

Rote’s Curious Identity on Self-Stresses: Independence of the Two Points for All Graphs, Existence at Generic Positions, and a Volume Interpretation

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October 20, 2026

Abstract

In Problem 8 of Oberwolfach Report 13/2015, G. Rote gives a self-stress ω of a wheel in general position in the plane with $\sum_{ij \in E} \omega_{ij} [a p_i p_j] [b p_i p_j] = 1$ for all points a, b , and asks whether other graphs with n vertices and $2n - 2$ edges carry such a self-stress (the triangular prism and $K_{3,3}$, each with one added edge, are named) and what the identity means. We show that for every graph, every placement of its vertices and every self-stress, the left-hand side does not depend on a and b . With the brackets replaced by determinants (twice the signed areas), as Rote, Santos and Streinu do, it equals $N(\omega) = \sum \omega_{ij} \det(p_i, p_j)^2$, and all these edge functions, with Rote’s line-integral variant, represent one class in the cokernel of the rigidity map, the dual of the stress space. With a projective form of this identity we prove that every graph that is dependent in the generic rigidity matroid of the plane, in particular every graph with $n \geq 2$ vertices and $2n - 2$ edges, carries at generic positions a self-stress satisfying the identity; it is unique exactly when the generic stress space is one-dimensional, which for $2n - 2$ edges means that the graph is generically rigid. The two named graphs get exact certificates. This answers the first question at generic positions, which is how we read the problem’s “general position”; “no three points collinear” does not suffice. For a non-crossing drawing, $N(\omega)$ is six times the volume under the Maxwell–Cremona lifting of ω , and Rote’s line integral is Green’s formula for this volume. For the two named graphs the normalised stress is not a monomial in polygon areas, unlike the wheel; sums of products of areas occur, and a closed-formula reading of the first question stays open. The second question is open-ended, and whether these readings are the intended meaning is a judgment. We also correct three points of the printed statement. The case of four points is due to Rote, Santos and Streinu; our identities are elementary and may be folklore, and no printed answer to the questions was found in the sources searched. This is an unrefereed note.

1 Introduction

1.1 The problem

A *framework* (G, p) is a finite graph $G = (V, E)$ with points $p_i \in \mathbb{R}^2$, $i \in V$. A *self-stress* is a map $\omega: E \rightarrow \mathbb{R}$ with

$$\sum_{j: ij \in E} \omega_{ij} (p_j - p_i) = 0 \quad \text{for every vertex } i \quad (1)$$

[BKS15, p. 722]. The self-stresses form the left kernel $\mathcal{S}(p)$ of the rigidity matrix $R(p)$, whose row ij carries $p_i - p_j$ in the columns of i and $p_j - p_i$ in those of j ; equivalently [RSS03, Lemma 2.4],

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ω is a self-stress if and only if $\sum_{ij} \omega_{ij} \langle p_j - p_i, v_j - v_i \rangle = 0$ for all velocities $v \in (\mathbb{R}^2)^V$. We write $\det(u, v) = u_1 v_2 - u_2 v_1$ and

$$D(a, p, q) := \det(p - a, q - a),$$

twice the signed area of the triangle apq ; this is the determinant $\det(a, p, q)$ of [RSS03]. Rote writes $[q_1 q_2 \dots q_k]$ for the signed area of the polygon $q_1 \dots q_k$.

In Problem 8 of [BKS15], pp. 722–723, Rote considers the wheel W_n ($n \geq 3$): a hub p_0 joined to $p_1, \dots, p_n$, which form the rim cycle, in general position; it has $n + 1$ vertices and $2n = 2(n + 1) - 2$ edges. He defines, with indices of the rim modulo n (the minus sign is not in the printed text; see correction (a) below),

$$\omega_{i,i+1} = \frac{1}{[p_i p_{i+1} p_0] [p_1 \dots p_n]}, \quad \omega_{0,i} = -\frac{[p_{i-1} p_i p_{i+1}]}{[p_{i-1} p_i p_0] [p_i p_{i+1} p_0] [p_1 \dots p_n]}, \quad (2)$$

states that this is a self-stress, and that for two arbitrary points a, b the function $f_{ij} = [a p_i p_j][b p_i p_j]$ satisfies

$$\sum_{ij \in E} \omega_{ij} f_{ij} = 1, \quad (3)$$

which, as the problem says, was used and proved for $n = 3$ in [RSS03, Lemma 3.10]. A second function satisfying (3) is [BKS15, p. 723], [RSS03, (10)]

$$f'_{ij} = \frac{3}{2} |p_i - p_j| \int_{p_i}^{p_j} |x|^2 ds = \frac{1}{2} |p_i - p_j|^2 (|p_i|^2 + |p_j|^2 + \langle p_i, p_j \rangle). \quad (4)$$

Question 1 asks whether there are other graphs with n vertices and $2n - 2$ edges for which a self-stress satisfying (3) “can be defined”; the next candidates named are the triangular prism with one additional edge and $K_{3,3}$ with one additional edge. *Question 2* asks for the meaning of (3), whether it is an instance of a more general phenomenon, and for its connections to homology and cohomology. The problem is record OWR-13498-012 of the UnsolvedMath collection [ula26], where it is listed as open.

Three corrections to the printed statement.

- (a) *The spoke stress needs a minus sign.* The printed formula for $\omega_{0,i}$ has no minus sign, both in the text extraction and in a rendering of p. 722 of the publisher’s file, in which other minus signs of the page are present. Without it the function is not a self-stress: for a convex counter-clockwise rim with the hub inside, every bracket in (2) is positive, so all printed values are positive, while at each rim vertex p_i the vectors $p_0 - p_i$, $p_{i-1} - p_i$, $p_{i+1} - p_i$ lie in an open half-plane, so no positive combination of them vanishes. With the minus sign of (2) the function is a self-stress and satisfies (3) (Proposition 2.4, which also shows that the printed sign fails at every rim vertex for all positions in general position).
- (b) *The second bracket of f_{ij} .* The printed $f_{ij} = [a p_i p_j][b p_j p_j]$ vanishes identically; the intended function, and that of [RSS03, (9)], is $[a p_i p_j][b p_i p_j]$.
- (c) *The area convention.* Replacing every bracket by D multiplies the ω of (2) by $\frac{1}{4}$ and f by 4, so (3) is the same statement in both conventions. The function f' contains no brackets, and $\sum \omega_{ij} f'_{ij} = 1$ holds only in the determinant convention of [RSS03]; with brackets read as signed areas, $\sum \omega_{ij} f'_{ij} = 4$.

From now on we use the determinant convention: brackets in (2) are replaced by D , the rim bracket by $D_{\text{rim}} := \sum_{i=1}^n \det(p_i, p_{i+1}) = 2[p_1 \dots p_n]$, and $f_{ij}(a, b) = D(a, p_i, p_j)D(b, p_i, p_j)$.

Reading of Question 1. Identity (3) is required for all a, b at once, for positions in general position. We read “general position” as *generic*: the statement is to hold outside a proper algebraic subset of the configuration space. For the wheel the self-stress is unique up to a scalar factor, as already noted in [RSS03, p. 37]; so the content of (3) is that the bi-affine function $(a, b) \mapsto \sum \omega_{ij} f_{ij}(a, b)$ is constant and that the constant is not zero. A further possible reading asks for an explicit formula of the type (2); Section 6 says what we have on it.

1.2 Results

Theorem A: for *every* framework (no genericity) and every self-stress, $\sum \omega_{ij} D(a, p_i, p_j) D(b, p_i, p_j)$ does not depend on a, b and equals $N(\omega) := \sum \omega_{ij} \det(p_i, p_j)^2$, and $\sum \omega_{ij} f'_{ij}$ has the same value; all these edge functions represent one class κ_p in $\text{coker } R(p) = \mathbb{R}^E / \text{im } R(p)$, the dual of $\mathcal{S}(p)$, and (3) reads $\langle \omega, \kappa_p \rangle = 1$ (Corollary 2.2). For the wheel this gives a short proof of Rote’s claim (Proposition 2.4). Theorem A_h is a projective form of Theorem A, whose constant is not identically zero (Lemma D). Theorem C answers Question 1: every graph that is dependent in the generic rigidity matroid of the plane, in particular every graph with $n \geq 2$ vertices and $2n - 2$ edges, carries at generic positions a self-stress satisfying (3); it is unique exactly when the generic stress space is one-dimensional, which for $2n - 2$ edges means that the graph is generically rigid; the two named graphs have exact certificates (Table 1), and “no three points collinear” is not enough (Proposition 4.1). Theorem B gives the meaning: for a non-crossing drawing, $N(\omega)$ is six times the volume under the Maxwell–Cremona lifting of ω , and $\sum \omega f'$ is Green’s formula for this volume; drawings with crossings reduce to this case (Corollary 5.1), and K_4 in convex position becomes the wheel W_4 . Proposition E: for the two named graphs the normalised stress is not a monomial in signed areas of polygons.

1.3 What is known, what is new, what stays open

The questions and the wheel identity are Rote’s [BKS15]. The case of four points is [RSS03, Lemma 3.10], proved there by bi-affinity in a and b ; the closing remarks of [RSS03], p. 37, state the wheel case without proof and suggest that the lemma is an instance of a more general phenomenon that the authors do not fully understand. The argument of [RSS03, p. 21] for f' ([RSS03, Lemma 2.4] with velocities $|p_i|^2 p_i / 2$) does not depend on the graph; it is our identity (9). The remark of [RSS03, p. 36], based on [RSS03, Lemma 5.1], that adding an element of the image of the rigidity map to a valid choice of f keeps it valid, is the four-point form of the cokernel reading. The survey [RSS08, Section 8] repeats the four-point statement. The vanishing of $\sum \omega_e d_e d_e^\top$ is the classical stress-tensor identity; the transfer of self-stresses under projective maps is the classical projective invariance of statics, which Izestiev attributes to Rankine [Ran63], see [Izm17, Theorem B and Lemma 4.5]; the Maxwell–Cremona correspondence [Max64, Whi82, HK92], [RSS08, Theorem 5.1] is cited and only sketched; the facts on the generic rigidity matroid that we use are standard. Not found in the sources searched (listed in the paragraph “Scope and priority”) are: the independence of a, b for general graphs (Theorem A and (8)), the constant in Theorem A_h and Lemma D, Theorem C, the volume interpretation Theorem B, and Proposition E. Theorems A and B are elementary and may well be folklore in rigidity theory; a search that finds nothing is not a proof of novelty, and we make no priority claim. Open: whether the two named graphs, or graphs with $2n - 2$ edges in general, admit a closed formula of another kind than a monomial, and any link with stress homology of cell complexes [CW94, Ryb99].

2 The identity for every self-stress

For an edge $e = ij$ put $d_e = p_j - p_i$ and $c_e = \det(p_i, p_j)$. Exchanging i and j changes the sign of both, so c_e^2 , $c_e d_e$ and $d_e d_e^\top$ depend only on the unordered edge. For $x \in \mathbb{R}^2$ let $x^\perp = (-x_2, x_1)$, so that $x^\perp \cdot y = \det(x, y)$, and let $\hat{p} = (x_1, x_2, 1) \in \mathbb{R}^3$. For an edge ij let $L_{ij} = \hat{p}_i \times \hat{p}_j = (d_e^\perp, c_e)$ (cross product in $\mathbb{R}^3$), the homogeneous coordinates of the line through p_i, p_j ; then

$$D(a, p_i, p_j) = c_e - \det(a, d_e) = \hat{a} \cdot L_{ij}. \quad (5)$$

For a self-stress ω define $N(\omega) = N(\omega; p) := \sum_{ij \in E} \omega_{ij} \det(p_i, p_j)^2$.

Theorem A. *Let (G, p) be any framework in the plane (no genericity, coincident points allowed) and ω any self-stress. Then for all $a, b \in \mathbb{R}^2$*

- (i) $\sum_{ij \in E} \omega_{ij} D(a, p_i, p_j) D(b, p_i, p_j) = N(\omega)$;
- (ii) (line coordinates) $\sum_{ij \in E} \omega_{ij} L_{ij} L_{ij}^\top = N(\omega) e_3 e_3^\top$;
- (iii) $\sum_{ij \in E} \omega_{ij} f'_{ij} = N(\omega)$.

In particular $N(\omega) = \sum \omega_{ij} D(o, p_i, p_j)^2$ for every base point o .

Proof. We use two facts. First,

$$\sum_e \omega_e d_e d_e^\top = 0. \quad (6)$$

Indeed $d_e d_e^\top = (p_j - p_i)(p_j - p_i)^\top + (p_i - p_j)(p_i - p_j)^\top$, so the sum over unordered edges equals the sum over ordered pairs (i, j) with $ij \in E$ of $\omega_{ij} (p_i - p_j)(p_i - p_j)^\top$, that is, $\sum_i [\sum_j \omega_{ij} (p_i - p_j)] p_i^\top = 0$ by (1). Second,

$$\sum_e \omega_e c_e d_e = 0. \quad (7)$$

Indeed $c_e d_e = \det(p_i, p_j) p_j + \det(p_j, p_i) p_i$, so the sum equals

$$\sum_{(i,j)} \omega_{ij} \det(p_i, p_j) p_j = \sum_j \det(\sum_i \omega_{ij} p_i, p_j) p_j,$$

and by (1) at j we have $\sum_i \omega_{ij} p_i = (\sum_i \omega_{ij}) p_j$, so every determinant vanishes. By (5),

$$D(a, p_i, p_j) D(b, p_i, p_j) = c_e^2 - c_e (a^\perp + b^\perp) \cdot d_e + (a^\perp)^\top d_e d_e^\top b^\perp;$$

summing with weights ω_e , the last two terms vanish by (7) and (6), which proves (i). For (ii), with J the rotation by $+90^\circ$ we have $d^\perp = Jd$, so the upper left 2×2 block of $\sum \omega L L^\top$ is $J(\sum \omega d d^\top) J^\top = 0$, the off-diagonal block is $J \sum \omega c d = 0$, and the $(3, 3)$ entry is $N(\omega)$. Part (iii) follows from (9) below, and the last claim is (i) with $a = b = o$. $\square$

Conversely (ii) implies (i), since by (5) the left side of (i) is $\hat{a}^\top (\sum \omega L L^\top) \hat{b}$. The trace of (6) is the classical identity $\sum \omega_e |d_e|^2 = 0$.

Velocity fields and a second proof. For a map $U: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (a velocity field) let $(RU)_{ij} := \langle p_j - p_i, U(p_j) - U(p_i) \rangle$. This is the rigidity matrix applied to the velocities $v_i = U(p_i)$, so $RU \in \text{im } R(p)$, and a self-stress is orthogonal to it [RSS03, Lemma 2.4].

Lemma 2.1. *For all $a, b, p, q \in \mathbb{R}^2$, with $M = \frac{1}{2}(a^\perp (b^\perp)^\top + b^\perp (a^\perp)^\top)$,*

$$D(a, p, q) D(b, p, q) - \det(p, q)^2 = \langle q - p, U(q) - U(p) \rangle, \quad U(x) = Mx - \det(a + b, x) x^\perp, \quad (8)$$

$$f'(p, q) - \det(p, q)^2 = \langle q - p, V(q) - V(p) \rangle, \quad V(x) = \frac{1}{2}|x|^2 x, \quad (9)$$

where $f'(p, q) = \frac{1}{2}|p - q|^2(|p|^2 + |q|^2 + \langle p, q \rangle)$.

Proof. Let $d = q - p$ and $c = \det(p, q)$. By (5) the left side of (8) is $-c \det(a + b, d) + (a^\perp \cdot d)(b^\perp \cdot d)$. The quadratic term is $d^\top M d = \langle d, M q - M p \rangle$. For the linear term, $d \cdot q^\perp = \det(q, d) = c$ and $d \cdot p^\perp = \det(p, d) = c$, so with $k = a + b$ we get $\langle d, -\det(k, q)q^\perp + \det(k, p)p^\perp \rangle = -c \det(k, d)$. For (9) put $s = |p|^2$, $t = |q|^2$, $u = \langle p, q \rangle$; then $|d|^2 = s + t - 2u$ and $\det(p, q)^2 = st - u^2$, so the left side is $\frac{1}{2}(s + t - 2u)(s + t + u) - st + u^2 = \frac{1}{2}(s^2 + t^2 - (s + t)u)$, and the right side is $\frac{1}{2}\langle q - p, tq - sp \rangle$, the same. $\square$

Identity (9) is the computation of [RSS03, p. 21]. Summing (8) and (9) over the edges gives

$$f(a, b) = f(o, o) + RU, \quad f' = f(o, o) + RV, \quad (10)$$

on every framework, where $f(o, o) = (\det(p_i, p_j)^2)_{ij}$; pairing with a self-stress gives Theorem A(i) and (iii) again. We use only that the vectors RU lie in $\text{im } R(p)$.

Corollary 2.2. *Let $\kappa_p \in \text{coker } R(p) = \mathbb{R}^E / \text{im } R(p)$ be the class of $(\det(p_i, p_j)^2)_{ij \in E}$. Every $f(a, b)$ and f' represent κ_p . The pairing $\langle \omega, [g] \rangle = \sum \omega_{ij} g_{ij}$ between $\mathcal{S}(p)$ and $\text{coker } R(p)$ is well defined and identifies $\text{coker } R(p)$ with the dual of $\mathcal{S}(p)$; and $N(\omega) = \langle \omega, \kappa_p \rangle$. So a self-stress satisfies (3) for all a, b if and only if $\langle \omega, \kappa_p \rangle = 1$.*

Proof. The first claim is (10). $\mathcal{S}(p)$ is the left kernel of $R(p)$, that is, the annihilator of $\text{im } R(p)$ in $\mathbb{R}^E$, which gives the duality by linear algebra. $\square$

Corollary 2.3. (a) *If A is an affine map of the plane with linear part A_0 , then the self-stresses of (G, p) and (G, Ap) coincide and $N(\omega; Ap) = \det(A_0)^2 N(\omega; p)$.* (b) *Let an edge uv with stress ω_{uv} be replaced by the edges uw, wv through a new vertex $p_w = (1 - t)p_u + tp_v$, $0 < t < 1$, with stresses ω_{uw}/t and $\omega_{wv}/(1 - t)$. The result is a self-stress, with the same value of N .*

Proof. (a) The forces transform by A_0 , and D is multiplied by $\det A_0$; use Theorem A(i). (b) The force exerted at u is $(\omega_{uv}/t)(p_w - p_u) = \omega_{uv}(p_v - p_u)$, as before, and at w the two forces $-\omega_{wv}(p_v - p_u)$ and $\omega_{uv}(p_v - p_u)$ cancel. Since $\det(p_u, p_w) = t c_{uv}$ and $\det(p_w, p_v) = (1 - t) c_{uv}$, the contribution to N is $(t^2/t + (1 - t)^2/(1 - t)) \omega_{uv} c_{uv}^2 = \omega_{uv} c_{uv}^2$. $\square$

Proposition 2.4 (Rote's wheel identity). *Let $p_0, \dots, p_n \in \mathbb{R}^2$ with $D(p_i, p_{i+1}, p_0) \neq 0$ for all i . Let $\sigma_{i,i+1} = 1/D(p_i, p_{i+1}, p_0)$ and $\sigma_{0,i} = -D(p_{i-1}, p_i, p_{i+1})/(D(p_{i-1}, p_i, p_0)D(p_i, p_{i+1}, p_0))$. Then σ is a self-stress of W_n with $N(\sigma) = D_{\text{rim}}$. Hence, if $D_{\text{rim}} \neq 0$, the stress $\omega = \sigma/D_{\text{rim}}$ of (2) (determinant convention) satisfies (3) for all a, b and $\sum \omega f' = 1$. With the printed sign of $\omega_{0,i}$, the equilibrium at a rim vertex p_i fails whenever p_{i-1}, p_i, p_{i+1} are not collinear.*

Proof. Fix a rim vertex p_i and let $u = p_{i-1} - p_i$, $w = p_{i+1} - p_i$, $z = p_0 - p_i$. Then $D(p_{i-1}, p_i, p_0) = \det(z, u)$, $D(p_i, p_{i+1}, p_0) = \det(w, z)$ and $D(p_{i-1}, p_i, p_{i+1}) = \det(w, u)$. The force at p_i , multiplied by $\det(z, u) \det(w, z)$, is

$$\det(w, z) u + \det(z, u) w + \det(u, w) z = 0,$$

a classical identity for three vectors in the plane (both sides are linear in z , and the left side vanishes for $z = u$ and $z = w$; if u, w are parallel, it vanishes identically). With the printed sign the last term is $\det(w, u)z$ and the sum is $2\det(w, u)z \neq 0$. At the hub, equilibrium follows because the forces at all vertices add up to zero for any function on the edges. For N we use the base point $o = p_0$ (Theorem A): the spokes contribute nothing, and the rim edges give $\sum_i D(p_0, p_i, p_{i+1})^2/D(p_i, p_{i+1}, p_0) = \sum_i D(p_0, p_i, p_{i+1}) = D_{\text{rim}}$. The rest follows from Theorem A. $\square$

For $n = 3$ the wheel is K_4 , and (2) is a multiple of the stress of [RSS03, Lemma 2.6] (both span the one-dimensional stress space; both satisfy (3), so they are equal).

3 A projective form of the identity

Theorem A(ii) says that the stress-weighted sum of the squared lines of the bars is a multiple of the square of the line at infinity e_3 . With the weights divided by $h(p_i)h(p_j)$, the same holds for every other line h . Call an affine function $h(x) = l \cdot \hat{x}$, $l \in \mathbb{R}^3 \setminus \{0\}$, *admissible* for p if $h_i := h(p_i) \neq 0$ for all vertices i .

Theorem A_h. *Let ω be a self-stress of (G, p) and $h = l \cdot \hat{x}$ admissible. Choose $S \in GL_3(\mathbb{R})$ with $S^\top l = e_3$ and define p'_i by $\hat{p}'_i = S^{-1}(\hat{p}_i/h_i)$. Then $\mu_{ij} := \omega_{ij}h_ih_j$ is a self-stress of (G, p') , the map $\omega \mapsto \mu$ is a linear bijection $\mathcal{S}(p) \rightarrow \mathcal{S}(p')$, and*

$$\sum_{ij \in E} \frac{\omega_{ij}}{h_i h_j} L_{ij} L_{ij}^\top = C_h(\omega) ll^\top, \quad C_h(\omega) = \det(S)^2 N(\mu; p').$$

Evaluated at $\hat{a}, \hat{b}$: $\sum \omega_{ij} D(a, p_i, p_j) D(b, p_i, p_j) / (h_i h_j) = C_h(\omega) h(a) h(b)$.

The point p' is the image of p under a projective transformation (the third coordinate of $S^{-1}(\hat{p}_i/h_i)$ is $l \cdot \hat{p}_i/h_i = 1$), and the transfer $\omega \mapsto \mu$ is the classical projective invariance of self-stresses, which, according to [Izm17, p. 17], goes back to Rankine [Ran63]; see [Izm17, Theorem B and Lemma 4.5]. We include the short proof because we need the constant; the formula for $C_h(\omega)$ and the proportionality to $ll^\top$ are what we did not find in the sources searched.

Proof. Equilibrium in homogeneous form. Since $\hat{p}_j - \hat{p}_i = (d_{ij}, 0)$,

$$\sum_j \omega_{ij} \hat{p}_i \times \hat{p}_j = \hat{p}_i \times \left(\sum_j \omega_{ij} d_{ij}, 0 \right),$$

and a vector $(v, 0)$ is parallel to $\hat{p}_i$ only if $v = 0$. So ω is a self-stress if and only if $\sum_j \omega_{ij} \hat{p}_i \times \hat{p}_j = 0$ at every vertex i . *Transfer.* Put $P_i = \hat{p}_i/h_i = S\hat{p}'_i$. With the rule $(AX) \times (AY) = \text{cof}(A)(X \times Y)$, $\text{cof}(A) = \det(A)A^{-\top}$,

$$\sum_j \mu_{ij} \hat{p}'_i \times \hat{p}'_j = \text{cof}(S^{-1}) \sum_j \omega_{ij} h_i h_j P_i \times P_j = \text{cof}(S^{-1}) \sum_j \omega_{ij} \hat{p}_i \times \hat{p}_j = 0.$$

The map is injective, and the inverse transformation is of the same kind (with $\tilde{h}(x) = e_3 \cdot S\hat{x}$, which takes the value $1/h_i$ at p'_i , and with S^{-1} in place of S), and maps μ back to ω ; so it is a bijection. *Constant.* We have $L'_{ij} := \hat{p}'_i \times \hat{p}'_j = \text{cof}(S^{-1}) L_{ij} / (h_i h_j)$. Theorem A(ii) for (p', μ) gives

$$N(\mu; p') e_3 e_3^\top = \sum \mu_{ij} L'_{ij} L'_{ij}{}^\top = \text{cof}(S^{-1}) \left[\sum \frac{\omega_{ij}}{h_i h_j} L_{ij} L_{ij}{}^\top \right] \text{cof}(S^{-1})^\top.$$

Since $\text{cof}(S^{-1})^{-1} = \det(S) S^{-\top}$ and $S^{-\top} e_3 = l$, the bracket equals $\det(S)^2 N(\mu; p') ll^\top$. $\square$

Theorem A is the case $h \equiv 1$. The constant $C_h(\omega)$ can vanish for a particular h , but not for all of them.

Lemma D. *Let the points p_i be pairwise distinct, and at every vertex v let the directions d_{vj} of the incident edges be pairwise non-parallel (both hold if no three points are collinear). If $\omega \neq 0$ is a self-stress, then $C_h(\omega) \neq 0$ for some admissible h .*

Proof. Suppose that $C_h(\omega) = 0$, that is, $T(h) := \sum \omega_{ij} L_{ij} L_{ij}^\top / (h_i h_j) = 0$, for all admissible h . Fix a vertex v and $\nu \in \mathbb{R}^2$ with $\nu \cdot (p_j - p_v) \neq 0$ for all $j \neq v$ (this excludes finitely many lines), and let

$g(x) = \nu \cdot (x - p_v)$. For small $t \neq 0$, $h = g + t$ is admissible. In $tT(g + t)$ the terms of the edges vj are $\omega_{vj}L_{vj}L_{vj}^\top/(g(p_j) + t)$, and all other terms carry a factor t . Letting $t \rightarrow 0$,

$$\sum_{j \sim v} \frac{\omega_{vj}}{\nu \cdot d_{vj}} L_{vj}L_{vj}^\top = 0$$

for all ν outside finitely many lines. The functions $\nu \mapsto 1/(\nu \cdot d_{vj})$, $j \sim v$, are linearly independent: if $\sum_j \alpha_j/(\nu \cdot d_j) = 0$, then $\sum_j \alpha_j \prod_{k \neq j} (\nu \cdot d_k) = 0$ as a polynomial, and at a vector $\nu \neq 0$ orthogonal to d_j only the j -th term survives, with a nonzero product since the d_k are not parallel to d_j ; so $\alpha_j = 0$. Applying this to each matrix entry gives $\omega_{vj}L_{vj}L_{vj}^\top = 0$, and $L_{vj} \neq 0$ because $p_v \neq p_j$; so $\omega_{vj} = 0$ for every edge at v . As v was arbitrary, $\omega = 0$, a contradiction. $\square$

4 Existence at generic positions: the answer to Question 1

Let r_G be the generic rank of $R(p)$, that is, its maximum over all $p \in (\mathbb{R}^2)^V$; it is attained on the nonempty Zariski-open set where some $r_G \times r_G$ minor does not vanish. Put $d_G = |E| - r_G$. The graph G is *dependent* (in the generic rigidity matroid of the plane) if $d_G \geq 1$, and *generically rigid* if $r_G = 2|V| - 3$. A graph with $n \geq 2$ vertices and $2n - 2$ edges (the graph with one vertex and no edge is excluded) has $n \geq 4$ and is dependent, because $r_G \leq 2n - 3$: the two translations and the rotation $v_i = Jp_i$ are independent infinitesimal motions in $\ker R(p)$ when the points are not all equal. For such a graph $d_G = 1$ if and only if G is generically rigid.

Theorem C. *Let G be dependent. There is a nonzero polynomial Ψ_G on $(\mathbb{R}^2)^V$ such that for every p with $\Psi_G(p) \neq 0$: (a) $\dim \mathcal{S}(p) = d_G$, and (b) the linear functional $N_p: \mathcal{S}(p) \rightarrow \mathbb{R}$, $\omega \mapsto N(\omega; p)$, is not zero. For such p the self-stresses satisfying (3) for all a, b form the nonempty affine hyperplane $\{N_p = 1\}$ of $\mathcal{S}(p)$. It is a single stress if and only if $d_G = 1$, and then it is $\omega/N(\omega)$ for every nonzero $\omega \in \mathcal{S}(p)$. In particular every graph with $n \geq 2$ vertices and $2n - 2$ edges carries, at generic positions, a self-stress satisfying (3), and it is unique if and only if the graph is generically rigid.*

Proof. Choose p^0 with $\text{rank } R(p^0) = r_G$ and no three points collinear (both conditions define nonempty Zariski-open sets), and $\omega \neq 0$ in $\mathcal{S}(p^0)$. By Lemma D there is an admissible h with $C_h(\omega) \neq 0$, and by Theorem A_h the projective image p' carries the self-stress μ with $N(\mu; p') = C_h(\omega)/\det(S)^2 \neq 0$; moreover $\dim \mathcal{S}(p') = \dim \mathcal{S}(p^0) = d_G$, so $\text{rank } R(p') = r_G$. Choose rows I and columns J with $|I| = |J| = r_G$ and $m(p') := \det R(p')_{I,J} \neq 0$. For each edge $e \notin I$ let $w^{(e)}(q) \in \mathbb{R}^E$ have the entries $w_e^{(e)} = m(q)$, $w_I^{(e)} = -R(q)_{e,J} \text{adj}(R(q)_{I,J})$ and 0 elsewhere; it is polynomial in q . Where $m(q) \neq 0$, $\text{rank } R(q) = r_G$ (it is at least r_G , and r_G is the maximum), so every row of $R(q)$ is the combination of the rows I whose coefficients are read off on the columns J ; hence $w^{(e)}(q)$, which annihilates the columns J by construction, annihilates all columns and lies in $\mathcal{S}(q)$. These d_G vectors are independent (look at their entries outside I), so they form a basis of $\mathcal{S}(q)$. At $q = p'$ some $w^{(e)}(p')$ has $N \neq 0$, since otherwise $N_{p'}$ would vanish on $\mathcal{S}(p')$, contradicting $N(\mu; p') \neq 0$. Put $\Psi_G(q) = m(q) N(w^{(e)}(q); q)$ for this e . It is a polynomial with $\Psi_G(p') \neq 0$, and (a), (b) hold where $\Psi_G \neq 0$. By Theorem A, a self-stress satisfies (3) for all a, b if and only if $N_p(\omega) = 1$. $\square$

Exact certificates for the two named graphs. Let the prism have the triangles 012 and 345 and the rungs 03, 14, 25, with the added edge 04; let $K_{3,3}$ have the sides $\{0, 1, 2\}$ and $\{3, 4, 5\}$, with the added edge 01. Each of the two graphs is unique up to isomorphism: the non-edges of the prism are the diagonals of its square faces, and those of $K_{3,3}$ lie inside one side; in both cases the automorphisms

prism + 04		$K_{3,3} + 01$	
edge	$\hat{\omega}_e$	edge	$\hat{\omega}_e$
01	187/847800	03	-1037/4017585
12	17/47100	04	1457/4504565
02	7/47100	05	42/316537
34	-1/37680	13	549/1339195
45	-1/628	14	-961/4504565
35	-1/9420	15	343/1582685
03	-1/9420	23	61/243490
14	-17/14130	24	31/121745
25	-1/785	25	-14/121745
04	-11/28260	01	212249/644152795

Table 1: The unique self-stresses satisfying (3) (determinant convention) at the certificate positions (11). In the area convention multiply by 4.

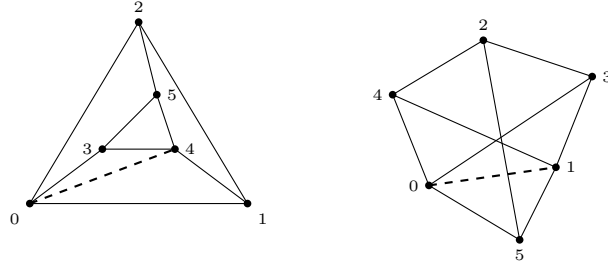


Figure 1: The certificate positions (11): the prism with the added edge 04 (dashed), drawn without crossings, and $K_{3,3}$ with the added edge 01 (dashed), with four crossings.

act transitively on them. Take

$$\begin{aligned}
 \text{prism} + 04 : \quad p &= (0, 0), (12, 0), (6, 10), (4, 3), (8, 3), (7, 6), \\
 K_{3,3} + 01 : \quad p &= (0, 0), (7, 1), (3, 8), (9, 6), (-2, 5), (5, -3)
 \end{aligned} \tag{11}$$

(Figure 1). In exact rational arithmetic, for both graphs: no three points are collinear; $\text{rank } R(p) = 9 = 2n - 3$, so the stress space has dimension 1; the stress is nonzero on all ten edges, and every nine-edge subgraph has rank 9 (the graphs are rigidity circuits); the normalised stress $\hat{\omega} = \omega/N(\omega)$ is given in Table 1; and $\sum \hat{\omega}_e f_e(a, b) = 1$ identically in a, b and $\sum \hat{\omega}_e f'_e = 1$. A reader can check Table 1 directly: equilibrium is twelve linear equations with rational data, and by Theorem A identity (3) reduces to $N(\hat{\omega}) = 1$. These certificates prove Theorem C for the two named graphs without Lemma D: rank 9 is the maximum, so the second half of the proof of Theorem C, run at the certificate position instead of p' , gives a polynomial Ψ_G that does not vanish there.

Proposition 4.1. *At the following positions no three points are collinear, the rank of $R(p)$ is maximal (7 for W_4 , 9 for the other two), the stress space has dimension 1, its generator is nonzero on all edges, and $N = 0$; so no self-stress satisfies (3):*

- (a) W_4 with $p_0 = (0, 0)$, $p_1 = (4, 1)$, $p_2 = (-3, 2)$, $p_3 = (-1, -5)$, $p_4 = (5, \frac{58}{5})$, where $D_{\text{rim}} = 0$;
- (b) $\text{prism} + 04$ with $p = (0, 0), (1, 0), (0, 1), (-\frac{83}{70}, \frac{2}{7}), (\frac{3}{5}, \frac{1}{5}), (\frac{2}{9}, \frac{5}{9})$;
- (c) $K_{3,3} + 01$ with $p = (0, 0), (1, 0), (0, 1), (\frac{3}{2}, -\frac{25}{29}), (-\frac{1}{2}, \frac{7}{5}), (\frac{5}{4}, -\frac{2}{3})$.

Proof. Exact computation. For (a) the generator is the σ of Proposition 2.4, with $N(\sigma) = D_{\text{rim}} = 0$. $\square$

For the wheel this is the general picture: where the triangles $p_i p_{i+1} p_0$ are nondegenerate and the stress space is spanned by σ , the normaliser N_p is nonzero if and only if $D_{\text{rim}} \neq 0$, and Rote's formula, with its factor $1/D_{\text{rim}}$, is undefined exactly where no normalised stress exists. The points (b) and (c) lie on the quartic factors Q_P and Q_K of N found in Section 6. In all three cases Lemma D still applies, and a projective image has $N \neq 0$; for (a), $h(x) = x_1 + 2x_2 + 30$ gives $C_h(\sigma) \neq 0$. So the hypothesis “general position” must be read as “generic”: no three points collinear does not suffice.

5 The meaning of the identity: a lifting volume

Setting. Let G be connected and drawn in the plane without crossings by straight segments: p is injective, and two edges meet only in a common endpoint. The faces are the connected components of the complement of the drawing; F_0 is the unbounded one. We use the Maxwell–Cremona correspondence [Max64, Whi82, HK92], [RSS08, Theorem 5.1] in the following form: for every self-stress ω there is a unique continuous function $z: \mathbb{R}^2 \rightarrow \mathbb{R}$, affine on every face, with $z = 0$ on F_0 , such that for every edge ij , with face L on the left and face R on the right of the directed segment $p_i \rightarrow p_j$,

$$\nabla z_L - \nabla z_R = \omega_{ij} J(p_j - p_i), \quad J = \text{rotation by } +90^\circ. \quad (12)$$

With this sign convention a fold along which z is convex carries a positive stress. (Sketch: define the gradients face by face along paths in the dual graph starting from F_0 ; around a vertex i the jumps add up to $J \sum_j \omega_{ij}(p_j - p_i) = 0$, and the cycles of the dual graph of a connected plane graph are generated by those around the vertices, so the gradients are well defined; the jump (12) is orthogonal to $p_j - p_i$, so the constants can be chosen to make z continuous. We do not reprove the correspondence.)

Theorem B. *In this setting, for every self-stress ω and all $a, b \in \mathbb{R}^2$,*

$$\sum_{ij \in E} \omega_{ij} D(a, p_i, p_j) D(b, p_i, p_j) = \sum_{ij \in E} \omega_{ij} f'_{ij} = 6 \int_{\mathbb{R}^2} z \, dA.$$

Proof. The first equality is Theorem A. Let $\varphi(x) = |x|^2/4$, so $\Delta\varphi = 1$. Since z is Lipschitz with compact support,

$$\int z \, dA = \int z \Delta\varphi \, dA = - \int \nabla z \cdot \nabla \varphi \, dA = - \sum_F \nabla z_F \cdot \int_F \nabla \varphi \, dA = - \sum_F \nabla z_F \cdot \oint_{\partial F} \varphi \, \nu_F \, ds,$$

where F runs over the bounded faces ($\nabla z = 0$ on F_0), each a bounded polygonal domain whose boundary is traversed once from each side (an edge with the same face on both sides is counted twice), and ν_F is the outer normal. Let $t = d/|d|$ be the unit direction of $p_i \rightarrow p_j$, $d = p_j - p_i$. The outer normal of L along the edge is $-Jt$, that of R is $+Jt$, so the edge contributes

$$(\nabla z_L - \nabla z_R) \cdot Jt \int_{[p_i, p_j]} \varphi \, ds = \omega_{ij} |d| \int_{[p_i, p_j]} \varphi \, ds = \frac{\omega_{ij} |d|^2}{12} (|p_i|^2 + |p_j|^2 + \langle p_i, p_j \rangle) = \frac{\omega_{ij} f'_{ij}}{6},$$

using (12), $Jd \cdot Jt = |d|$ and $\int_{[p_i, p_j]} |x|^2 ds = |d| (|p_i|^2 + |p_j|^2 + \langle p_i, p_j \rangle)/3$. (If $L = R$, then $\omega_{ij} = 0$ by (12), and the contribution is 0 on both sides.) Summing over the edges gives $\int z \, dA = \frac{1}{6} \sum \omega_{ij} f'_{ij}$. $\square$

So Rote's line-integral formula (4) is literally Green's formula for the volume under the lifting, and (3) says that the lifting of ω encloses the signed volume $\frac{1}{6}$ (z may change sign). For the planar certificate of the prism +04 (Figure 1) and the stress of Table 1, the lifting has $z = 0$ on the outer triangle 012 and $z(p_3) = \frac{77}{23550}$, $z(p_4) = \frac{187}{23550}$, $z(p_5) = \frac{119}{23550}$, and the volume under it is exactly $\frac{1}{6}$; both independent verification runs built this lifting with their own programs.

Corollary 5.1 (drawings with crossings). *Let G be connected and let its straight-line drawing be in general position: no edge passes through a vertex other than its endpoints, no two edges overlap in a segment, and no point other than a vertex lies on three or more edges. Insert every crossing point as a new vertex and give the piece of an edge between the parameters $s < t$ (with $p_i + s d$ and $p_i + t d$ its ends) the stress $\omega_{ij}/(t - s)$. This gives a non-crossing connected framework with a self-stress $\tilde{\omega}$ and $N(\tilde{\omega}) = N(\omega)$, so $N(\omega)$ is six times the volume under the lifting of $\tilde{\omega}$.*

Proof. Without the general-position hypothesis the subdivided drawing need not be non-crossing; with it, it is. The stress property and N are kept by Corollary 2.3(b), applied once for every new vertex on every edge (at a crossing point the two forces of each subdivided edge cancel). Then apply Theorem B. $\square$

For example, the drawing of $K_{3,3} + 01$ in Figure 1 has four crossings; after subdivision the lifting of the stress of Table 1 has volume exactly $\frac{1}{6}$. For four points in convex position, K_4 has one crossing, and the subdivided framework is the wheel W_4 with hub at the intersection of the diagonals; the subdivided stress of [RSS03, Lemma 2.6] equals Rote's stress (2) of this wheel, edge by edge. For four points with one inside the triangle of the others, K_4 is the wheel W_3 drawn without crossings. So [RSS03, Lemma 3.10] and Rote's wheel identity are the same statement in both cases.

The wheel. If the rim is a simple polygon that is star-shaped from the hub ($D(p_0, p_i, p_{i+1}) > 0$ for all i , and the rim winds once around p_0), the drawing of W_n is non-crossing, and the pyramid of height H over the rim, with apex over p_0 and $z = 0$ outside, is a continuous function, affine on the triangles $p_0 p_i p_{i+1}$. Since the rim vertices lie on the closure of the outer face, every lifting vanishes there and is such a pyramid; so by the correspondence the stress space is spanned by σ , and the pyramid is the lifting of a multiple of σ . Its volume is $H[p_1 \dots p_n]/3$, so by Theorem B the stress satisfying (3) has the pyramid with $H = 1/(2[p_1 \dots p_n]) = 1/D_{\text{rim}}$ as its lifting. Without star-shapedness $N = 6 \text{ Vol}$ still holds (after subdivision if the drawing has crossings), but the lifting need not be a pyramid.

What this gives for Question 2. *A more general phenomenon:* the left side of (3) is independent of a, b for every graph and every self-stress (Theorem A), and with the weights divided by $h(p_i)h(p_j)$, for any line h , it is a constant multiple of $h(a)h(b)$ (Theorem A_h). *Its meaning:* $N(\omega)$ is six times the volume enclosed by the Maxwell–Cremona lifting (Theorem B, Corollary 5.1), and Rote's f' is Green's formula for this volume. *Homology and cohomology:* the edge functions $f(a, b)$ for all a, b , and f' , differ by elements of the image of the rigidity map, given by explicit velocity fields (10); they define one class κ_p in $\text{coker } R(p)$, the dual of the stress space, and (3) is the statement $\langle \omega, \kappa_p \rangle = 1$ (Corollary 2.2). We have not attempted a link with the stress homology of cell complexes [CW94, Ryb99, ERR99]; the closest homological framework we found, the cosheaf theory of reciprocal figures and lifts of Cooperband, Ghrist and Hansen [CGH23], is not used here. Whether these readings are the meaning Rote had in mind is a judgment that the text of the problem does not decide.

6 No monomial formula for the two named graphs

For the wheel, (2) is a monomial (a constant times a product of integer powers) in signed areas of polygons on the vertex set. For the two named graphs this is not possible. We work on the

prism + 04	$K_{3,3} + 01$
01 $-D_{025}D_{034}D_{124}D_{345}$	03 $-D_{014}D_{015}D_{123}D_{245}$
12 $-D_{014}D_{025}D_{034}D_{345}$	04 $D_{013}D_{015}D_{124}D_{235}$
02 $D_{014}D_{034}D_{125}D_{345}$	05 $-D_{013}D_{014}D_{125}D_{234}$
34 $-D_{014}D_{035}D_{245}$	13 $D_{014}D_{015}D_{023}D_{245}$
45 $-D_{014}D_{034}D_{235}$	14 $-D_{013}D_{015}D_{024}D_{235}$
35 $D_{014}D_{034}D_{245}$	15 $D_{013}D_{014}D_{025}D_{234}$
03 $-D_{014}D_{245}D_{345}$	23 $-D_{013}D_{014}D_{015}D_{245}$
14 $D_{025}D_{034}D_{345}$	24 $D_{013}D_{014}D_{015}D_{235}$
25 $-D_{014}D_{034}D_{345}$	25 $-D_{013}D_{014}D_{015}D_{234}$
04 $D_{345}(D_{014}D_{023}D_{125} - D_{013}D_{025}D_{124})$	01 $D_{023}w_{03} + D_{024}w_{04} + D_{025}w_{05}$

Table 2: Polynomial self-stresses w on the slice $p_0 = (0, 0)$, $p_1 = (1, 0)$, $p_2 = (0, 1)$; $D_{ijk} = D(p_i, p_j, p_k)$. The entry w_{01} of $K_{3,3} + 01$ is determined by the equilibrium at p_0 .

slice $p_0 = (0, 0)$, $p_1 = (1, 0)$, $p_2 = (0, 1)$, with the free coordinates $x_3, y_3, x_4, y_4, x_5, y_5$, and write $D_{ijk} = D(p_i, p_j, p_k)$; on the slice $D_{012} = 1$, $D_{01j} = y_j$, $D_{02j} = -x_j$ and $D_{245} = x_4(y_5 - 1) - x_5(y_4 - 1)$. Table 2 gives a polynomial self-stress w on the slice for each graph; both were obtained from signed 9×9 minors of the rigidity matrix, and both satisfy (1) as polynomial identities. Their normalisers are

$$N(w) = -D_{014}D_{034}D_{345} Q_P \quad (\text{prism} + 04), \quad N(w) = -D_{013}D_{014}D_{015} Q_K \quad (K_{3,3} + 01), \quad (13)$$

where, with $C_P = -x_4y_3y_5 + x_4y_3 + x_4y_5 + x_5y_3y_4 - x_5y_3 - x_5y_4 + x_5 + y_4 - 1$ and $E_K = x_4 - x_5 + y_4 - y_5$,

$$\begin{aligned} Q_P &= x_3y_5D_{245} + x_5C_P \\ &= x_3x_4y_5^2 - x_3x_4y_5 - x_3x_5y_4y_5 + x_3x_5y_5 - x_4x_5y_3y_5 + x_4x_5y_3 + x_4x_5y_5 \\ &\quad + x_5^2y_3y_4 - x_5^2y_3 - x_5^2y_4 + x_5^2 + x_5y_4 - x_5, \\ Q_K &= A_Ky_3 + B_K, \quad A_K = x_3D_{245} + x_4x_5E_K, \\ B_K &= x_3^2D_{245} - x_3(x_4(x_4 + y_4)(y_5 - 1) - x_5(x_5 + y_5)(y_4 - 1)) - x_4x_5E_K. \end{aligned}$$

Expanded, Q_P has 13 terms and Q_K has 24 terms, both of degree 4.

Lemma 6.1. Q_P and Q_K are irreducible over $\mathbb{C}$.

Proof. A polynomial $Ay + B$ with A, B free of the variable y is irreducible if A and B have no common factor: a factorisation has one factor free of y , which divides A and B .

Q_P is linear in x_3 with $A = y_5D_{245}$ and $B = x_5C_P$. D_{245} is irreducible (linear in x_4 , with coprime coefficients $y_5 - 1$ and $x_5(y_4 - 1)$). y_5 does not divide B , since $B|_{y_5=0} = x_5(x_4y_3 + x_5y_3y_4 - x_5y_3 - x_5y_4 + x_5 + y_4 - 1) \neq 0$. D_{245} does not divide C_P : substituting $x_4 = x_5(y_4 - 1)/(y_5 - 1)$ into C_P gives x_5 times a rational function of y_3, y_4, y_5 , plus $y_4 - 1$, which is not zero. Since D_{245} is not associate to x_5 , it does not divide B .

Q_K is linear in y_3 with $A = A_K$, $B = B_K$. A_K is linear in y_4 : $A_K = y_4x_5(x_4 - x_3) + R$ with $R = x_3x_4(y_5 - 1) + x_3x_5 + x_4x_5(x_4 - x_5 - y_5)$, and A_K is irreducible because x_5 does not divide R ($R|_{x_5=0} = x_3x_4(y_5 - 1)$) and $x_4 - x_3$ does not divide R ($R|_{x_4=x_3} = x_3(x_3(y_5 - 1) + x_5 + x_5(x_3 - x_5 - y_5))$). Finally A_K does not divide B_K : at $(x_3, x_4, x_5, y_5) = (\frac{1}{2}, \frac{3}{4}, -\frac{2}{5}, \frac{1}{3})$ and $y_4 = -\frac{139}{20}$ we have $A_K = 0$ and $B_K = -\frac{552}{125}$. $\square$

Proposition E. Let G be the prism + 04 or $K_{3,3} + 01$, and let $\hat{\omega}(p)$ be the unique self-stress satisfying (3), defined for generic p . There are no real constants c_e , polygons Π_k with vertices among the p_i and integers $m_{e,k}$ such that $\hat{\omega}_e = c_e \prod_k [\Pi_k]^{m_{e,k}}$ holds on a nonempty open set of positions, for any edge e .

Proof. Suppose the identity holds for an edge e on a nonempty open set U . Every position with p_0, p_1, p_2 not collinear is $A(s)$ for a unique slice point s and a unique affine map A , and $(A, s) \mapsto A(s)$ is a diffeomorphism onto this open set; so there are an affine A and a nonempty open set U_0 of the slice with $A(U_0) \subset U$. By Corollary 2.3(a), $\hat{\omega}_e(As) = \det(A_0)^{-2} \hat{\omega}_e(s)$, while $[\Pi_k](As) = \det(A_0)[\Pi_k](s)$; so $\hat{\omega}_e = c'_e \prod_k [\Pi_k]^{m_{e,k}}$ on U_0 with another constant c'_e . On the nonempty open subset of U_0 where $N(w) \neq 0$ we have $\hat{\omega} = w/N(w)$, with w from Table 2. Both sides are rational functions of $(x_3, \dots, y_5)$, so the identity $w_e/N(w) = c'_e \prod_k [\Pi_k]^{m_{e,k}}$ holds in $\mathbb{R}(x_3, \dots, y_5)$. On the slice every polygon area is a polynomial of degree at most 2 (shoelace formula), not identically zero (an area that vanishes on the slice vanishes everywhere, by the same covariance, and an area that vanishes everywhere cannot occur, since $\hat{\omega}_e \neq 0$ at generic positions by Table 1), so all irreducible factors of the numerator and the denominator of the right side, in lowest terms, have degree at most 2. On the left, Q ($= Q_P$ or Q_K) divides $N(w)$ by (13) and is irreducible of degree 4 (Lemma 6.1); it does not divide w_e , because at the point of Proposition 4.1(b), respectively (c), which lies on the slice, $Q = 0$ and $w_e \neq 0$ for all ten edges. So Q divides the denominator of $w_e/N(w)$ in lowest terms, which contradicts unique factorisation in $\mathbb{R}[x_3, \dots, y_5]$. $\square$

The factorisations (13), and the fact that the reduced denominator of every entry $w_e/N(w)$ contains the quartic factor, were also obtained by exact factorisation over $\mathbb{Q}$ with a computer algebra system in the original computation, and recomputed independently by both verification runs, each with its own polynomial stress; as a control, the same computation for the wheel W_5 finds only factors of degree at most 2, as (2) predicts. On the slice, N factors as $(\deg 1)(\deg 2)(\deg 2)(\deg 4)$ for the prism +04 and as $(\deg 1)^3(\deg 4)$ for $K_{3,3} + 01$; the factors of degree at most 2 are collinearity conditions.

Remark 6.2 (sums of products of areas). Proposition E excludes monomials only. Sums of products of areas do occur: on the slice

$$Q_P = D_{025} - D_{014}D_{025} + D_{024}D_{035} + D_{025}^2 - D_{025}D_{035} - D_{025}D_{045} + D_{035}D_{045}$$

(recall $D_{012} = 1$ there), and the entries of Table 2 for the prism are products of triangle areas, so on the slice $\hat{\omega}$ of the prism +04 is a quotient of polynomials in triangle areas, with a sum in the denominator. We have not looked for such an expression for Q_K . By Corollary 2.3(a), $\hat{\omega}$ is invariant under affine maps of determinant one; we do not pursue the invariant-theoretic consequences. Whether a rational expression of this kind is a formula in the sense of the problem is a matter of interpretation, and we leave the closed-formula reading of Question 1 open.

7 Computations and reproducibility

The proofs use computation only for finite exact statements: the certificates of Table 1, the positions of Proposition 4.1, the identities (13) and the witnesses in the proofs of Lemma 6.1 and Proposition E. All of them are checked by `check_note.py` in the source archive, which uses exact rational arithmetic and only the Python standard library and runs in about a second. It also checks Proposition 2.4 and corrections (a)–(c) for $n = 3, \dots, 8$, Lemma 2.1 as polynomial identities, Theorem A on 175 stresses of 48 random frameworks (a quarter of them with collinear or coincident points), Theorem A_h on 71 random cases, the liftings and volumes of Section 5 (including $K_{3,3} + 01$ subdivided at its four crossings and K_4 against W_4), and the area expression of Q_P . The programs of the original computation (`original/`, with `sympy` and `networkx`) include the factorisations of Section 6 over $\mathbb{Q}$ and a test of Theorem C on the 149 graphs of the `networkx` graph atlas with $n \leq 7$ vertices and $2n - 2$ edges, which is corroboration only and not part of any proof. The programs of the first two verification runs are in `verification_run_A/` and `verification_run_B/`, those of the verification run of the written note in `independent_run_2/`.

Verification. The result was first obtained, with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each read the sources and wrote its own programs. The first run compared the statement with the report and with [RSS03], re-derived all proofs line by line (Theorem A with a proof of its own), and checked in exact arithmetic Rote’s formula with both signs and conventions, Theorem A with degenerate positions, the certificates and their twenty values, Proposition 4.1, Theorem A_h , the lifting of the planar certificate and the factorisations of Section 6, with its own programs and polynomial stress. The second run checked Theorem A on 778 stresses of 120 random frameworks, Rote’s formula, the certificates, and Theorem B and Corollary 5.1 with its own face tracing on 187 stresses of plane frameworks and on 80 stresses of drawings with 682 crossing points in all, with negative controls; it recomputed the factorisations, found the area expression of Q_P , re-ran the original programs with identical outputs, and checked the bibliography with Crossref and zbMATH. Neither run found a mathematical error. Their corrections were made (among them the hypothesis of Corollary 5.1, the half-plane argument in correction (a), the restriction of the pyramid picture to star-shaped rims, the citations for the projective invariance, [CGH23], and Remark 6.2). When the note was written, `check_note.py` was added, and a rendering of p. 722 of the report confirmed correction (a).

A further independent verification run, also AI-assisted, then checked the written note line by line. It read the sources again (the report, with a new rendering of p. 722; [RSS03]; [RSS08, Theorem 5.1 and Section 8]; [Izm17, Section 4.1]; the abstract and introduction of [CGH23]), re-derived every proof, and recomputed with its own programs, in exact arithmetic: corrections (a)–(c) for $n = 3, \dots, 6$; Lemma 2.1 symbolically; Theorem A on 150 stresses of 60 frameworks, degenerate positions included; Theorem A_h on 35 random cases; both certificates and their twenty values from the positions alone, with a mutation test; Proposition 4.1; the limit in the proof of Lemma D; the liftings and volumes of Section 5 (the planar certificate, $K_{3,3} + 01$ subdivided, star-shaped wheels, K_4 against W_4 and W_3 , and 27 random plane frameworks); and the normalised stresses on the slice with the factorisations of Section 6, by a kernel computation over the field of rational functions. It re-ran the whole package from the source archive with identical outputs (up to lines with running times) and checked the bibliography with Crossref and the arXiv interface. It found no mathematical error. Its corrections were made: the uniqueness statement in the abstract and in Section 1.2 (for a general dependent graph uniqueness means a one-dimensional stress space, and generic rigidity only for $2n - 2$ edges), the convention of the constant $N(\omega)$ in the abstract, the exclusion of the one-vertex graph, and the attribution of the projective invariance to Rankine, which is Izvestiev’s. Its programs and outputs are in `independent_run_2/`.

Scope and priority. The note answers Question 1 of Problem 8 of [BKS15] for generic positions, which is how we read its “general position” (Theorem C); Proposition 4.1 shows that “no three points collinear” does not suffice. Question 2 is open-ended; we answer it by Theorems A, A_h and B and Corollary 2.2, and whether they give the meaning Rote intended is a judgment. The closed-formula reading of Question 1 stays open: Proposition E excludes monomials only, and sums of products of areas occur (Remark 6.2). The credits are those of Section 1.3: the questions and the wheel identity are Rote’s, the case of four points and the graph-independent argument for f' are in [RSS03], the projective transfer is classical [Ran63, Izm17], and the Maxwell–Cremona correspondence is cited, not reproved. Only the independence of a, b for general graphs, the constant of Theorem A_h with Lemma D, Theorem C, Theorem B and Proposition E are, as far as we found, not in print; Theorems A and B are elementary and may be folklore.

What was read: pp. 722–723 of [BKS15] in the publisher’s PDF (text and rendering); [RSS03] in the arXiv version v2, to whose numbering we refer (Lemmas 2.4–2.6, Theorem 3.9, Lemma 3.10 with its proof and the remarks on pp. 20–21, Lemma 5.1, and the closing remarks on pp. 36–37), the

book version not being compared; of [RSS08], Sections 5.1 and 8 (around Lemma 8.5) of the arXiv version; of [Izm17], Section 4.1; of [CGH23], the abstract and a text search, which finds no volume or area identity; of [KMMS23, KSS25, KMMS24, ERR99], the abstracts; of [BS15], the title. We did not read [Max64, Whi82, HK92, Ran63, CW94, Ryb99]. The searches were made on 20 October 2026: queries to the arXiv interface (self-stresses with identities, volume, wheels or cohomology; equilibrium stresses and volume; Maxwell–Cremona liftings; pseudo-triangulations and stresses; works of Rote), some refused with HTTP status 429 and retried; keyword searches in zbMATH Open (self-stresses, Maxwell–Cremona, 2015–2026); the 116 works citing [RSS03] in Semantic Scholar, by title; Crossref and zbMATH for the bibliography; and seven web searches. The verification run of the written note repeated, on the same day, queries to the arXiv interface (self-stresses with volume, wheels or identities; Maxwell–Cremona liftings and volume; works of Rote on stresses; rigidity circuits; cokernels), keyword searches in zbMATH Open and one web search, and found nothing new. OpenAlex refused the lists of citing works (HTTP status 429, also in that run), so the four works it counts as citing the report could not be listed, and zbMATH has no citation search. The older rigidity literature (for instance Crapo and Whiteley, and the journal *Structural Topology*) was not checked. No printed answer to the questions was found in these sources; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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