

Sharp Two-Sided Root Selection for Common-Interlacing Polynomials with Fixed Trace

Alper Ferudun*

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Abstract

Let a finite family of monic real-rooted polynomials have a common interlacer and the same sum of roots. If a positive weighted average has all its roots in an interval with center c and radius R , then one member has all its roots in the concentric interval of radius $2R/\sqrt{3}$. The factor $2/\sqrt{3}$ is optimal in every degree at least three, even for characteristic polynomials of zero-diagonal real symmetric matrices. The optimal factor in degrees one and two is one. The proof expresses the polynomials through nonnegative residues, converts the two endpoint constraints into a probability measure with a prescribed mean, and uses an exact affine majorant. Weak interlacing and repeated roots are included. A corollary selects a principal submatrix of a constant-diagonal Hermitian matrix using the root interval of the derivative of its characteristic polynomial. The result concerns a single globally common-interlacing family; it does not assert the same factor for the leaves of an arbitrary interlacing tree.

1 Statement and context

A monic polynomial g of degree $n - 1$ *interlaces* a monic real-rooted polynomial p of degree n if their ordered roots satisfy

$$\lambda_1(p) \leq \gamma_1(g) \leq \lambda_2(p) \leq \cdots \leq \gamma_{n-1}(g) \leq \lambda_n(p).$$

All inequalities may be weak. The *trace* of p is the sum of its roots, namely the negative of its coefficient of x^{n-1} .

The common-interlacing selection lemma bounds one extreme root of a member by that of the average [2, Lemma 4.2]. Applying it to both ends need not select the same member. Marcus asked whether additional structure permits simultaneous control of the largest and smallest eigenvalues [1, p. 965, question 4(3)]. The following sharp theorem supplies such control for a family with a common trace.

Theorem 1. *Let $p_1, \dots, p_m$ be monic real-rooted polynomials of degree $n \geq 1$. Suppose that they have a common interlacer and the same trace. Let $t_i > 0$, $\sum_i t_i = 1$, and put $f = \sum_i t_i p_i$. If every root of f belongs to $[c - R, c + R]$, where $R \geq 0$, then there is an index i such that every root of p_i belongs to*

$$\left[c - \frac{2R}{\sqrt{3}}, c + \frac{2R}{\sqrt{3}} \right].$$

For every $n \geq 3$, the factor $2/\sqrt{3}$ cannot be decreased uniformly over these families. For $n = 1, 2$, the optimal factor is one.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

Weights equal to zero may simply be discarded. In particular, c and R may be the midpoint and half-length of the smallest interval containing the roots of f . The assumption is that *all* the p_i share one interlacer. This is stronger than the recursive sibling condition defining an interlacing family in [2, Definition 4.3]. The theorem therefore does not imply a depth-independent dilation factor for an arbitrary interlacing tree.

There are already substantial two-sided results under other structures. Ravichandran and Srivastava's mixed-determinant method gives asymptotically optimal multi-paving [3]. In particular, it addresses the paving exponent in the adjacent question 4(4) of [1]. Recent spectral-recovery work also uses positive measures for mixed determinantal polynomials [4]. Those results concern different polynomial constructions and matrix hypotheses. Here the contribution is the optimal $2/\sqrt{3}$ factor for a single common-interlacing family with a fixed trace and its elementary residue proof. The dataset record OWR-13674-019 [5] also includes a separate question about distance from real-rootedness; that question is not resolved in this note.

2 Residues and the two endpoint constraints

We first explain the reductions needed for weak interlacing. If a root γ occurs r times in an interlacer g , then it occurs at least $r - 1$ times in every p_i : the $r - 1$ intervening roots of p_i must equal γ . Remove these common factors from all polynomials and from g . The quotient interlacer has simple roots and still interlaces all the quotient polynomials. Removing the same roots from each polynomial preserves equality of their traces.

For a simple interlacer $g(x) = \prod_{j=1}^s (x - \gamma_j)$, polynomial division and partial fractions give

$$\frac{p_i(x)}{g(x)} = x - \alpha_i - \sum_{j=1}^s \frac{c_{ij}}{x - \gamma_j}, \quad c_{ij} = -\frac{p_i(\gamma_j)}{g'(\gamma_j)} \geq 0. \quad (1)$$

The sign follows directly from the interlacing inequalities. The polynomial part has $\alpha_i = \text{tr}(p_i) - \text{tr}(g)$, so fixed trace means that $\alpha_i = \alpha$ is independent of i .

This representation also verifies that the given interlacer interlaces every positive average. Indeed, averaging preserves the nonnegative residues. When all the residues are positive, the quotient is strictly increasing between its poles, with the appropriate infinite limits, and has one zero in each intervening and exterior interval. Zero residues follow by continuity after adding a positive number tending to zero to each residue. Restoring the removed factors proves the assertion for a repeated interlacer as well.

Normalize a containing interval of positive radius to $[-1, 1]$. The roots of the interlacer then lie in $[-1, 1]$. After the preceding reduction, write

$$c_j = \sum_i t_i c_{ij}, \quad \frac{f(x)}{g(x)} = x - \alpha - \sum_j \frac{c_j}{x - \gamma_j}.$$

If a simple root γ_j of g is an endpoint of this interval, interlacing forces $f(\gamma_j) = 0$. Hence $c_j = 0$, and positivity of the weights gives $c_{ij} = 0$ for every i . Thus $x - \gamma_j$ is another common factor and can be removed. All removed factors have roots in $[-1, 1]$. We are left with simple poles in $(-1, 1)$, or with no poles.

Because the quotient for f has no zero outside $[-1, 1]$, its values at the two endpoints satisfy

$$\sum_j \frac{c_j}{1 - \gamma_j} \leq 1 - \alpha, \quad \sum_j \frac{c_j}{1 + \gamma_j} \leq 1 + \alpha. \quad (2)$$

For example, on the right the quotient is strictly increasing and tends to $+\infty$; a negative value at 1 would force a zero larger than 1. The left endpoint is analogous. In particular, $|\alpha| \leq 1$.

Put

$$w_j = \frac{c_j}{1 - \gamma_j^2}, \quad S = \sum_j w_j, \quad M = \sum_j w_j \gamma_j.$$

Then (2) becomes $S + M \leq 1 - \alpha$ and $S - M \leq 1 + \alpha$. Consequently the two numbers

$$w_+ = \frac{1 - S - \alpha - M}{2}, \quad w_- = \frac{1 - S + \alpha + M}{2} \quad (3)$$

are nonnegative. The measure

$$\mu = \sum_j w_j \delta_{\gamma_j} + w_+ \delta_1 + w_- \delta_{-1} \quad (4)$$

is a probability measure on $[-1, 1]$ with mean $-\alpha$. The endpoint masses account for a containing interval whose endpoints need not both be roots of the reduced average.

3 An exact affine majorant

Set $K = 2/\sqrt{3}$. For $|\alpha|, |y| \leq 1$, define

$$\Phi_\alpha(y) = (1 - y^2) \max \left\{ \frac{1}{(K - \alpha)(K - y)}, \frac{1}{(K + \alpha)(K + y)} \right\}.$$

Lemma 2. *For every $|\alpha|, |y| \leq 1$,*

$$\Phi_\alpha(y) \leq L_\alpha(y) := 1 + \frac{2\alpha(y + \alpha)}{K^2 - \alpha^2}. \quad (5)$$

Hence every probability measure μ on $[-1, 1]$ with mean $-\alpha$ satisfies $\int \Phi_\alpha d\mu \leq 1$.

Proof. The identities

$$\begin{aligned} L_\alpha(y)(K - \alpha)(K - y) - (1 - y^2) \\ = \frac{K - \alpha}{K + \alpha} \left(y - \frac{K - \alpha}{2} \right)^2 + \frac{\alpha^2}{4}, \end{aligned} \quad (6)$$

$$\begin{aligned} L_\alpha(y)(K + \alpha)(K + y) - (1 - y^2) \\ = \frac{K + \alpha}{K - \alpha} \left(y + \frac{K + \alpha}{2} \right)^2 + \frac{\alpha^2}{4} \end{aligned} \quad (7)$$

follow by expansion using $K^2 = 4/3$. Every denominator is positive since $K > 1$. Each of the two branches defining Φ_α is therefore at most $L_\alpha(y)$, proving (5). Finally, integration gives

$$\int L_\alpha(y) d\mu(y) = 1 + \frac{2\alpha}{K^2 - \alpha^2} \left(\int y d\mu(y) + \alpha \right) = 1. \quad \square$$

Proof of Theorem 1, the upper bound. For a positive-radius interval, perform the normalization and common-factor reductions above. Define

$$Q_i = \sum_j c_{ij} \max \left\{ \frac{1}{(K - \alpha)(K - \gamma_j)}, \frac{1}{(K + \alpha)(K + \gamma_j)} \right\}. \quad (8)$$

For the measure (4), the endpoint masses contribute zero to Φ_α . Thus Lemma 2 gives

$$\sum_i t_i Q_i = \sum_j w_j \Phi_\alpha(\gamma_j) = \int \Phi_\alpha d\mu \leq 1.$$

Choose an index i with $Q_i \leq 1$. Both separate sums obtained by dropping the maximum in (8) are at most one. Substitution into (1) gives

$$\frac{p_i(K)}{g(K)} \geq 0, \quad \frac{p_i(-K)}{g(-K)} \leq 0.$$

On either exterior interval the derivative of the quotient is

$$1 + \sum_j \frac{c_{ij}}{(x - \gamma_j)^2} > 0.$$

It has no zero to the right of K or to the left of $-K$. The interlacer has no root there either, so every root of p_i lies in $[-K, K]$. This reasoning also covers an empty sum. Restore the removed factors, whose roots are in $[-1, 1]$, and undo the normalization.

If $R = 0$, apply the result to $[c - \varepsilon, c + \varepsilon]$ along a sequence $\varepsilon \downarrow 0$. Since there are finitely many indices, one occurs infinitely often and all roots of its polynomial equal c . $\square$

The proof is a selection rule once the common interlacer and the containing interval are known: compute the residues, then choose a member minimizing Q_i . This is an exact mathematical rule; no bit-complexity bound for recovering arbitrary algebraic interlacers or endpoints is asserted.

4 Optimality and a matrix corollary

Consider

$$\begin{aligned} p_+(x) &= (x - 2)(x + 1)^2 = x^3 - 3x - 2, \\ p_-(x) &= (x + 2)(x - 1)^2 = x^3 - 3x + 2. \end{aligned}$$

Both have trace zero and are interlaced by $x^2 - 1$. Their equal-weight average is $x^3 - 3x$, with root interval $[-\sqrt{3}, \sqrt{3}]$, while each member has a root of absolute value 2. This proves optimality of K in degree three. For degree $n > 3$, multiply both polynomials and their common interlacer by x^{n-3} . Weak interlacing is preserved, and the same extremal roots remain.

The cubics are the characteristic polynomials of the adjacency matrix of K_3 and its negative. Thus the obstruction persists for zero-diagonal real symmetric matrices. In degree one, a common trace makes all polynomials identical. In degree two, write

$$p_i(x) = (x - h)^2 - r_i^2, \quad r_i \geq 0,$$

with common h . The average has roots $h \pm \sqrt{\sum_i t_i r_i^2}$, and some r_i is no larger than that square root. This member lies inside the average's root interval. Identical polynomials show that the factor one is optimal. These observations complete the proof of Theorem 1.

For a Hermitian matrix A , let $A^{(i)}$ denote the principal submatrix obtained by deleting row and column i .

Corollary 3. *Let A be a Hermitian $N \times N$ matrix with constant diagonal, where $N \geq 2$, and let $P(x) = \det(xI - A)$. If all roots of P' belong to $[c - R, c + R]$, then for some i ,*

$$\|A^{(i)} - cI\| \leq \frac{2R}{\sqrt{3}}.$$

For $N \leq 3$, the factor on the right can be replaced by one.

Proof. Put $p_i(x) = \det(xI - A^{(i)})$. If $\lambda_1 \leq \dots \leq \lambda_N$ are the eigenvalues of A , Cauchy interlacing shows that $\prod_{j=2}^{N-1}(x - \lambda_j)$ interlaces every p_i . For $N = 2$ this is the constant polynomial. The traces $\text{tr}(A) - A_{ii}$ are equal. Differentiating the determinant with respect to its diagonal variable gives $\sum_i p_i = P'$, so the monic average is P'/N . Theorem 1 applies in degree $N - 1$. For Hermitian matrices, the operator norm equals the largest absolute eigenvalue. $\square$

The corollary does not claim optimality of $2/\sqrt{3}$ within the narrower class of these principal-submatrix families. Likewise, the theorem’s global common-interlacer hypothesis cannot be replaced here by a tree hypothesis without a further argument. These distinctions keep the result separate from general signing and multi-paving bounds.

The accompanying standard-Python checker verifies the two square-completion identities as polynomial identities after clearing denominators, and checks 780 rational residue families using exact arithmetic in $\mathbb{Q}(\sqrt{3})$. It also verifies the sharp cubics and their matrix realizations. These checks support the written proof; finite sampling does not establish the general theorem.

AI systems assisted with literature searches, derivation, code, and preparation of this manuscript. The mathematical and computational checks are documented in the accompanying verification report. This preprint has not undergone independent peer review, and the bounded literature search does not certify absolute priority.

References

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