

Dihedral Angles Do Not Determine Face Angles of Nonconvex Polyhedra: An Explicit Pair of Triangulated Bipyramids

Alper Ferudun*

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Abstract

At the 2015 Oberwolfach workshop on discrete differential geometry N. Amenta asked whether Stoker’s conjecture extends to nonconvex polytopes: for a polytope with given combinatorics, do the dihedral angles, all different from 0 and π , determine the face angles? One may assume that the polytope is triangulated and homeomorphic to a sphere (Oberwolfach Report 13/2015, Problem 4). We show by one explicit example that the answer is no. Two embedded nonconvex polyhedra P and P' realize the same labelled triangulation of the sphere, the boundary of the triangular bipyramid. No two adjacent faces are coplanar, all nine interior dihedral angles of P' are equal to those of P (one edge is reflex in both, with angle about 273.82°), and twelve of the eighteen face angles differ. The coordinates lie in a real quadratic field, and the proof is an exact finite certificate, checked by a short program in rational arithmetic, together with elementary lemmas. The pair also refutes, for this mesh, the conjecture of Amenta and Rojas (2018, restated in the journal version of 2020) that a vector of dihedral angles is consistent with at most one set of inner face angles of an embedded triangle mesh. For convex polyhedra the answer is yes, by a theorem in a preprint of Wang and Xie, and by that theorem no pair of the kind found here can have a convex member. We claim one explicit pair only: no family of pairs and nothing on continuous deformations with constant dihedral angles; the offset question of Rote, merged with Amenta’s question in the problem record, is not addressed. The papers of Stoker and of Karcher (1968), the closest prior sources, and the later papers of Milka (1970) and Pogorelov (2002) on Stoker’s problem could not be read. No printed example was found in the sources searched; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The question

The open problems of the Oberwolfach workshop on discrete differential geometry of March 2015, collected by G. Rote, contain the following problem of N. Amenta [BKS15, p. 721, Problem 4], titled “Are face angles determined by dihedral angles?” Stoker’s conjecture says that, for a convex 3-polytope with given combinatorics, the dihedral angles, if all of them are specified and different from 0 and π , determine all face angles. The question is whether this is also true for a non-convex polytope; the source adds that one may assume the polytope to be triangulated and homeomorphic to a sphere. In the UnsolvedMath collection this problem is part of record OWR-13498-009 [ula26], which merges it with the next problem of the same page, G. Rote’s question on the existence of offset polytopes [BKS15, p. 721, Problem 5]. This note treats only Amenta’s question.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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1.2 Known results

The convex statement is Stoker’s conjecture; according to Pak [Pak10, §26.9] it is stated in [Sto68, §6]. Wang and Xie gave a proof for convex Euclidean polyhedra in all dimensions [WX22, Theorem 1.6]: two convex polyhedra of the same combinatorial type with equal corresponding dihedral angles have equal corresponding face angles. Their paper is an arXiv preprint; we found no journal version (Crossref and OpenAlex, 20 October 2026). Earlier, Karcher verified the conjecture “for a class of 3-dimensional convex polyhedra with 5 vertices and 6 faces”, as Wang and Xie describe [Kar68]; Pak’s hints attribute to the same paper the cases of simple and of circumscribed polytopes [Pak10, Exercise 26.1 and its hint]. Mazzeo and Montcouquiol [MM11, §1] list Karcher’s paper, a paper of Milka [Mil70] and one of Pogorelov [Pog02] among the works on the problem, with results for specific classes of polyhedra. Amenta and Rojas [AR18, §2] describe Pogorelov’s paper as a proof of the conjecture for triangulated convex polyhedra, and Pak attributes to it the statement that a deformation of a simplicial convex polytope that preserves the combinatorics and the dihedral angles is a rigid motion [Pak10, Exercise 32.2 and its hint]. We could not read the papers of Milka and Pogorelov. Polyhedra with 5 vertices and 6 faces are triangular bipyramids (Remark 6.2), the combinatorial type of our example. Infinitesimal versions for convex polyhedra were proved by Mazzeo and Montcouquiol [MM11, Theorem 1] and by Winter [Win26]; Belletti related the conjecture to the volume conjecture [Bel23]. The spherical analogue is false, by examples of Schlenker [Sch00] as reported in [WX22] and [Pak10, §26.9]. Rigidity of nonconvex polyhedra with respect to the dihedral angles *together with* the edge lengths, different data, was studied by Stoker [Sto68, Theorem III] (as described in [CK25, §1.1]), by Biedl and Genç for orthogonal polyhedra [BG11] and by Cho and Kim [CK25].

Amenta and Rojas [AR18, §1] called a deformation that keeps all dihedral angles and changes inner angles a *dihedral flex*, noted that it is not known whether dihedral-flexible polyhedra exist, and proved that a generic immersion of a triangle mesh homeomorphic to the sphere is dihedral infinitesimally rigid and hence dihedral rigid [AR18, Theorem 2 and §3]. They conjecture [AR18, §8, Conjecture 1] that for a given vector of dihedral angles there is at most one set of inner face angles consistent with an embedding of a mesh; the author manuscript of the journal version [AR20] repeats this in its Section 7.

1.3 The result

Let $\mathcal{K}$ be the boundary complex of the triangular bipyramid, with vertices y, x, a_1, a_2, a_3 , the nine edges $xa_k, ya_k, a_k a_l$, and the six triangles

$$xa_1a_2, \quad xa_2a_3, \quad xa_3a_1, \quad ya_1a_2, \quad ya_2a_3, \quad ya_3a_1.$$

Polyhedra, interior dihedral angles and face angles are defined in Section 2.

Theorem 1.1. *There are two polyhedra P and P' in $\mathbb{R}^3$ whose boundaries are embeddings of $\mathcal{K}$, linear on each triangle, with the following properties.*

- (a) *Each of P, P' is homeomorphic to a ball, its boundary is an embedded polyhedral sphere, and no two adjacent faces are coplanar; so no interior dihedral angle equals 0 or π .*
- (b) *Both are nonconvex: the edge a_2a_3 is reflex in both, with interior dihedral angle about 273.8239° ; the other eight edges are convex.*
- (c) *At each of the nine edges the interior dihedral angles of P and P' are equal.*
- (d) *Twelve of the eighteen face angles of P' differ from the corresponding face angles of P , namely all angles at a_1, a_2, a_3 ; for example, in the triangle xa_1a_2 the angle at a_2 is about 124.64° in P and 113.90° in P' . The six angles at x and y agree.*

The coordinates of P and P' lie in $\mathbb{Q}(\sqrt{D})$ for an explicit 55-digit integer D that is not a square (Section 4).

Corollary 1.2. *The answer to Amenta’s question [BKS15, p. 721, Problem 4] is negative: for non-convex polytopes with given combinatorics, even triangulated ones homeomorphic to a sphere with five vertices, the dihedral angles (all different from 0 and π) do not determine the face angles.*

Corollary 1.3. *The conjecture of Amenta and Rojas [AR18, §8, Conjecture 1], restated in [AR20, §7], fails for the mesh $\mathcal{K}$: the dihedral vector of P is consistent with two different sets of inner face angles of embeddings of $\mathcal{K}$.*

Amenta and Rojas measure the dihedral angle at an edge as the angle of the rotation about the edge that takes one adjacent normal to the other [AR18, §6]; for polyhedra with the same orientation of the labelled triangles, as P and P' (check [C5] in Section 4), equal interior angles give equal rotation angles. Corollary 1.3 is a consequence of Theorem 1.1, not a priority argument: the conjecture suggests that no example was known to its authors in 2018, but this is circumstantial and is not used here as evidence of novelty.

1.4 What is claimed and what is not

We claim one explicit pair: no family, although numerical searches find many pairs (Section 5), and nothing on continuous dihedral flexes; two isolated realizations of one dihedral vector do not contradict the generic dihedral rigidity of [AR18, Theorem 2]. By the theorem of Wang and Xie, no pair as in Theorem 1.1 can have a convex member (Remark 6.1). Rote’s offset question is not addressed; the record cites Aurenhammer and Walzl [AW16] for it, whose paper we did not read in full and did not check against every requirement of that problem. Without triangulation there are elementary examples; those we know (Remark 6.3) have straight face angles, or dihedral angles π after triangulation.

We could not read the two papers of 1968 closest to the question: Stoker’s [Sto68], in whose Section 6 the conjecture is stated, and Karcher’s [Kar68], on a class of convex bipyramids. The publisher refused automated access, and no open copy was found. Nor could we read the later papers of Milka [Mil70] and Pogorelov [Pog02] on Stoker’s problem. So we claim no more than this: no printed example was found in the sources searched, listed in the paragraph “Scope and priority”. A search that finds nothing is not a proof of novelty.

2 Definitions and conventions

Definition 2.1. Let K be a triangulation of the 2-sphere. A *realization* of K is a map X from the vertex set of K to $\mathbb{R}^3$; it defines, for every triangle abc of K , the triangle $\text{conv}(X_a, X_b, X_c)$, and for every edge the segment between the images of its ends. We call the realization a *polyhedral sphere* if the triangles are non-degenerate and the piecewise linear map $|K| \rightarrow \mathbb{R}^3$ is injective, and we then call the compact region bounded by its image (Jordan–Brouwer) a *polyhedron* with the combinatorics K . Two polyhedra have the *same combinatorics* if they are realizations of the same labelled K .

Let P be a polyhedron and let e be an edge with adjacent triangles f and g . Near an interior point of e , P coincides with one of the two closed wedges bounded by the half-planes of f and g at e . The *interior dihedral angle* $\theta_e \in (0, 2\pi)$ is the angle of that wedge; e is *convex* if $\theta_e < \pi$ and *reflex* if $\theta_e > \pi$. If f and g are not coplanar, then $\theta_e \notin \{0, \pi\}$. The *face angles* are the angles of the triangles at their vertices, in $(0, \pi)$. The condition of the source, that the dihedral angles be different from 0 and π , thus excludes coplanar adjacent faces, and in a triangulated polyhedron it excludes nothing else.

Remark 2.2 (conventions). (i) Equal interior angles $\theta_e(P) = \theta_e(P')$ imply equality under any coarser convention, for instance for the angle in $[0, \pi]$ between the face planes or for the exterior angle. They also imply the same type (convex or reflex) at every edge. (ii) A similarity, and in particular a reflection, preserves all interior dihedral angles and all face angles. So a pair as in Theorem 1.1 is not related by a similarity, and the image of P' under any similarity is again a valid partner of P (check [C10] in Section 4). Our pair has, in addition, the same orientation; requiring this, as check [C5] does, is stricter than needed. (iii) The question concerns the existence of two different polyhedra with the same dihedral angles, as in Stoker's conjecture; it is not a question about continuous deformations.

3 Lemmas

For a realization X of $\mathcal{K}$ we write x, y, a_1, a_2, a_3 also for the image points, H for the plane through a_1, a_2, a_3 when these are affinely independent, and $T_x = \text{conv}(x, a_1, a_2, a_3)$, $T_y = \text{conv}(y, a_1, a_2, a_3)$.

Lemma 3.1 (embedded bipyramids). *Let a_1, a_2, a_3 be affinely independent, and let x and y lie strictly on opposite sides of H . Then the realization of $\mathcal{K}$ is a polyhedral sphere, and the polyhedron it bounds is $T_x \cup T_y$, a ball, with $T_x \cap T_y = \text{conv}(a_1, a_2, a_3)$.*

Proof. T_x and T_y are non-degenerate tetrahedra, so the six triangles are non-degenerate. T_x lies in the closed half-space of H containing x and meets H exactly in its face $\Delta = \text{conv}(a_1, a_2, a_3)$; the same holds for T_y on the other side. Hence $T_x \cap T_y = \Delta$, and $T_x \cup T_y$ is a ball, as the union of two balls glued along a disc in their boundaries; its boundary is the union of the six lateral triangles. Two lateral triangles of the same tetrahedron meet in their common edge. A lateral triangle F of T_x and a lateral triangle G of T_y meet only in H , and $F \cap H, G \cap H$ are edges of Δ ; so $F \cap G$ is the intersection of two edges of Δ , which is their common edge or common vertex, as in $\mathcal{K}$. Thus the six triangles meet only as prescribed by $\mathcal{K}$, and the map $|\mathcal{K}| \rightarrow \mathbb{R}^3$ is injective. $\square$

Near an interior point of an equatorial edge $a_k a_l$ the polyhedron of Lemma 3.1 is the union of the two dihedral wedges of T_x and of T_y at that edge; near an interior point of an edge $x a_k$ it coincides with T_x , because T_y meets the segment $x a_k$ only in a_k ; the same holds for $y a_k$ and T_y .

For the exact tests we compare angles through squared cosines. If u, w are non-zero vectors, put $c(u, w) = u \cdot w$ and $q(u, w) = |u|^2 |w|^2$, so that $\cos \angle(u, w) = c/\sqrt{q}$. Since $\cos$ is injective on $[0, \pi]$,

$$\angle(u, w) = \angle(u', w') \iff c^2 q' = c'^2 q \text{ and } \text{sign } c = \text{sign } c', \quad (1)$$

where c, q belong to (u, w) and c', q' to (u', w') . These are polynomial identities and sign conditions in the coordinates.

Lemma 3.2 (angle tests with outward normals). *Let P be a polyhedron, with triangles oriented coherently so that the signed volume $\frac{1}{6} \sum_{abc} \det(X_a, X_b, X_c)$ is positive, and let $N_f = (X_b - X_a) \times (X_c - X_a)$ for $f = abc$. Let $e = ij$ be an edge with adjacent triangles f, g , let l be the vertex of g opposite to e , and let $\varphi \in [0, \pi]$ be the angle between N_f and N_g . If f and g are not coplanar, then e is reflex if and only if $(X_l - X_i) \cdot N_f > 0$, and*

$$\theta_e = \pi - \varphi \text{ if } e \text{ is convex,} \quad \theta_e = \pi + \varphi \text{ if } e \text{ is reflex.}$$

Proof. For an injective polyhedral sphere, the coherent orientation with positive signed volume is the outward one: by the divergence theorem applied to the field $p \mapsto p/3$, the signed volume of the outward orientation is the volume of P . So N_f points out of P near interior points of f . Let W_1 be the wedge of angle less than π bounded by the half-planes of f and g at e , and W_2 the other one; the angle of W_1 is $\pi - \varphi$. Near f , the solid lies on the side $-N_f$. If the half-plane of g lies on the side

$-N_f$ of the plane of f , that is, $(X_l - X_i) \cdot N_f < 0$, then W_1 lies on that side near f , so P coincides with W_1 near e and $\theta_e = \pi - \varphi$. Otherwise W_1 lies on the side $+N_f$, P coincides with W_2 near e , and $\theta_e = 2\pi - (\pi - \varphi) = \pi + \varphi$. The value 0 of $(X_l - X_i) \cdot N_f$ would mean that f and g are coplanar. $\square$

The reflex test of Lemma 3.2 is valid only for an embedded surface with its true outward orientation. For the pair of Theorem 1.1, embeddedness comes from Lemma 3.1, and the outward orientation from the sign of the signed volume, which is computed exactly. The following test needs no orientation at all.

Lemma 3.3 (orientation-free angle tests). *Let the realization satisfy the hypotheses of Lemma 3.1. For an edge with direction d and a vector w put $w^\perp = (d \cdot d)w - (w \cdot d)d$.*

- (a) *At an edge xa_k , with $\{k, l, m\} = \{1, 2, 3\}$ and $d = a_k - x$, the edge is convex and $\theta = \angle(u_l, u_m)$, where $u_l = (a_l - x)^\perp$, $u_m = (a_m - x)^\perp$; similarly at ya_k .*
- (b) *At an edge $a_k a_l$, with $d = a_l - a_k$, let $u_x = (x - a_k)^\perp$, $u_y = (y - a_k)^\perp$, $u_m = (a_m - a_k)^\perp$, and assume that u_x and u_y are not parallel (the faces are not coplanar). Then $u_m = \alpha u_x + \beta u_y$ with α, β of the same sign. The edge is convex with $\theta = \angle(u_x, u_y)$ if $\alpha, \beta > 0$, and reflex with $\theta = 2\pi - \angle(u_x, u_y)$ if $\alpha, \beta < 0$.*

Proof. All the vectors u lie in the plane $d^\perp$, and each points from the edge into the corresponding triangle. (a) Near the edge xa_k the polyhedron coincides with T_x , whose wedge at that edge has angle $\angle(u_l, u_m) < \pi$. (b) Near the edge $a_k a_l$ the polyhedron is the union of the wedge of T_x , between the directions u_x and u_m , and the wedge of T_y , between u_m and u_y ; each has angle less than π . Since x and y lie on opposite sides of H , and the plane of directions of H meets $d^\perp$ in the line $\mathbb{R}u_m$, the vectors u_x and u_y lie on opposite sides of this line; so either u_m or $-u_m$ lies in the open convex cone of u_x, u_y , which means that α, β have the same sign. The interior angle is $\theta = \angle(u_x, u_m) + \angle(u_m, u_y)$. If $\alpha, \beta > 0$ this is $\angle(u_x, u_y) < \pi$. If $\alpha, \beta < 0$, the direction u_m is not on the shorter arc from u_x to u_y , and $\theta = 2\pi - \angle(u_x, u_y) > \pi$. $\square$

The signs of α and β are those of $(u_m \times u_y) \cdot (u_x \times u_y)$ and $(u_x \times u_m) \cdot (u_x \times u_y)$. By Lemmas 3.2 and 3.3 together with (1), the equality of an interior dihedral angle in two such realizations is decided by the equality of the type and one identity with one sign condition; the same holds, by (1), for face angles.

Lemma 3.4 (the apices). *Let P, P' satisfy the hypotheses of Lemma 3.1 and have equal interior dihedral angles at the three edges xa_1, xa_2, xa_3 . Then the three face angles at x are equal in P and P' . The same holds at y .*

Proof. Near x the polyhedron coincides with the tetrahedron T_x . The unit vectors from x towards a_1, a_2, a_3 form a spherical triangle whose angles are the three interior dihedral angles at xa_1, xa_2, xa_3 and whose sides are the three face angles at x . A spherical triangle is determined by its angles, through the polar triangle; explicitly, $\cos a = (\cos A + \cos B \cos C) / (\sin B \sin C)$ for the side a opposite to the angle A . $\square$

So in a pair as in Theorem 1.1 the angles at the apices must agree, and the difference must sit at the vertices a_1, a_2, a_3 of degree four, whose links are spherical quadrilaterals; a spherical quadrilateral is not determined by its angles.

4 The pair and the proof of the theorem

4.1 The data

Let

$$D = 3062866728592379129617243123692916332759782649517324864 = 344^2 \cdot m,$$

$$m = 25882797530695469929837438511466640183543322822449,$$

where D has 55 digits and is not a perfect square. Let $\lambda^* \approx 1.1277699453$ be the smaller root of

$$\begin{aligned} c_2 &= 2370332412579475586518069420, \\ c_2 t^2 + c_1 t + c_0, \quad c_1 &= -7096484086952680799587220308, \\ c_0 &= 4988458519232521555387437375, \end{aligned}$$

whose discriminant $c_1^2 - 4c_2c_0$ is exactly D ; the other root is about 1.8661. The polyhedron P has the vertices

$$y = 0, \quad a_1 = \left(\frac{3}{20}, -\frac{2}{5}, -\frac{11}{5}\right), \quad a_2 = \left(\frac{49}{50}, \frac{17}{20}, -\frac{7}{100}\right), \quad a_3 = \lambda^* \left(-\frac{22}{25}, -\frac{27}{100}, -\frac{22}{25}\right),$$

and an upper apex $x \in \mathbb{Q}(\sqrt{D})^3$; the polyhedron P' has $y' = 0$ and vertices $x', a'_1, a'_2, a'_3 \in \mathbb{Q}(\sqrt{D})^3$. The exact coordinates of x, x' and a'_k are pairs (r, s) of rationals, standing for $r + s\sqrt{D}$, with numerators of up to 120 and denominators of up to 122 digits. They are not printed here; they form the certificate file `certificate/pair_A_exact.json` of the reproducibility package, whose SHA-256 checksum is

$$\text{ca6a9b6465c960dfbb6c77e1f95b285b210d5ccefcb31a064dcf0cfcc59ba9609}.$$

Table 1 gives all coordinates to six decimals, and Table 2 the angles.

vertex	P	P'
y	(0, 0, 0)	(0, 0, 0)
x	(0.059685, 2.166665, 0.779494)	(-6.118500, 8.053372, 3.203931)
a_1	(0.15, -0.4, -2.2)	(0.699203, -1.864542, -10.254981)
a_2	(0.98, 0.85, -0.07)	(1.846494, 1.601551, -0.131892)
a_3	(-0.992438, -0.304498, -0.992438)	(-6.116763, -1.876734, -6.116763)

Table 1: The vertices of P and P' , rounded to six decimals. In both, x and y lie on opposite sides of the plane H of $a_1a_2a_3$, and the line xy meets H outside the triangle $a_1a_2a_3$, beyond the edge a_2a_3 .

4.2 The certificate

The program `check_note.py` of the reproducibility package (Section 7) uses only the standard library of Python. It represents an element $r + s\sqrt{D}$ of $\mathbb{Q}(\sqrt{D})$ by the pair (r, s) of fractions and multiplies by $(r + s\sqrt{D})(r' + s'\sqrt{D}) = (rr' + ss'D) + (rs' + sr')\sqrt{D}$. The sign of $r + s\sqrt{D}$ is decided exactly: it is the common sign if r and s have the same sign or one of them is zero, and otherwise the sign of the term of larger absolute value, found by comparing r^2 with s^2D , which are never equal since D is not a square. The program runs in under a second and checks the following, all in exact arithmetic except the last item.

- [C1] The certificate file has the checksum above; D has 55 digits and is not a square; $c_1^2 - 4c_2c_0 = D$; the value λ^* in the file is a root of $c_2t^2 + c_1t + c_0$, the smaller one; $y = y' = 0$, and a_1, a_2, a_3 are as stated.

edge	θ_e in P and P'	corner	in P	in P'
ya_1	145.7736°	y in $ya_1a_2, ya_2a_3, ya_3a_1$	90.7675, 128.5177, 47.9368	same
ya_2	32.2559°	x in $xa_1a_2, xa_2a_3, xa_3a_1$	33.0198, 49.5871, 24.3942	same
ya_3	45.9569°	a_1 in ya_1a_2, ya_3a_1	29.9066, 39.8173	13.1467, 55.5007
xa_1	120.8279°	a_1 in xa_1a_2, xa_3a_1	22.3380, 52.9510	33.0821, 44.8633
xa_2	27.7621°	a_2 in ya_1a_2, ya_2a_3	59.3259, 27.1247	76.0858, 41.0243
xa_3	37.9213°	a_2 in xa_1a_2, xa_2a_3	124.6423, 96.2609	113.8981, 79.3490
a_1a_2	71.6167°	a_3 in ya_2a_3, ya_3a_1	24.3576, 92.2459	10.4580, 76.5626
a_1a_3	38.7133°	a_3 in xa_2a_3, xa_3a_1	34.1520, 102.6548	51.0639, 110.7425
a_2a_3	273.8239° (reflex)			

Table 2: Left: the interior dihedral angles, equal in P and P' (exactly, by check [C6] and [C7]). Right: the face angles in degrees; the six at the apices agree exactly, the twelve at a_1, a_2, a_3 differ (check [C8]).

- [C2] For P and for P' : a_1, a_2, a_3 are affinely independent, and x, y lie strictly on opposite sides of H (the hypotheses of Lemma 3.1).
- [C3] For P and for P' : no triangle is degenerate, and at each of the nine edges the two adjacent triangles are not coplanar (a non-zero 3×3 determinant).
- [C4] For P and for P' , a test of embeddedness that does not use Lemma 3.1: every edge is disjoint from every triangle with which it shares no vertex (exact segment–triangle test), and is not contained in the plane of a triangle with which it shares one vertex. Then two triangles meet exactly in their common face: otherwise an extreme point of their intersection outside the common face would lie on an edge of one of them and in the other triangle, which these tests exclude.
- [C5] The labelled triangles are coherently oriented, and the signed volumes of P and P' have the same sign.
- [C6] Lemma 3.2: at each edge, P and P' have the same type and equal $\cos \varphi$ in the sense of (1); the types agree with those of [C7] at all 18 edge instances.
- [C7] Lemma 3.3: at each edge, P and P' have the same type and equal angle in the sense of (1); a_2a_3 is the only reflex edge, in both.
- [C8] Face angles, by (1): exactly twelve of the eighteen differ, and the six at x and y are equal.
- [C9] P' is not similar to P : the ratios $|a_1a_2|^2 : |a_1a_3|^2$ differ (this also follows from [C8]).
- [C10] The mirror image of P' and a similar copy of P' (scale $\frac{3}{2}$, a rational rotation and a translation) pass the tests [C2], [C3], [C7], [C8] against P ; the mirror image has the opposite orientation.
- [C11] Controls: P against itself, against a similar copy and against its mirror image gives nine equal dihedral angles and no differing face angle; moving one coordinate of a'_3 by 10^{-6} leaves only the angle at a_1a_2 , whose triangles do not contain a_3 , unchanged.
- [C12] The data of Section 5: the matrix R in the file is the rotation of the quaternion $(100, 5, -18, 7)$, with $R^\top R = I$ and $\det R = 1$; $a'_k = t_k a_k$ and $a'_k - x' = s_k R(a_k - x)$ with $t_k, s_k > 0$; and the three conditions (2) hold.
In floating point: the apex angles computed from the dihedral angles by the formula of Lemma 3.4 agree with the computed face angles to 10^{-12} degrees.

A second program, `mutation_test.py`, hands six deliberately wrong certificates to the first one: x' moved by 10^{-30} , a'_3 moved by 10^{-6} , D replaced by $D+1$, the labels a_1, a_2 exchanged in P' , P' replaced by P , and x' reflected through y . Each of them is rejected, by checks other than the checksum.

4.3 Proof of Theorem 1.1

(a) By [C2] and Lemma 3.1, P and P' are polyhedral spheres that bound balls; [C4] confirms the embeddedness independently. By [C3] no two adjacent faces are coplanar, so no interior dihedral angle is 0 or π . (b) By [C7] (and [C6]) the edge a_2a_3 is reflex in both. A polyhedron with a reflex edge is not convex: points of P close to an interior point of the edge, on the two sides of it, are joined by a segment that leaves P . (c) By [C7], Lemma 3.3 and (1), the interior dihedral angles are equal at all nine edges; [C6] gives the same by Lemma 3.2, which applies by (a) and [C5]. (d) By [C8] and (1). The numerical values in Theorem 1.1 and in the tables are floating-point evaluations of the exact quantities. $\square$

Corollary 1.2 is a restatement. For Corollary 1.3, P and P' are embeddings of the triangle mesh $\mathcal{K}$ realizing the same vector of dihedral angles, with the same orientation by [C5] (so the convention of [AR18] and the interior angles agree), and with different inner face angles by [C8].

5 How the pair was found

This section explains where the pair comes from; the proof of Theorem 1.1 does not use it. By Lemma 3.4 a partner P' of P has the same vertex cones at x and at y , up to congruence. So we put $y = y' = 0$, keep the cone at y , and move the cone at x by a rotation:

$$a'_k = t_k a_k, \quad a'_k - x' = s_k R(a_k - x), \quad t_k, s_k > 0, \quad R \in SO(3).$$

Then the six dihedral angles at the edges xa_k and ya_k do not change. With $N_{kl} = (a_k - x) \times (a_l - x)$ and $M_{kl} = a_l \times a_k$, the normals of the triangles $x'a'_k a'_l$ and $y'a'_k a'_l$ are positive multiples of RN_{kl} and M_{kl} , and $|RN_{kl}| = |N_{kl}|$. Hence the dihedral angle at $a_k a_l$ is unchanged (for the same type) if and only if

$$A_{kl}(x) := ((R - I)N_{kl}) \cdot M_{kl} = 0, \quad (2)$$

three conditions that are affine-linear in x . The points x' and t_k, s_k exist if and only if the three planes spanned by a_k and $R(a_k - x)$ have a common line, that is, if $\det(a_1 \times R(a_1 - x), a_2 \times R(a_2 - x), a_3 \times R(a_3 - x)) = 0$.

A floating-point search over random nonconvex bipyramids and rotations, followed by Newton's method, found four numerical pairs, each with one reflex edge, with largest face-angle differences between 18° and 39° . To make one of them exact we took R to be the rational rotation of the quaternion $(100, 5, -18, 7)$, with denominator 5199, the rational points a_1, a_2 above, and $a_3 = \lambda(-\frac{22}{25}, -\frac{27}{100}, -\frac{22}{25})$ with a parameter λ . For each λ the conditions (2) determine x by Cramer's rule, and the determinant above becomes a rational function of λ . Its numerator factors over $\mathbb{Q}$ into λ , three further linear factors, with roots $-\frac{4637560805}{313396006}$, $\frac{489410995}{393950054}$ and $\frac{23003637315}{13356891878}$, and the irreducible quadratic $c_2 t^2 + c_1 t + c_0$ of Section 4. The rational roots give degenerate configurations (some a_k is parallel to $R(a_k - x)$); the root λ^* gives the pair.

A heuristic parameter count, not a statement of this note, suggests a family of dimension $8+3-4=7$ (the shape of P up to similarity, the rotation R , four conditions). A numerical search of the second verification run with a different solver found 21 further floating-point pairs among 3000 nonconvex starts, each with exactly one reflex edge, and none with a convex member among 3000 convex starts.

6 Remarks

Remark 6.1 (no convex member). Equal interior dihedral angles force the same type at every edge (Remark 2.2). If P were convex, all interior dihedral angles of P' would be less than π ; an embedded

polyhedral sphere with this property bounds a convex body (local convexity implies convexity; compare [Pak10, Exercise 27.3 and its hint]), so P' would be convex, and by [WX22, Theorem 1.6] all face angles would agree. Hence, granted that theorem of a preprint, no pair as in Theorem 1.1 has a convex member, and a reflex edge is unavoidable.

Remark 6.2 (five vertices and six faces). A polyhedron with $V = 5$ vertices and $F = 6$ faces has $E = 9$ edges by Euler’s formula, and $2E = 18 = 3F$ forces all faces to be triangles. A triangulation of the sphere with five vertices has degrees between 3 and 4 summing to 18, so it has a vertex v of degree 3. Its link uwz is not a triangle of the triangulation (otherwise the three triangles at v together with uwz would already form a sphere); removing v and adding uwz gives a triangulation with four vertices, the boundary of a tetrahedron, so the original one is the boundary of the triangular bipyramid $\mathcal{K}$. Thus Karcher’s class [Kar68], as described in [WX22], consists of convex realizations of $\mathcal{K}$; we do not know whether it contains all of them.

Remark 6.3 (non-triangulated examples). Without triangulation the phenomenon is elementary. Let $B_1 = [-2, 2] \times [-1, 1] \times [0, 1]$ and $B_2 = \rho_\vartheta([-1, 1] \times [-2, 2]) \times [1, 2]$, where ρ_ϑ is the rotation of the plane by a small angle ϑ . The union is a ball with 20 vertices, 32 edges and 14 simply connected faces, and for small $|\vartheta|$ its combinatorics does not depend on ϑ . All faces are horizontal or vertical, so all dihedral angles are $\pi/2$ or $3\pi/2$, while the face angles at the four points where the boundaries of the two rectangles cross, in the horizontal faces at height 1, are $\pi/2 \pm \vartheta$. But the four long vertical faces have the straight angle π at these points, and triangulating would create edges with dihedral angle π , which the source excludes. The same mechanism is implicit in [BDK⁺13, §3, Figure 1]: of a non-orthogonal polyhedron whose dihedral angles are multiples of $\pi/2$, the authors note that its graph is also realized orthogonally after rotating the top part. As far as we can tell from the drawing, that polyhedron is made of two stacked boxes as above, with straight face angles at the points where the box boundaries cross; the authors do not state the example as an answer to the nonconvex Stoker question. Alexandrov [Ale14, §1] describes, in hyperbolic and spherical space, a convex polytope with a small tetrahedron attached inside one of its faces and rotated within that face; in $\mathbb{R}^3$ the same works, but the enlarged face is not simply connected, and triangulating it without new vertices, as in his footnote, creates edges with dihedral angle π .

Remark 6.4 (edge lengths). Cho and Kim give non-congruent nonconvex pairs with equal edge lengths and equal dihedral angles [CK25, Examples 3.2–3.4]. As far as we can tell from their text (we did not examine the figures in detail), the corresponding faces are congruent, so the face angles agree and these are not examples for Amenta’s question; Examples 3.2 and 3.3 also use dihedral angles π . In our pair the edge lengths differ, as they must.

Remark 6.5 (open questions). We do not know a pair with rational coordinates; with the base points and the rotation above, the rational roots of the factorization are degenerate. Whether the pairs form a family of the expected dimension, whether there is a smallest example in some sense, and whether an embedded triangulated polyhedron can admit a continuous dihedral flex (open according to [AR18, AR20]) are not addressed here.

7 Computations and reproducibility

The proof of Theorem 1.1 consists of Lemmas 3.1 to 3.3 and the exact checks [C1]–[C8]. Besides the certificate and the two programs of Section 4, the reproducibility package contains, in `original/`, the programs with which the pair was found and first verified: the exact verifier `verify_pair.py`, based on Lemmas 3.1 and 3.2; the cross-check `crosscheck_mp.py`, which shares no code with it, works with 120 digits, decides interior and exterior by a ray-parity test instead of an orientation, and runs a segment–triangle self-intersection test (largest difference of corresponding dihedral angles

$7.7 \cdot 10^{-121}$ radians); four controls, all rejected by the verifier; and the search and construction programs of Section 5. The folders `verification_run_A/`, `verification_run_B/` and `independent_run_2/` contain the programs and outputs of the three verification runs described next. Supporting evidence, such as the cross-check and the numerical searches, is not part of the proof. All recorded outputs were reproduced from an extracted copy of the source archive on 20 October 2026.

Verification. The pair was found, and first verified, with the programs in `original/`. Three independent verification runs, all AI-assisted, followed. The first two examined the finder’s write-up; each checked the statement against the Oberwolfach report, re-derived Lemmas 3.1, 3.2 and 3.4, and recomputed the whole certificate with an exact program of its own that uses no orientation (in-face vectors orthogonal to the edge and exact cone membership, as in Lemma 3.3). The first run confirmed every exact sign by 80-digit arithmetic and re-ran the original programs and controls with identical results. The second run wrote its verifier before reading the original programs, confirmed every sign by 400-digit interval arithmetic, added an exact embeddedness test that does not use Lemma 3.1 and a typing of every edge by the generalized winding number, recomputed the data of Section 5, and found the conjecture of Amenta and Rojas. Deliberately wrong inputs were rejected in both runs (the four controls in the first, ten mutated certificates in the second, of which the eight wrong or trivial ones were rejected and the mirror image and a similar copy of P' accepted as partners). Both runs confirmed the certificate, and all corrections they required were made. The programs `check_note.py` and `mutation_test.py` were added when this note was written; they repeat the exact checks of both runs with the standard library only.

The third run examined the text of this note. It re-read the Oberwolfach page and the sources of Section 1, checked Lemmas 3.1 to 3.4, (1), Remarks 6.1 and 6.2 and the proof of Theorem 1.1 line by line, and wrote, before reading `check_note.py`, an exact verifier of its own in $\mathbb{Q}(\sqrt{D})$ with the standard library only. That verifier decides embeddedness by an exact triangle–triangle test for all fifteen pairs of triangles, and the type of every edge by an exact point-in-polyhedron test (a point in the smaller wedge at the edge and the parity of exact ray crossings), which uses neither an orientation nor Lemma 3.1; it also types every edge by Lemma 3.2 and by Lemma 3.3, and all three typings agree. It confirmed all claims of Theorem 1.1, every number printed in Theorem 1.1 and in Tables 1 and 2, the position of the line xy stated in Table 1, and, with `sympy`, the construction data and the factorization of Section 5. Ten wrong or trivial inputs were rejected by its geometric checks alone, and four valid ones (among them the mirror image and a similar copy of P') were accepted. It re-ran every program of the package from the source archive with identical outputs, except for one line of a recorded output of the second run that its program does not print, which was moved to a separate file. It found the papers of Milka and Pogorelov and the preprint status of [WX22], which this note now records.

Scope and priority. The note answers Amenta’s question [BKS15, p. 721, Problem 4], the first clause of record OWR-13498-009, in the negative, by one explicit pair. It claims no family of pairs and nothing about continuous dihedral flexes. The second clause of the record, Rote’s question on offset polytopes [BKS15, p. 721, Problem 5], is not addressed; for it the record points to Aurenhammer and Walzl [AW16], whose paper we did not read in full and did not check against every requirement of that problem. The question is Amenta’s; the convex statement is Stoker’s conjecture [Sto68, §6], proved for convex Euclidean polyhedra in all dimensions in a preprint of Wang and Xie [WX22, Theorem 1.6]; Karcher [Kar68] verified it for a class of convex polyhedra with five vertices and six faces, as described in [WX22]; Milka [Mil70] and Pogorelov [Pog02] wrote on the problem, as described in [MM11, AR18, Pak10]; the infinitesimal convex versions are due to Mazzeo and Montcouquiol [MM11] and to Winter [Win26]. Amenta and Rojas [AR18, AR20] proved generic dihedral rigidity, noted that

the existence of dihedral-flexible polyhedra is not known, and stated the conjecture that Corollary 1.3 refutes. Rigidity with respect to dihedral angles and edge lengths (different data) is due to Stoker [Sto68, Theorem III], Biedl and Genç [BG11] and Cho and Kim [CK25]. The non-triangulated mechanism of Remark 6.3 is implicit in [BDK⁺13, §3, Figure 1], not stated there as an answer; Alexandrov [Ale14] constructs nonconvex deformations preserving all dihedral angles in hyperbolic and spherical space and leaves the Euclidean case aside; Schlenker’s spherical counterexamples [Sch00] are cited as reported in [WX22, Pak10], a secondary source. Only the pair of Theorem 1.1 with its certificate, and Corollaries 1.2 and 1.3, are, to our knowledge, not in print.

What was read: the open-problems page of [BKS15] in the open PDF of the publisher; [AR18] in full (arXiv); of [AR20], the author manuscript in the NSF Public Access Repository (Sections 1, 2 and 7; the version of record on the publisher’s site refused automated access and was not compared); the introduction and Theorem 1.6 of [WX22]; Sections 1 and 3 of [CK25] and all of [BDK⁺13], including its Figure 1 as a drawing (arXiv versions); Section 1 of [Ale14] (arXiv); the introduction and Theorem 1 of [MM11] (arXiv); the abstracts or introductions of [Win26, Bel23, Hem15a, FI17]; the summary and the introduction of the thesis [Hem15b]; the workshop summary [CMP]; and, in Pak’s notes [Pak10], Sections 26.7–26.9, Exercises 26.1, 26.2, 27.3 and 32.2 with their hints, and Sections 32.6–32.7. Not read: [Sto68] and [Kar68], which the publisher’s site refused to automated requests with a bot challenge that we did not try to get around, and for which OpenAlex and Semantic Scholar list no open copy (a search on scholar.archive.org met a bot check, which we did not try to get around; zbMATH has no review text); [Mil70] and [Pog02], for which we found no open copy (zbMATH has no review text); the version of record of [AR20]; any thesis or later paper of Rojas; [AW16] beyond its abstract; [Sch00]. What we say about [Sto68, Kar68, Mil70, Pog02] is taken from [Pak10, WX22, CK25, MM11, AR18].

The searches were made on 19 and 20 October 2026: queries to the arXiv interface (for Stoker’s conjecture, for dihedral and face angles, for nonconvex polyhedra with given dihedral angles, for dihedral rigidity and dihedral flexes, and for the work of Amenta), Crossref, zbMATH, OpenAlex and Semantic Scholar; the lists of works citing [Sto68] (77 titles in Semantic Scholar, 49 in OpenAlex), [Kar68] (9 titles) and [AR18] (3 titles), scanned by title; and ten web searches. No printed answer to the question was found in the sources searched (the Oberwolfach page, the papers and notes listed above, the AIM summary and these citation lists): none of the works read gives two embedded triangulated nonconvex polyhedra with the same combinatorics, equal dihedral angles and different face angles, and none claims that no such pair exists. A search that finds nothing is not a proof of novelty, and we make no priority claim.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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