

The Ford-Circle Packing Has Maximum Area: An Answer to a Question of Propp and Kenyon

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Abstract

Two disks of radius 1 centred at $(\pm 1, 1)$ and the x -axis enclose a curvilinear triangle. In the open problem session of the 2015 Oberwolfach workshop on discrete differential geometry, Propp and Kenyon asked whether, among all packings of this triangle by disks that touch the x -axis, the greedy packing has the largest total area. The greedy packing places each new disk in an interstice so that it touches the two disks bounding the interstice. It is a rescaled copy of the Ford circles, and its area is $\pi(\zeta(3)/\zeta(4) - 1) \approx 0.34754$. We prove that the answer is yes. More generally, let A and B be tangent disks resting on a line, and let $\mathcal{G}$ be the greedy packing of the gap between them. For every packing P of this gap by disks resting on the line, and for every $\alpha > 1$, we show that $\sum_{D \in P} r_D^\alpha \leq \sum_{D \in \mathcal{G}} r_D^\alpha$, where r_D is the radius of D . The main step is the same inequality for the weight $1/(e^{1/\sqrt{r}} - 1)$, for which the greedy value of the gap is the product of the weights of A and B . Powers of the radius are superpositions of rescaled copies of this weight. An optimal finite packing contains a chain of tangent disks from A to B , and we bound the value of such a chain by moving two consecutive disks at a time. Along such a move the value is, up to an additive constant, a product of two log-convex functions. The log-convexity reduces to the positivity of an explicit function of two variables whose double power series has non-negative coefficients. We do not discuss uniqueness of the maximizer. This is an unrefereed note.

1 Introduction

Problem 7 of the open problem session of the 2015 Oberwolfach workshop *Discrete Differential Geometry*, posed by Jim Propp and Richard Kenyon and recorded by Rote [Rot15, p. 722], reads as follows. Two disks of radius 1 with centres $(\pm 1, 1)$ enclose, together with the x -axis, a curved triangular region T . One wants to place infinitely many non-overlapping disks that touch the x -axis into T , so that their total area is as large as possible. The question is whether “the greedy method of successively placing each new circle into the interstices”, each new circle touching two previously placed ones, gives the maximum area. The problem is record OWR-13498-011 of the UnsolvedMath dataset [ula26].

A disk *resting on the x -axis* is a closed disk in the closed upper half-plane that is tangent to the x -axis. A *packing* is a family of disks with pairwise disjoint interiors; it is at most countable. A packing of T is a packing by disks resting on the x -axis whose interiors lie in T . In an interstice bounded by the x -axis and two tangent disks resting on it, exactly one disk resting on the axis touches both of them, and it is the largest disk that fits (Lemma 2.4). So the greedy method is unambiguous. It produces the packing $\mathcal{G}^*$ of T by the disks tangent to the axis at $2p/q - 1$ with radius $1/q^2$, for

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coprime integers $0 < p < q$. This is the image under $z \mapsto 2z - 1$ of the Ford circles [For38] at the fractions in $(0, 1)$. Its area is

$$\text{area}(\mathcal{G}^*) = \pi \sum_{q \geq 2} \frac{\varphi(q)}{q^4} = \pi \left(\frac{\zeta(3)}{\zeta(4)} - 1 \right) = 0.3475435 \dots,$$

about 81% of the area $2 - \pi/2$ of T .

Theorem 1.1. *Every packing P of T by disks resting on the x -axis satisfies $\sum_{D \in P} \text{area}(D) \leq \pi(\zeta(3)/\zeta(4) - 1) = \text{area}(\mathcal{G}^*)$. So the greedy packing has maximum area, and the answer to the question of Propp and Kenyon is yes.*

Theorem 1.1 is a special case of the following result. Let A and B be tangent disks resting on the x -axis. A *packing of the gap* (A, B) is a packing by disks resting on the x -axis whose base points lie strictly between those of A and B and whose interiors are disjoint from A and B . (One may think of the packings of the curvilinear triangle bounded by A , B and the axis; for the triangle T this is made precise in Lemma 2.2.) The greedy packing $\mathcal{G}(A, B)$ is defined as above. For a disk D we write r_D for its radius, and for $s > 0$ we put

$$w(s) = \frac{1}{e^{1/s} - 1}.$$

Theorem 1.2. *Let A and B be tangent disks resting on the x -axis, let $\mathcal{G} = \mathcal{G}(A, B)$, and let P be a packing of the gap (A, B) .*

$$(a) \quad \sum_{D \in P} w(\sqrt{r_D}) \leq w(\sqrt{r_A}) w(\sqrt{r_B}) = \sum_{D \in \mathcal{G}} w(\sqrt{r_D}).$$

$$(b) \quad \text{For every } \alpha > 1, \quad \sum_{D \in P} r_D^\alpha \leq \sum_{D \in \mathcal{G}} r_D^\alpha < \infty.$$

For $\alpha \leq 1$ the greedy sum in (b) is infinite. For the two unit disks the right-hand side of (b) is $\zeta(2\alpha - 1)/\zeta(2\alpha) - 1$, and the right-hand side of (a) is $\sum_{q \geq 2} \varphi(q)/(e^q - 1) = 1/(e - 1)^2$. Theorem 1.1 is (b) with $\alpha = 2$ for the two unit disks. Corollary 3.1 extends (a) to all superpositions of rescaled copies of w , and Remark 3.2 shows that some such restriction on the weight is needed.

Idea of the proof. Part (b) follows from part (a) applied to rescaled copies of the gap, because r^α is a superposition of the functions $w(\sqrt{r}/t)$, $t > 0$ (Section 3). For (a) we describe a disk by its base point x and its *size* $s = \sqrt{r}$. Two disks have disjoint interiors if and only if $|x - x'| \geq 2ss'$. The greedy value of a tangent gap of sizes a, b is $w(a)w(b)$, and the Farey step is the identity $w(a)w(b) = w(c)(1 + w(a) + w(b))$ for $1/c = 1/a + 1/b$. If a packing with n disks has maximal weight among the packings with at most n disks, then each of its disks touches a neighbour on each side. So the packing contains a chain of disks from A to B in which consecutive disks are tangent (Section 4). By induction on n , the disks between two consecutive members of the chain contribute at most the greedy value of that gap, and it remains to show that the resulting value Φ of any chain is at most $w(a)w(b)$. We prove this by induction on the length of the chain (Section 5). If two non-consecutive disks of the chain are tangent, the chain splits into two shorter chains. Otherwise we move two consecutive disks and keep all others fixed. Along this one-parameter family, Φ is a constant plus a product of two log-convex functions. So it is maximal at an end of the family, where the chain splits. The log-convexity (Lemma 5.1) is the only analytic input. It reduces to the positivity of an explicit function of two variables whose double power series has non-negative coefficients (Section 6). The weight $w(s) = \sum_{d \geq 1} e^{-d/s}$ turns sums over primitive lattice points (m, n) , which describe the greedy packing, into sums over all lattice points; this is where the arithmetic of the Ford circles enters.

Related work. The greedy packing of a tangent gap is the image of the Ford circles [For38] at the fractions in $(0,1)$ under a real Möbius transformation, and it is organised by the Stern–Brocot tree [GKP94, §4.5]. Without the requirement that the disks touch the line, the greedy (osculatory) packing of the curvilinear triangle is an Apollonian packing. Boyd studied the exponents of osculatory packings [Boy71, Boy72]. Kocik [Koc19] expressed the area of the disks tangent to a given disk of an Apollonian packing through a zeta function, and Rudnick and Zhang [RZ17] determined gap distributions of tangency points in circle packings. Dumitrescu and Tóth [DT15] (Section 3.2 of the arXiv version) use the same greedy packing of the gap between two tangent disks resting on a line, which they identify with the Ford disks, to show that a greedy packing of n disks touching the boundary of a square has total perimeter $\Omega(\log n)$. Aguilar Martín [Agu26] recently studied when greedy packings of nested rings are optimal. These works address different questions. We found no earlier solution of the problem of Propp and Kenyon.

2 Preliminaries

A disk resting on the x -axis with base point x and radius $r > 0$ has centre (x, r) . We write it as $D = (x, s)$, where $s = \sqrt{r}$ is its *size*.

Lemma 2.1. *Two disks (x, s) and (x', s') resting on the x -axis have disjoint interiors if and only if $|x - x'| \geq 2ss'$, and they are tangent if and only if equality holds.*

Proof. With $r = s^2$ and $r' = s'^2$, the condition $(x - x')^2 + (r - r')^2 \geq (r + r')^2$ on the distance of the centres is equivalent to $(x - x')^2 \geq 4rr'$, with equality in one if and only if there is equality in the other. $\square$

A *tangent gap* (A, B) is a pair of tangent disks $A = (x_A, a)$ and $B = (x_B, b)$ resting on the axis with $x_A < x_B$, so that $x_B - x_A = 2ab$. By Lemma 2.1, a disk (x, s) with $x_A < x < x_B$ can belong to a packing of the gap if and only if $x - x_A \geq 2as$ and $x_B - x \geq 2bs$. Adding the two inequalities gives $s \leq ab/(a + b)$. Horizontal translations preserve all these notions, so only the sizes a, b of a tangent gap matter below.

Lemma 2.2. *Let $A^* = (-1, 1)$ and $B^* = (1, 1)$ be the unit disks centred at $(\mp 1, 1)$; they form a tangent gap. Let*

$$T = \{(u, v) : |u| < 1, 0 < v < \beta(u)\}, \quad \beta(u) = 1 - \sqrt{1 - (1 - |u|)^2},$$

the open curvilinear triangle bounded by A^ , B^* and the x -axis. A disk D resting on the x -axis has its interior in T if and only if its base point x satisfies $|x| < 1$ and the interior of D is disjoint from A^* and B^* . Hence the packings of T by disks resting on the x -axis are exactly the packings of the gap (A^*, B^*) .*

Proof. For $0 \leq u < 1$ the vertical line through $(u, 0)$ meets $A^* \cup B^*$ first at height $\beta(u)$, on the boundary of B^* (for $u = 0$ at the common point $(0, 1)$); the case $u < 0$ is symmetric. So T is the part of the strip $|u| < 1$ strictly between the axis and $A^* \cup B^*$, and $T \cap (A^* \cup B^*) = \emptyset$.

Let $D = (x, s)$ with radius $r = s^2$. If the interior of D lies in T , then its centre (x, r) lies in T , so $|x| < 1$, and the interior of D is disjoint from A^* and B^* . Conversely, suppose that $|x| < 1$ and that the interior of D is disjoint from the interiors of A^* and B^* . By Lemma 2.1 the distance between the centres of D and A^* is at least $r + 1$, so the open disk $\text{int } D$ does not meet the closed disk A^* ; the same holds for B^* . Let H be the open upper half-plane and $U = H \setminus (A^* \cup B^*)$. The set T is open and connected. Since β is continuous on $(-1, 1)$ and $\beta(u) \rightarrow 0$ as $|u| \rightarrow 1$, the boundary points of T in H lie on the graph of β , which is contained in $A^* \cup B^*$. So T is closed in U , and hence T is a connected component of U . The connected set $\text{int } D \subseteq U$ contains the points (x, ε) with $0 < \varepsilon < \min\{2r, \beta(x)\}$, which lie in T . Therefore $\text{int } D \subseteq T$. $\square$

2.1 The weight

We put $w(0) = 0$ and, for $s > 0$,

$$w(s) = \frac{1}{e^{1/s} - 1} = \sum_{d \geq 1} e^{-d/s}.$$

The function w is continuous and strictly increasing on $[0, \infty)$ and smooth on $(0, \infty)$.

Lemma 2.3. (i) If $a, b > 0$ and $1/c = 1/a + 1/b$, then $w(a)w(b) = w(c)(1 + w(a) + w(b))$.

(ii) For $p > 1$ and $s > 0$, $\int_0^\infty t^{p-1} w(s/t) dt = \Gamma(p) \zeta(p) s^p$.

Proof. (i) Put $\sigma = e^{1/a}$ and $\tau = e^{1/b}$, so that $e^{1/c} = \sigma\tau$. Then

$$w(c)(1 + w(a) + w(b)) = \frac{1}{\sigma\tau - 1} \cdot \frac{(\sigma - 1)(\tau - 1) + (\sigma - 1) + (\tau - 1)}{(\sigma - 1)(\tau - 1)} = \frac{1}{(\sigma - 1)(\tau - 1)} = w(a)w(b).$$

(ii) We have $w(s/t) = \sum_{d \geq 1} e^{-dt/s}$ and $\int_0^\infty t^{p-1} e^{-dt/s} dt = \Gamma(p)(s/d)^p$. Summing over d is allowed by Tonelli's theorem. $\square$

2.2 The greedy packing

Lemma 2.4. Let (A, B) be a tangent gap with sizes a and b .

(i) Exactly one disk C resting on the axis with base point between x_A and x_B is tangent to both A and B . Its size c satisfies $1/c = 1/a + 1/b$, no disk of a packing of the gap is larger, and (A, C) and (C, B) are tangent gaps.

(ii) The greedy packing $\mathcal{G}(A, B)$, obtained by inserting C and repeating the construction in (A, C) and (C, B) indefinitely, consists of one disk for each pair (m, n) of coprime positive integers, with size s given by $1/s = m/a + n/b$ and base point $x_A + 2ans$, and of no other disks. It is a packing of the gap (A, B) .

(iii) $\sum_{D \in \mathcal{G}(A, B)} w(s_D) = w(a)w(b)$. For $p > 0$,

$$\sum_{D \in \mathcal{G}(A, B)} s_D^p = G_p(a, b) := \sum_{\substack{m, n \geq 1 \\ \gcd(m, n) = 1}} \left(\frac{m}{a} + \frac{n}{b} \right)^{-p},$$

which is finite if and only if $p > 2$. For $p > 2$, $G_p(1, 1) = \sum_{q \geq 2} \varphi(q) q^{-p} = \zeta(p-1)/\zeta(p) - 1$.

Proof. (i) By Lemma 2.1, $C = (x, c)$ is tangent to A and B if and only if $x - x_A = 2ac$ and $x_B - x = 2bc$. Adding gives $2ab = 2c(a + b)$, so $c = ab/(a + b)$, and then x is determined. Every disk of a packing of the gap has size at most $ab/(a + b) = c$, as noted above. The pairs (A, C) and (C, B) are tangent by construction.

(ii) For a pair $v = (m, n) \neq (0, 0)$ of non-negative integers let D_v be the disk resting on the axis with size s_v and base point x_v given by $1/s_v = m/a + n/b$ and $x_v = x_A + 2ans_v$. Then $D_{(1,0)} = A$ and $D_{(0,1)} = B$, since $x_B - x_A = 2ab$. For $v' = (m', n')$ a direct computation gives

$$x_{v'} - x_v = 2(mn' - m'n) s_v s_{v'}.$$

Suppose that $mn' - m'n = 1$. By this identity and Lemma 2.1, $(D_v, D_{v'})$ is a tangent gap. The pairs $(v, v + v')$ and $(v + v', v')$ satisfy the same condition, so $D_{v+v'}$ is tangent to D_v and $D_{v'}$ and its base point lies between theirs; by (i), $D_{v+v'}$ is the disk inserted into the gap $(D_v, D_{v'})$. Starting from $(1, 0)$ and $(0, 1)$, this is the construction of the Stern–Brocot tree by mediants, which produces every pair of coprime positive integers exactly once [GKP94, §4.5]. So $\mathcal{G}(A, B)$ consists of the disks D_v with v a pair of coprime positive integers. If $v \neq v'$ are two such pairs, or if v' is $(1, 0)$ or $(0, 1)$, then $mn' - m'n$ is a non-zero integer. By the identity and Lemma 2.1, D_v and $D_{v'}$ then have disjoint interiors, and they are tangent if and only if $|mn' - m'n| = 1$. Moreover $x_v - x_A = 2nas_v > 0$ and $x_B - x_v = 2mbs_v > 0$. Hence $\mathcal{G}(A, B)$ is a packing of the gap (A, B) .

(iii) By (ii),

$$\sum_{D \in \mathcal{G}(A, B)} w(s_D) = \sum_{\gcd(m, n)=1} \sum_{d \geq 1} e^{-d(m/a+n/b)} = \sum_{M, N \geq 1} e^{-M/a-N/b} = w(a)w(b),$$

because $(d, m, n) \mapsto (dm, dn)$ is a bijection onto the pairs of positive integers. The formula for $\sum s_D^p$ is immediate from (ii). Since $(m+n)/\max(a, b) \leq m/a + n/b \leq (m+n)/\min(a, b)$, finiteness does not depend on a, b , and we may take $a = b = 1$. The pairs with $m+n = q \geq 2$ are the $(m, q-m)$ with $\gcd(m, q) = 1$, so there are $\varphi(q)$ of them and $G_p(1, 1) = \sum_{q \geq 2} \varphi(q)q^{-p}$. This series converges for $p > 2$ because $\varphi(q) < q$, and it diverges for $p \leq 2$ because $\varphi(q)/q^2 \geq 1/(2q)$ for primes q . For $p > 2$, the identity $\sum_{d|n} \varphi(d) = n$ gives $\zeta(p) \sum_{q \geq 1} \varphi(q)q^{-p} = \sum_{n \geq 1} n^{1-p} = \zeta(p-1)$ [HW08, Ch. XVII]; the term $q = 1$ equals 1. $\square$

For A^*, B^* the disk $D_{(m, n)}$ has radius $1/(m+n)^2$ and base point $2n/(m+n) - 1$; this is the description of $\mathcal{G}^*$ in the introduction. With $p = 4$ and $s^4 = r^2$, Lemma 2.4(iii) gives $\text{area}(\mathcal{G}^*) = \pi G_4(1, 1) = \pi(\zeta(3)/\zeta(4) - 1)$.

3 Reduction to the weight w

In this section we derive Theorem 1.2(b), Theorem 1.1 and a more general statement from Theorem 1.2(a), which is proved in Sections 4 to 6. For $t > 0$ the homothety $(u, v) \mapsto (u/t^2, v/t^2)$ maps a disk (x, s) resting on the axis to the disk $(x/t^2, s/t)$. It preserves the condition of Lemma 2.1, so it maps a packing of a tangent gap of sizes a, b to a packing of a tangent gap of sizes $a/t, b/t$. Applying Theorem 1.2(a) to the image gives, for every packing P of a tangent gap (A, B) and every $t > 0$,

$$\sum_{D \in P} w(s_D/t) \leq w(a/t)w(b/t). \quad (1)$$

Proof of Theorem 1.2(b), assuming (a). Let $p = 2\alpha > 2$, so that $r_D^\alpha = s_D^p$. Multiply (1) by t^{p-1} and integrate over $t \in (0, \infty)$. By Tonelli's theorem and Lemma 2.3(ii) the left-hand side becomes $\Gamma(p)\zeta(p) \sum_{D \in P} s_D^p$. On the right-hand side, expand $w(a/t)w(b/t) = \sum_{M, N \geq 1} e^{-t(M/a+N/b)}$ and integrate termwise:

$$\int_0^\infty t^{p-1} w(a/t) w(b/t) dt = \Gamma(p) \sum_{M, N \geq 1} \left(\frac{M}{a} + \frac{N}{b} \right)^{-p} = \Gamma(p) \zeta(p) G_p(a, b),$$

where we wrote $(M, N) = d(m, n)$ with $d \geq 1$ and $\gcd(m, n) = 1$. Dividing by $\Gamma(p)\zeta(p)$ and using Lemma 2.4(iii) gives $\sum_{D \in P} s_D^p \leq G_p(a, b) = \sum_{D \in \mathcal{G}} s_D^p < \infty$. $\square$

Proof of Theorem 1.1. By Lemma 2.2, the packings of T by disks resting on the axis are the packings of the gap (A^*, B^*) , and $\mathcal{G}^* = \mathcal{G}(A^*, B^*)$. Since $\text{area}(D) = \pi r_D^2$, the claim is Theorem 1.2(b) with $\alpha = 2$, together with $\sum_{D \in \mathcal{G}^*} r_D^2 = G_4(1, 1) = \zeta(3)/\zeta(4) - 1$. $\square$

Corollary 3.1. *Let μ be a positive Borel measure on $(0, \infty)$, and put $W(s) = \int_0^\infty w(s/t) d\mu(t) \in [0, \infty]$ for $s > 0$. Then every packing P of a tangent gap (A, B) satisfies*

$$\sum_{D \in P} W(s_D) \leq \int_0^\infty w(a/t) w(b/t) d\mu(t) = \sum_{D \in \mathcal{G}(A, B)} W(s_D).$$

Proof. Integrate (1) against μ and use Tonelli's theorem. For the equality, Lemma 2.4(ii) gives

$$\sum_{D \in \mathcal{G}(A, B)} w(s_D/t) = \sum_{\gcd(m, n)=1} w\left(\frac{1}{m t/a + n t/b}\right),$$

which equals $w(a/t)w(b/t)$ by the computation in the proof of Lemma 2.4(iii) with $a/t, b/t$ in place of a, b . $\square$

The point mass $\mu = \delta_1$ gives Theorem 1.2(a), and $d\mu(t) = t^{p-1}dt/(\Gamma(p)\zeta(p))$ gives $W(s) = s^p$ by Lemma 2.3(ii).

Remark 3.2. The greedy packing does not maximize every increasing weight. Let $W(s) = 1$ for $s \geq 1/3$ and $W(s) = 0$ otherwise. The four disks of radius $1/9$ (size $1/3$) with base points $-1/3, -1/9, 1/9, 1/3$ form a packing of the gap (A^*, B^*) by Lemma 2.1: consecutive base points differ by $2/9 = 2 \cdot \frac{1}{3} \cdot \frac{1}{3}$, and the outer ones are at distance $2/3 = 2 \cdot 1 \cdot \frac{1}{3}$ from ∓ 1 . This packing has weight 4. The greedy packing $\mathcal{G}^*$ contains only the $\varphi(2) + \varphi(3) = 3$ disks of size $1/q \geq 1/3$.

4 Chains, and the proof of Theorem 1.2(a)

Definition 4.1. Let (A, B) be a tangent gap with sizes a, b , and let $k \geq 1$. A k -chain in (A, B) is a sequence $A = C_0, C_1, \dots, C_k, C_{k+1} = B$ of disks resting on the axis, with base points $x_0 < x_1 < \dots < x_{k+1}$ and pairwise disjoint interiors, such that C_i and C_{i+1} are tangent for $0 \leq i \leq k$.

Let s_i be the size of C_i , so $s_0 = a$ and $s_{k+1} = b$. By Lemma 2.1, the tangencies mean $x_{i+1} - x_i = 2s_i s_{i+1}$. Hence $x_j - x_i = 2 \sum_{l=i}^{j-1} s_l s_{l+1}$ for $i < j$, so the base points are determined by the sizes, and the vector $s = (s_1, \dots, s_k)$ satisfies

$$\sum_{l=0}^k s_l s_{l+1} = s_0 s_{k+1}, \tag{2}$$

$$\sum_{l=i}^{j-1} s_l s_{l+1} \geq s_i s_j \quad \text{for } 0 \leq i < j \leq k+1, \quad j \geq i+2, \quad (i, j) \neq (0, k+1). \tag{3}$$

Condition (2) says $x_{k+1} - x_0 = 2ab$. Conditions (3) express the disjointness of non-consecutive members; for the pair $(0, k+1)$ it holds with equality by (2). Conversely, every $s \in (0, \infty)^k$ satisfying (2) and (3) defines a unique k -chain in (A, B) , with $x_i = x_A + 2 \sum_{l < i} s_l s_{l+1}$. Let $\mathcal{C}_k(a, b)$ be the set of these s (all $s_i > 0$). We call $s \in \mathcal{C}_k(a, b)$ *degenerate* if equality holds in (3) for some pair (i, j) , that is, if two non-consecutive members of the chain, other than the pair A, B , are tangent. The *value* of $s \in \mathcal{C}_k(a, b)$ is

$$\Phi(s) = \sum_{i=1}^k w(s_i) + \sum_{i=0}^k w(s_i) w(s_{i+1}).$$

By Lemma 2.4(iii) this is the total weight of the chain together with the greedy packings of the $k+1$ tangent gaps (C_i, C_{i+1}) .

Proposition 4.2 (chain inequality). *For all $a, b > 0$, $k \geq 1$ and $s \in \mathcal{C}_k(a, b)$ we have $\Phi(s) \leq w(a)w(b)$.*

Proposition 4.2 is proved in Section 5.

Proof of Theorem 1.2(a), assuming Proposition 4.2. The equality is Lemma 2.4(iii). For $n \geq 0$ and $a, b > 0$ let $V_n(a, b)$ be the supremum of $\sum_{D \in P} w(s_D)$ over all packings P with at most n disks of a tangent gap with sizes a, b . We show $V_n(a, b) \leq w(a)w(b)$ by induction on n , for all a, b at once. Clearly $V_0 = 0$. Let $n \geq 1$, and fix a tangent gap (A, B) with sizes a, b . Put $c = ab/(a + b)$.

A maximizer exists. Let K_n be the set of tuples $((x_i, s_i))_{i=1}^n$ in $([x_A, x_B] \times [0, c])^n$ with $x_i - x_A \geq 2as_i$, $x_B - x_i \geq 2bs_i$ and $|x_i - x_j| \geq 2s_i s_j$ for $i \neq j$. It is compact. For a point of K_n , the pairs with $s_i > 0$ are distinct disks, because their base points differ, and by Lemma 2.1 they form a packing of the gap. Every packing of the gap with at most n disks arises in this way, after padding with pairs $(x_A, 0)$. Since w is continuous and $w(0) = 0$, the function $\sum_i w(s_i)$ attains its maximum on K_n , so there is a packing P^* with at most n disks and weight $V_n(a, b)$. If P^* has fewer than n disks, then $V_n(a, b) \leq V_{n-1}(a, b) \leq w(a)w(b)$ by induction. So assume that P^* has exactly n disks.

Every disk of P^ is tangent to a neighbour on each side.* Let $D = (x_D, s_D) \in P^*$. We claim that some $E \in P^* \cup \{B\}$ with $x_E > x_D$ is tangent to D . If not, let $\eta > 0$ be the minimum of the slacks $x_E - x_D - 2s_D s_E$ over the finitely many $E \in P^* \cup \{B\}$ with $x_E > x_D$. Let S be the largest size among the members of $P^* \cup \{A, B\}$, and let $0 < \delta < \eta/2$. Replace D by $D' = (x_D + \delta, s_D + \delta/(2S))$. For $E \in P^* \cup \{A\}$ with $x_E < x_D$ we have $(x_D + \delta) - x_E \geq 2s_D s_E + \delta \geq 2(s_D + \delta/(2S))s_E$, since $s_E \leq S$. For $E \in P^* \cup \{B\}$ with $x_E > x_D$ we have $x_E - (x_D + \delta) \geq 2s_D s_E + \eta - \delta > 2(s_D + \delta/(2S))s_E$. In particular $x_A < x_D + \delta < x_B$. By Lemma 2.1 the new family is a packing of the gap with n disks, and its weight is larger because w is strictly increasing. This contradicts the maximality of P^* . For the left side, move D to $(x_D - \delta, s_D + \delta/(2S))$ instead, with η the minimal slack over the $E \in P^* \cup \{A\}$ with $x_E < x_D$.

A chain through P^ .* Start from any $D \in P^*$ and repeatedly pass to a member of $P^* \cup \{A\}$ to the left that is tangent to the current disk. The base points decrease strictly and P^* is finite, so the process stops, and it can only stop at A . Doing the same to the right ends at B . The disks visited form a k -chain in (A, B) with $1 \leq k \leq n$; its members have pairwise disjoint interiors, since $P^* \cup \{A, B\}$ is a packing. Let $s \in \mathcal{C}_k(a, b)$ be its size vector.

The remaining disks. Each of the $n - k$ disks of P^* that are not in the chain has its base point in exactly one of the intervals (x_i, x_{i+1}) , $0 \leq i \leq k$. The disks with base point in (x_i, x_{i+1}) form a packing of the tangent gap (C_i, C_{i+1}) with at most $n - 1$ disks, so by the induction hypothesis their weight is at most $w(s_i)w(s_{i+1})$. Summing over i and using Proposition 4.2,

$$V_n(a, b) = \sum_{D \in P^*} w(s_D) \leq \sum_{i=1}^k w(s_i) + \sum_{i=0}^k w(s_i)w(s_{i+1}) = \Phi(s) \leq w(a)w(b).$$

This completes the induction. Finally, the weight of a countable packing is the supremum of the weights of its finite subfamilies, which are packings of the same gap. $\square$

5 The chain inequality

The following lemma is proved in Section 6.

Lemma 5.1 (key lemma). *For every $\rho > 0$ the function $f_\rho(\lambda) = 1 + w(e^\lambda - \rho) + w(\rho)$ is log-convex on $(\log \rho, \infty)$, that is, $\log f_\rho$ is convex there.*

Proof of Proposition 4.2. We use induction on k , for all a, b at once. Throughout, $s_0 = a$ and $s_{k+1} = b$.

The case $k = 1$. Condition (3) is empty, and (2) reads $s_1(a + b) = ab$. So $1/s_1 = 1/a + 1/b$, and $\Phi(s) = w(s_1) + w(a)w(s_1) + w(s_1)w(b) = w(a)w(b)$ by Lemma 2.3(i).

Now let $k \geq 2$, assume the claim for all $k' < k$, and let $s \in \mathcal{C}_k(a, b)$.

Degenerate chains. Suppose equality holds in (3) for the pair (i, j) , so that C_i and C_j are tangent. The *inner* chain $C_i, C_{i+1}, \dots, C_j$ is a $(j - i - 1)$ -chain in the tangent gap (C_i, C_j) . The *outer* chain $C_0, \dots, C_i, C_j, \dots, C_{k+1}$ is a $(k - j + i + 1)$ -chain in (A, B) . For the outer chain, (2) follows from $s_i s_j = \sum_{l=i}^{j-1} s_l s_{l+1}$, and its conditions (3) are among those of s , because the base points do not change. Since $j \geq i + 2$ and $(i, j) \neq (0, k + 1)$, both chains have between 1 and $k - 1$ intermediate members. Their values satisfy

$$\Phi(s) = \Phi(\text{outer}) + \Phi(\text{inner}) - w(s_i)w(s_j),$$

because the two chain values together contain every term of $\Phi(s)$ exactly once, and in addition the term $w(s_i)w(s_j)$ of the outer chain, which belongs to its consecutive pair (C_i, C_j) . By the induction hypothesis $\Phi(\text{inner}) \leq w(s_i)w(s_j)$ and $\Phi(\text{outer}) \leq w(a)w(b)$, so $\Phi(s) \leq w(a)w(b)$. This holds at every degenerate point of $\mathcal{C}_k(a, b)$.

Non-degenerate chains: the two-disk move. Suppose that s is not degenerate. We keep C_0 and $C_3, \dots, C_{k+1}$ fixed and move C_1 and C_2 . Let $L = s_0 s_1 + s_1 s_2 + s_2 s_3$ and $K = L + s_0 s_3 = (s_1 + s_3)(s_2 + s_0)$. For θ in the open interval $I = (\log s_3, \log(K/s_0))$ put

$$s_1(\theta) = e^\theta - s_3, \quad s_2(\theta) = K e^{-\theta} - s_0.$$

Both are positive on I , and $s_0 s_1(\theta) + s_1(\theta) s_2(\theta) + s_2(\theta) s_3 = (s_1(\theta) + s_3)(s_2(\theta) + s_0) - s_0 s_3 = L$. Let $s(\theta)$ be the vector obtained from s by replacing s_1, s_2 with $s_1(\theta), s_2(\theta)$. Then $s(\theta^*) = s$ for $\theta^* = \log(s_1 + s_3)$. Along the curve, condition (2) and the conditions (3) for pairs (i, j) with $\{i, j\} \cap \{1, 2\} = \emptyset$ do not change: each such sum either contains all three terms of L (if $i = 0$ and $j \geq 3$) or none of them (if $i \geq 3$), and $s_i s_j$ does not involve s_1, s_2 . Let

$$F = \{\theta \in I : s(\theta) \in \mathcal{C}_k(a, b)\},$$

a closed subset of I . As $\theta \downarrow \log s_3$ we have $s_1(\theta) \rightarrow 0$ and $s_2(\theta) \rightarrow L/s_3 > 0$, so the condition $s_0 s_1 + s_1 s_2 \geq s_0 s_2$ for the pair $(0, 2)$ fails near the left end of I . As $\theta \uparrow \log(K/s_0)$ we have $s_2(\theta) \rightarrow 0$ and $s_1(\theta) \rightarrow L/s_0 > 0$, so the condition $s_1 s_2 + s_2 s_3 \geq s_1 s_3$ for the pair $(1, 3)$ fails near the right end. Both pairs occur in (3) because $k \geq 2$. Hence F is a compact subset of I , and the connected component of F containing θ^* is an interval $J = [\theta_-, \theta_+]$. If all conditions (3) involving index 1 or 2 were strict at θ_+ , they would remain strict near θ_+ by continuity; since the other conditions do not depend on θ , a neighbourhood of θ_+ would lie in F , contradicting the choice of θ_+ . So one of these conditions holds with equality at θ_+ . It concerns a pair $(i, j) \neq (0, k + 1)$, so $s(\theta_+)$ is degenerate. In the same way $s(\theta_-)$ is degenerate. By the degenerate case, $\Phi(s(\theta_\pm)) \leq w(a)w(b)$.

Along the curve, $\Phi(s(\theta)) - h(\theta)$ is constant, where

$$\begin{aligned} h(\theta) &= w(s_1) + w(s_2) + w(s_0)w(s_1) + w(s_1)w(s_2) + w(s_2)w(s_3) \\ &= (1 + w(s_1) + w(s_3))(1 + w(s_2) + w(s_0)) - (1 + w(s_0))(1 + w(s_3)), \end{aligned}$$

with $s_1 = s_1(\theta)$ and $s_2 = s_2(\theta)$. Here $1 + w(s_1(\theta)) + w(s_3) = f_{s_3}(\theta)$ and $1 + w(s_2(\theta)) + w(s_0) = f_{s_0}(\log K - \theta)$, where $\log K - \theta = \log(s_2(\theta) + s_0)$ lies in $(\log s_0, \infty)$. By Lemma 5.1 both factors are positive and log-convex functions of θ on I . Their product is log-convex, and hence convex, so h is convex on I . A convex function on J is bounded by its values at the end points, so

$$\Phi(s) = \Phi(s(\theta^*)) \leq \max\{\Phi(s(\theta_-)), \Phi(s(\theta_+))\} \leq w(a)w(b). \quad \square$$

6 Proof of the key lemma

Let $\rho > 0$ and $\lambda > \log \rho$, and put $s = e^\lambda - \rho > 0$ and $f = f_\rho$. Differentiating in λ gives $f' = e^\lambda w'(s)$ and $f'' = e^\lambda w'(s) + e^{2\lambda} w''(s)$. Since $e^\lambda = s + \rho$,

$$f f'' - f'^2 = e^\lambda \Psi(s, \rho), \quad \Psi(s, \rho) = (1 + w(s) + w(\rho))(w'(s) + (s + \rho)w''(s)) - (s + \rho)w'(s)^2.$$

So it suffices to show that $\Psi(s, \rho) > 0$ for all $s, \rho > 0$.

Put $q = 1/s$, $x = 1/\rho$, $E = e^q$ and $X = e^x$, so $E, X > 1$; in this section x does not denote a base point. Then $w(s) = 1/(E - 1)$ and $w(\rho) = 1/(X - 1)$. Since $d/ds = -q^2 d/dq$ and $dE/dq = E$,

$$w'(s) = \frac{q^2 E}{(E - 1)^2}, \quad w''(s) = \frac{q^3 E (q(E + 1) - 2(E - 1))}{(E - 1)^3}.$$

With $s + \rho = (q + x)/(qx)$ and $1 + w(s) + w(\rho) = (E(X - 1) + (E - 1))/((E - 1)(X - 1))$ one finds

$$w'(s) + (s + \rho)w''(s) = \frac{q^2 E P}{x(E - 1)^3}, \quad (s + \rho)w'(s)^2 = \frac{q^3 (q + x)E^2}{x(E - 1)^4},$$

where $P = x(E - 1) + (q + x)(q(E + 1) - 2(E - 1))$. Therefore

$$\Psi(s, \rho) = \frac{q^2 E \Omega}{(E - 1)^4 x (X - 1)}, \quad \Omega = (E(X - 1) + (E - 1))P - q(q + x)E(X - 1).$$

Now $P = x(q(E + 1) - (E - 1)) + q(q(E + 1) - 2(E - 1))$ and $P - q(q + x) = x(qE - (E - 1)) + q(qE - 2(E - 1))$. Writing $\Omega = E(X - 1)(P - q(q + x)) + (E - 1)P$, we obtain

$$\begin{aligned} \Omega &= (X - 1)E [x(qE - (E - 1)) + q(qE - 2(E - 1))] \\ &\quad + (E - 1) [x(q(E + 1) - (E - 1)) + q(q(E + 1) - 2(E - 1))], \end{aligned}$$

that is,

$$\Omega = x(X - 1)U_1 + (X - 1)U_2 + xU_3 + U_4,$$

where, as functions of q ,

$$\begin{aligned} U_1 &= E(qE - E + 1) = qe^{2q} - e^{2q} + e^q, \\ U_2 &= qE(qE - 2E + 2) = q^2 e^{2q} - 2qe^{2q} + 2qe^q, \\ U_3 &= (E - 1)(qE + q - E + 1) = qe^{2q} - e^{2q} + 2e^q - q - 1, \\ U_4 &= q(E - 1)(qE + q - 2E + 2) = q^2 e^{2q} - 2qe^{2q} + 4qe^q - q^2 - 2q. \end{aligned}$$

Taylor coefficients. For integers $m \geq j \geq 0$ and real γ , $m! [q^m] q^j e^{\gamma q} = \frac{m!}{(m-j)!} \gamma^{m-j}$. For example, $q^2 e^{2q}$ contributes $m(m-1)2^{m-2}$ and qe^{2q} contributes $m2^{m-1}$. Collecting terms, for $m \geq 2$,

$$\begin{aligned} m! [q^m] U_1 &= (m-2)2^{m-1} + 1, & m! [q^m] U_2 &= m((m-5)2^{m-2} + 2), \\ m! [q^m] U_3 &= (m-2)2^{m-1} + 2, & m! [q^m] U_4 &= m((m-5)2^{m-2} + 4) \quad (m \geq 3), \end{aligned}$$

while $[q^2]U_4 = 0$ and all four coefficients vanish for $m \leq 1$. Since $X - 1 = \sum_{n \geq 1} x^n/n!$ and $x(X - 1) = \sum_{n \geq 2} n x^n/n!$, we have

$$\Omega = \sum_{n \geq 0} \omega_n(q) \frac{x^n}{n!}, \quad \omega_0 = U_4, \quad \omega_1 = U_2 + U_3, \quad \omega_n = nU_1 + U_2 \quad (n \geq 2).$$

For $m \geq 2$ the Taylor coefficients of these functions satisfy the following.

- $m! [q^m] \omega_0 = m((m-5)2^{m-2} + 4)$ for $m \geq 3$, which is 0 for $m = 3, 4$ and positive for $m \geq 5$; and $[q^2] \omega_0 = 0$.
- $m! [q^m] \omega_1 = (m-2)2^{m-1} + 2 + m((m-5)2^{m-2} + 2) = (m+1)((m-4)2^{m-2} + 2)$, which is 0 for $m = 2, 3$ and positive for $m \geq 4$.
- For $n \geq 2$, since $[q^m] U_1 \geq 0$, $m! [q^m] \omega_n \geq m! [q^m] \omega_2 = 2^{m-2}(m^2 - m - 8) + 2(m+1)$, which is 0 for $m = 2$, equals 4 for $m = 3$, and is positive for $m \geq 4$.

So every ω_n is an entire function of q with non-negative Taylor coefficients, and $[q^5] \omega_0 = 20/5! = 1/6$. For $q, x > 0$ this gives $\Omega(q, x) \geq \omega_0(q) \geq q^5/6 > 0$, hence $\Psi(s, \rho) > 0$. This proves Lemma 5.1, and with it Proposition 4.2 and Theorem 1.2. $\square$

7 Concluding remarks

The chain inequality is an inequality between elementary functions of finitely many variables, and Lemma 5.1 is the only analytic input. We have not addressed uniqueness of the maximizer, gaps between disjoint disks that are not tangent, or packings whose disks need not touch the line. Remark 3.2 shows that the conclusion depends on the weight; Corollary 3.1 gives the class of weights covered by the method.

Verification. The claims were checked as follows. All proofs are by hand; the computations are cross-checks and are not used in the proofs.

1. A program by the author, `verify_owr13498011.py` (standard library only, about 15 seconds), does the following.
 - In exact rational arithmetic it checks Lemma 2.3(i), the identity for h in Section 5, the formulas for w' and w'' (by formal differentiation in q and E), the polynomial identity between the three expressions for Ω , and the formula for Ψ at 1000 random rational points, with E and X treated as independent variables.
 - It builds the greedy packing of (A^*, B^*) exactly down to size $1/40$, confirms the description of Lemma 2.4(ii) and the pairwise disjointness of these 489 disks, and checks Remark 3.2.
 - It compares the closed forms of the coefficients of $U_1, \dots, U_4$ with exact series expansions for $m \leq 160$, and finds no negative coefficient of Ω for $m, n \leq 160$.
 - Numerically (Decimal or double precision) it checks $\sum_{\gcd(m,n)=1} w(1/(m/a + n/b)) = w(a)w(b)$ for three gaps, the value $\zeta(3)/\zeta(4) - 1$, the second derivative of $\log f_\rho$ at 300 points, the convexity of h on 400 random two-disk moves, the chain inequality on 625 random chains ($k \leq 6$) after hill climbing, and the integral in the proof of Theorem 1.2(b) for $p = 4$.

A second program, `bose_chain_opt.py` (numpy and scipy), maximizes $\Phi(s)/(w(a)w(b))$ over k -chains with SLSQP for $k \leq 6$ and six gaps with a, b between 0.02 and 40. The maxima are attained at Farey chains, where $\Phi = w(a)w(b)$. Excesses of order 10^{-9} occur only at points that violate a constraint by a similar amount, within the solver's tolerance. The program also evaluates 900 random recursive packings (chains inside chains) without finding a violation.

2. The finder's programs check the coefficient formulas and the non-negativity of the coefficients of Ω in exact arithmetic up to degree 80, and, in double precision, the formula for Ψ , the integral representation of the greedy value, the convexity along two-disk moves and the chain inequality for random chains, for the power weights s^p with several p .

3. Two independent verifiers checked all proofs by hand, in an earlier version of the argument that uses Lemma 2.3(ii) inside the convexity argument rather than at the end. They fetched the source anonymously and wrote their own code: exact tests of the formula for Ψ (400 and 500 random points); exact expansions of Ω , one from its bracket form, with no negative coefficient up to degree 120 in q and 60 in x ; finite differences of $\log f_\rho$ with 80 to 200 digits; the greedy value by Hurwitz zeta sums and by brute force; convexity on random two-disk moves for $2.1 \leq p \leq 8$; and local maximization of the chain value (SLSQP, Nelder–Mead, random two-disk moves) for $k \leq 7$, several gaps and $p \in \{2.02, 2.5, 4, 6, 12\}$. No chain value exceeded the greedy value by more than numerical errors of relative size about 10^{-11} . In most runs the optimizer converged to a Farey chain, where the chain value equals the greedy value. They also reran the finder’s scripts and reproduced the recorded outputs.
4. An independent referee checked all proofs of the previous version of this note line by line and found no mathematical error. The referee’s programs, written from the text before the other programs were read, are in the folder `referee/referee2` of the release.
 - In exact arithmetic they derive Ψ from its definition, confirm every displayed formula of Section 6, and find no negative coefficient $m!n! [q^m x^n] \Omega$ for $m, n \leq 400$.
 - They confirm the base points of the greedy disks and their pairwise disjointness in six rational gaps, and they check the identities and values of Sections 2 and 3, exactly or numerically.
 - In 1225 runs of SLSQP over k -chains ($2 \leq k \leq 8$, seven gaps with sizes between 0.05 and 100), each output was re-evaluated in 50-digit arithmetic after (2) was restored exactly. No chain satisfying (3) had a value above $w(a)w(b)$. Numerical maximization also found no violation of a large-scale limit of Proposition 4.2, or of the chain inequality for the area weight.
 - The referee also reran the author’s two programs and five of the finder’s programs and reproduced their recorded outputs.

After the referee’s report, and as the referee suggested, we added the proof that $\mathcal{G}(A, B)$ is a packing in Lemma 2.4(ii) and the description of the greedy packing by a Möbius transformation in the related-work paragraph. The two formulas of that proof are among the referee’s exact checks, and both additions were checked in exact arithmetic by a further program of the author, `revision2_checks.py`.

Scope and priority. Theorem 1.1 answers Problem 7 of [Rot15] as posed, and the question as stated in the dataset record [ula26]. Some text extractions of the source PDF drop or garble the sign $\pm$ in the centres; the rendered page shows $(\pm 1, 1)$, and this is the only reading in which two disks and the axis enclose a curved triangle. In any case Theorem 1.2 covers every tangent gap. We have not proved that the greedy packing is the only maximizer. We searched arXiv, Crossref, zbMATH, OpenAlex, OpenCitations and the web. We found no statement or solution of the problem apart from the source; the closest works [Boy71, Boy72, Koc19, RZ17, DT15, Agu26] address different questions. The four works that OpenAlex lists as citing the report (on defects in digital images, on response maps of alternating-current networks, on T-splines, and on developable approximation of surfaces; the last is also the only one listed by OpenCitations) do not concern this problem. The referee repeated these searches independently and found no statement or solution of the problem either. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the proof, to write and run the verification programs, and to help prepare this manuscript. The author has

reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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