\Pi)) = \frac{1}{3N}$ for

$N \leq 16$, the identity and the bounds of Remark 3.5 on 64 random rational checkerboard copulas with all divisibility patterns of M and N , the values for M , W and $(M + \Pi)/2$, Lemma 3.1 on 70 pairs and the estimate of Remark 3.6(d) on 50 pairs, together with deliberately wrong inputs; and a floating-point one, which checked the cell estimate (2) at every node and Lemma 3.3(c) for 12 random three-dimensional checkerboard copulas, the value $D_1(\Pi_3, M_3) = \frac{19}{60}$, Lemma 3.1 for $d = 3$ on 32 pairs and Proposition 5.1 for $d = 2, 3$. It re-ran the whole package of the first version from the archive: every recorded output was reproduced, two of them up to a line that records the running time. It examined what [MT10] contains, as far as open sources allow, and required the corrections made in this version: the statement of priority for Theorem 1.1 in every dimension (Section 1.3), Remark 3.8, the credit to [TF14], the version of the corpus record, and the wording of the citations of [DT25] and [Kim68]. The exact program of this note, `check_note.py`, verifies the statements (A)–(F) listed above. No proof depends on a computation.

Scope and priority. The note answers the question of [Dur15, p. 1126] for every $d \geq 2$ (Theorem 1.1, which gives co-meagerness, more than the second category asked for) and adds the statement on conditional distributions (Theorem 1.2, Corollary 1.3). The question, the conventions and the results quoted in Section 1 are due to Durante, Fernández-Sánchez and Trutschnig [Dur15, DFST15], and the metric D_1 to Trutschnig [Tru11] and Fernández-Sánchez and Trutschnig [FT15]. Theorem 1.1 follows in a few lines from known results (Section 1.3): for $d = 2$ from [DFST15, Theorem 4.1] and the D_1 -density of check-min copulas, which is classical [Bro66, Kim68, Iwa80, KFT21b] and is stated and used in [DFST15]; for $d \geq 3$ the density may be in [MT10] (Remark 3.8). We claim no priority for Theorem 1.1. To our knowledge Theorem 1.2 and Corollary 1.3 are not in print; [TF14] constructs copulas with singular atomless conditional distributions. Baire category results in the strong operator topology are known for other properties of doubly stochastic operators, for example ergodicity [Iwa80, Proposition 4], mixing [Iwa92] and zero entropy [FH19], and for subclasses of copulas with respect to D_1 [DFST16, KFT21a]; we found no statement on singular conditional distributions there, but our search of the operator literature was limited.

What was read: [Dur15], pp. 1124–1127, in the publisher’s file; [DFST15] in full, in the publisher’s version; [Kim68, Bro66] in full, and [Iwa80] in full in the scan of the publisher; of [MT09] Sections 1 and 3.4; of [KFT21b] (arXiv version 2) Sections 2 and 3; of [DT25] (arXiv version 1) Sections 2 and 3; of [MFT21] (arXiv version 2) pp. 3–5; of [FT26] Sections 1, 2 and 4.1; of [AW26] Section 2.3 and of [AF26] Section 3.3. Of [Iwa92] only the zbMATH review, and of [KFT21a, FH19] only the abstracts were read. Of [MT10] we read only the abstract (the zbMATH review is not available) and the descriptions in [MT09, DFST15, KFT21b, AW26, AF26]. We could not read [FT15, Tru11, DFST16, TF14]: the publishers’ sites refused automated access or showed no abstract; what we say about [Tru11, FT15] is taken from their restatements in [DFST15, KFT21b, MFT21, FT26], and what we say about [TF14] from [DFST15, Section 5]. We could not read [DFSQMÚF15], which by its abstract proves convergence of patchwork approximations in D_1 . The proofs of the cited results were read only where Remark 3.6 relies on them [Kim68, Lemma 2.2, Theorem 2.2], [KFT21b, Theorem 3.2], [DFST15, Theorem 4.1].

The searches were made on 14 October 2026: queries to arXiv (copulas with Baire, typical, meager, first category or generic; doubly stochastic measures and Markov operators with Baire category; the works of the proposers since 2015), to zbMATH (copulas with Baire category; doubly stochastic operators with residual, category or typical; the record of [MT10]), Crossref, OpenAlex (the 14 works citing [DFST15], the 39 citing [FT15] and the 48 citing [MT10]), Semantic Scholar (citation contexts of [DFST15] and of the 62 works citing [MT10] listed there), and nine web searches. No text answering the question was found. The most recent paper of two of the proposers on $(\mathcal{C}_2, D_1)$ [FT26] cites the singular-copula result only for d_∞ . This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A Completeness of $(\mathcal{C}_d, D_1)$

Completeness is proved in [Tru11] for $d = 2$ and stated for all d in [MFT21]; we include a short proof so that Theorem 1.1 does not depend on these references.

Proposition A.1. *$(\mathcal{C}_d, D_1)$ is a complete metric space.*

Proof. Let (A_n) be a D_1 -Cauchy sequence. Then (H_{A_n}) is Cauchy in $L^1(\lambda^d)$ and converges to some H . The copulas are 1-Lipschitz (Lemma 2.1(b)) and bounded by 1, and groundedness, uniform margins and nonnegative box volumes V_A pass to pointwise limits; so $(\mathcal{C}_d, d_\infty)$ is compact by the Arzelà–Ascoli theorem, and a subsequence A_{n_k} converges uniformly to some $A \in \mathcal{C}_d$. Fix $x \in I$. The functions $y \mapsto A_{n_k}(x, y) = \int_0^x H_{A_{n_k}}(t, y) dt$ converge pointwise to $A(x, y) = \int_0^x H_A(t, y) dt$, and in $L^1(\lambda^{d-1})$ to $y \mapsto \int_0^x H(t, y) dt$, since the L^1 -distance is at most $\|H_{A_{n_k}} - H\|_1$. Hence $\int_0^x H(t, y) dt = \int_0^x H_A(t, y) dt$ for almost every y . Taking x rational, we get a single null set Z_0 such that for $y \notin Z_0$, with $H(\cdot, y)$ integrable (true for almost every y), the identity holds for all rational x , hence by continuity for all x ; so $H(\cdot, y) = H_A(\cdot, y)$ almost everywhere on I . By Tonelli’s theorem $H = H_A$ λ^d -almost everywhere, and $D_1(A_n, A) = \|H_{A_n} - H\|_1 \rightarrow 0$. $\square$

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