

All Incorrect Equilibria of the Planar Four-Agent Formation Law Are Unstable

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Abstract

Four agents in the plane move by the standard distance-based gradient law $\dot{p}_i = \sum_{j \neq i} e_{ij}(p_j - p_i)$, $e_{ij} = \|p_i - p_j\|^2 - \bar{d}_{ij}^2$, where all six desired distances come from one planar target. In an Oberwolfach report of 2015, B. D. O. Anderson asked whether, as for rectangular targets, all incorrect equilibria are saddle points when the target is a general quadrilateral or a triangle with an interior agent. We show that the answer is yes for every planar target: at every incorrect equilibrium the Hessian of $V = \frac{1}{4} \sum_{i < j} e_{ij}^2$ has a negative eigenvalue; all incorrect equilibria except the collocated one also have a positive eigenvalue, and the collocated one is a local maximum modulo translations. In the language of multidimensional scaling, the first statement says that the complete-graph s-stress of four points in the plane has no spurious second-order critical points, for every ground truth. At an incorrect equilibrium spanning the plane, $e_{ij} = \kappa \lambda_i \lambda_j$ with $\kappa > 0$, where λ is the affine dependency of the agents, and the smallest Hessian eigenvalue is at most $-\kappa \|\lambda\|^2/2 \leq -\sqrt{8V/3}$; both constants are sharp. The collinear case is known. For the spanning case we combine the classical self-stress of four planar points, a Schur-complement argument as in recent work of Criscitiello on the s-stress, and an inequality between two 2×2 matrices depending only on λ , proved by explicit identities in the elementary symmetric functions of λ . Consequently almost every initial condition converges to a formation congruent to the target, while collinear initial conditions show that literal global convergence fails. This is an unrefereed note.

1 Introduction

In distance-based formation control, mobile agents must reach a prescribed shape, specified by inter-agent distances, using relative measurements only. A standard law, due to Krick, Broucke and Francis [KBF09], moves each agent towards or away from its neighbours according to the errors in the squared distances. For four agents in the plane with all six distances prescribed (the complete graph K_4) it reads

$$\dot{p}_i = \sum_{j \neq i} e_{ij} (p_j - p_i), \quad e_{ij} = \|p_i - p_j\|^2 - \bar{d}_{ij}^2 \quad (i = 1, \dots, 4), \quad (1)$$

where $p_i \in \mathbb{R}^2$ is the position of agent i and $e_{ij} = e_{ji}$. We always assume that the desired distances are realizable in the plane: $\bar{d}_{ij} = \|\bar{q}_i - \bar{q}_j\|$ for a *target* configuration $\bar{q}_1, \dots, \bar{q}_4 \in \mathbb{R}^2$. Then (1) is the gradient flow $\dot{p} = -\nabla V(p)$ on $\mathbb{R}^8$ of

$$V = \frac{1}{4} \sum_{i < j} e_{ij}^2. \quad (2)$$

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(The report [And15] writes $V = \frac{1}{2}\|e\|^2$, for which $-\nabla V$ is twice the right side of (1); the factor changes neither the equilibria nor the signs of the Hessian eigenvalues.) Since all six distances are prescribed, every configuration with $e = 0$ is congruent to the target [And15, p. 649]. An equilibrium of (1) is *correct* if $e = 0$ and *incorrect* otherwise.

For rectangular targets, work of Anderson, Dasgupta, Summers and Yu [AYDS10, DAYS11], as summarised in [And15], showed that all incorrect equilibria (two incorrect rectangles, collocated and collinear configurations, and further configurations spanning the plane) are saddle points, so that from non-collinear initial conditions one converges in practice to the target shape. In the Oberwolfach report [And15, pp. 647–649], Anderson asked whether the same holds when the target is a general quadrilateral, or three agents forming a triangle with the fourth inside it: “Are all incorrect equilibria again saddle points?” This is record OWR-13497-006 of the UnsolvedMath dataset [ula26]. Records OWR-13497-001, -004 and -005 ask, in different words, for global convergence to the target shape under the same law.

Throughout, $P \in \mathbb{R}^{4 \times 2}$ is the matrix with rows $p_i^\top$, $\mathbf{1} \in \mathbb{R}^4$ is the all-ones vector, and $\|\cdot\|$ is the Euclidean or Frobenius norm. For weights $w = (w_{ij})_{i < j}$, $L(w)$ is the weighted Laplacian: $L(w)_{ij} = -w_{ij}$ for $i \neq j$ and $L(w)_{ii} = \sum_{j \neq i} w_{ij}$, so that $v^\top L(w)v = \sum_{i < j} w_{ij}(v_i - v_j)^2$. If the agents span the plane, their *affine dependency* is the vector $\lambda \in \mathbb{R}^4 \setminus \{0\}$, unique up to scale, with $\sum_i \lambda_i = 0$ and $\sum_i \lambda_i p_i = 0$.

Theorem 1.1. *Let the target be any configuration of four points in the plane, and let P be an incorrect equilibrium of (1). Then $\nabla^2 V(P)$ has a negative eigenvalue. More precisely:*

- (a) *If the agents are collinear or collocated, then $\lambda_{\min}(\nabla^2 V(P)) \leq \lambda_{\min}(L(e)) < 0$.*
- (b) *If the agents span the plane, then $e_{ij} = \kappa \lambda_i \lambda_j$ for all $i < j$, where λ is their affine dependency and $\kappa > 0$, and*

$$\lambda_{\min}(\nabla^2 V(P)) \leq -\frac{\kappa}{2} \|\lambda\|^2 \leq -\sqrt{8V(P)/3}.$$

- (c) *If the agents are not collocated, then $\nabla^2 V(P)$ also has a positive eigenvalue. If they are collocated, then $\nabla^2 V(P)$ is negative definite on the orthogonal complement of the translations, so P is a nondegenerate local maximum of V modulo translations.*

The constants $\frac{1}{2}$ and $\sqrt{8/3}$ in (b) cannot be improved (Proposition 6.1).

No genericity is assumed: the theorem covers convex and non-convex quadrilaterals, triangles with an interior agent, rectangles, and targets with three collinear or two coincident points. All incorrect equilibria except the collocated ones are saddle points in the strict sense (the Hessian is indefinite), and the collocated ones are local maxima modulo translations. The report [And15] counts the collocated equilibria of the rectangle case among the saddle points, so there “saddle point” means an unstable equilibrium; in the terminology of [LPP⁺19], every incorrect equilibrium is a strict saddle.

Corollary 1.2 (Almost global convergence). *For every planar target, every solution of (1) exists for all $t \geq 0$, has a constant centroid and converges to a single equilibrium. The set of initial conditions $p(0) \in \mathbb{R}^8$ whose solution converges to an incorrect equilibrium has Lebesgue measure zero; for all other initial conditions the formation converges to a configuration congruent to the target. If the target is not a single point, this exceptional set is not empty: it contains the incorrect equilibria, for instance the collocated configurations, and, if the target is not collinear, every collinear initial configuration.*

So (1) converges almost globally for every planar target, as was known for rectangles; this is the natural reading of records OWR-13497-001, -004 and -005. Global convergence in the literal sense fails for every target that is not a single point.

Earlier work. The instability in part (a) of Theorem 1.1 is known. Sun, Helmke and Anderson [SHA15, Lemma 6], as used in [PSAA18, Lemma 3], show that degenerate incorrect equilibria are unstable when the target is not degenerate, and for every target it follows from [CMRB26, Lemma 3.4], by which rank-deficient second-order critical points of the complete-graph s-stress are global minimizers; see also [AH14] for formations on a line. We include a short proof, which also gives the bound in part (a). According to its abstract, the paper [SYAD09] of Summers, Yu, Anderson and Dasgupta proves that every incorrect equilibrium is locally unstable for formations whose internal angles satisfy an acuteness condition; we have not seen that paper and do not know which targets the condition covers. In a later paper the same authors [SYDA13] note that even for four agents it was open whether locally stable incorrect equilibria can exist, and propose semidefinite certificates that can decide this, in principle, for a given target. Park, Sun, Anderson and Ahn [PSAA18] state that it is not known whether a general, non-rectangular K_4 formation in the plane is reached almost globally under (1); they, and [PST⁺16] before them, obtain almost global convergence for modified laws with virtual coordinates. Sahebsara and de Queiroz [SdQ24] embed planar formations in $\mathbb{R}^3$ with virtual body frames.

Up to a factor 2, V is the complete-graph squared stress (s-stress) of multidimensional scaling (see [CMRB26, Cri26]), for $n = 4$ points, ground truth in dimension $\ell = 2$ and optimization dimension $k = 2$. So Theorem 1.1 says that this landscape is benign (every second-order critical point is a global minimizer) for every ground truth. For $k = \ell$ this holds when $n \leq \ell + 1$, and spurious local minima exist for $\ell = 1$, $n \geq 7$ [SGJ⁺25]. Criscitiello, McRae, Rejock and Boumal [CMRB26, §6.2] construct strict spurious second-order critical points for $n = \max\{\ell + 2, 7\}$ and non-strict ones for $n = \max\{\ell + 2, 6\}$ ($\ell \geq 2$); they remark that their construction in the plane yields nothing for $n \leq 5$, and suspect that for $\ell \leq 4$ and $n = 6$ the landscape is benign for almost every ground truth. Criscitiello [Cri26] proves benignness for $k \geq 2(\ell + 1)$, and for $k = n - 2$ whenever $k \geq \ell + 1$ [Cri26, Thm. 6.1]; for $n = 4$, $k = 2$ the latter covers collinear targets ($\ell = 1$), but not targets spanning the plane.

Method. At a spanning equilibrium e is a multiple of the classical self-stress $\lambda_i \lambda_j$ (Lemma 3.2). Moving agent i by $\lambda_i z$ gives an explicit second variation, whose positive part is bounded through a Schur complement of the target Gram matrix that is singular because the target is planar; this is the mechanism of the Schur-companion directions of [Cri26, §6.2], which there serve the relaxed case $k \geq \ell + 1$. The new ingredient, which makes the unrelaxed case $k = \ell = 2$ work, is the sharp inequality $\lambda_{\max}(NK) \leq \|\lambda\|^4/2$ of Proposition 4.1 for two 2×2 matrices depending only on λ . For $n = \ell + 2 \geq 6$ agents the method fails, in agreement with [CMRB26] (Remark 5.2).

2 Preliminaries

Write $p_{ij} = p_i - p_j$ and, for $Q \in \mathbb{R}^{4 \times 2}$ with rows $q_i^\top$, $q_{ij} = q_i - q_j$. Let $\langle A, B \rangle = \text{tr}(A^\top B)$.

Lemma 2.1. *For $P, Q \in \mathbb{R}^{4 \times 2}$ and $e = e(P)$,*

$$\nabla V(P) = L(e)P, \quad H_P[Q] := \left. \frac{d^2}{dt^2} V(P + tQ) \right|_{t=0} = \sum_{i < j} \left(2\langle p_{ij}, q_{ij} \rangle^2 + e_{ij} \|q_{ij}\|^2 \right). \quad (3)$$

In particular (1) is $\dot{P} = -\nabla V(P)$, and P is an equilibrium if and only if $L(e)P = 0$.

Proof. We have $e_{ij}(P + tQ) = e_{ij} + 2t\langle p_{ij}, q_{ij} \rangle + t^2\|q_{ij}\|^2$. The first derivative of $V(P + tQ)$ at $t = 0$ is $\sum_{i < j} e_{ij} \langle p_{ij}, q_{ij} \rangle = \langle L(e)P, Q \rangle$, and the second is the sum in (3). $\square$

V is invariant under rigid motions of the plane. Replacing P by $P - \frac{1}{4}\mathbf{1}\mathbf{1}^\top P$ changes neither e nor $\nabla^2 V(P)$, so we may assume that equilibria are *centred*, $P^\top \mathbf{1} = 0$. By (3), H_P vanishes on the translations $Q = \mathbf{1}b^\top$. Let $J = I - \frac{1}{4}\mathbf{1}\mathbf{1}^\top$ and let $X^* = J\bar{Q}\bar{Q}^\top J$ be the centred Gram matrix of the target, where $\bar{Q}$ has rows $\bar{q}_i^\top$. So X^* is positive semidefinite, $X^*\mathbf{1} = 0$, $\text{rank } X^* \leq 2$, and $X_{ii}^* + X_{jj}^* - 2X_{ij}^* = \bar{d}_{ij}^2$; the rank condition is used only in Step 3 of Section 5.

Lemma 2.2. *If P is an equilibrium and $L(e) \succeq 0$, then $e = 0$.*

Proof. For symmetric 4×4 matrices X let $e_{ij}(X) = X_{ii} + X_{jj} - 2X_{ij} - \bar{d}_{ij}^2$ and $f(X) = \frac{1}{4} \sum_{i < j} e_{ij}(X)^2$. Then $f(PP^\top) = V(P)$, $f(X^*) = 0$, and f is convex, being a convex quadratic function of an affine map. Its derivative is $Df(X)[Y] = \frac{1}{2} \langle L(e(X)), Y \rangle$, since $\langle L(w), Y \rangle = \sum_{i < j} w_{ij}(Y_{ii} + Y_{jj} - 2Y_{ij})$. Put $X = PP^\top$, so $e(X) = e$. By convexity,

$$0 = f(X^*) \geq f(X) + \frac{1}{2} \langle L(e), X^* \rangle - \frac{1}{2} \langle L(e), PP^\top \rangle.$$

Here $\langle L(e), PP^\top \rangle = \text{tr}(P^\top L(e)P) = 0$ because $L(e)P = 0$, and $\langle L(e), X^* \rangle \geq 0$ because both matrices are positive semidefinite. Hence $V(P) = f(X) \leq 0$, so $e = 0$. $\square$

This is the standard argument showing that rank-deficient second-order critical points of low-rank factorizations of convex problems are global minimizers; for the s-stress see [CMRB26, Lemma 3.4] and the references there.

3 The two kinds of incorrect equilibria

The agents of an equilibrium are collocated, collinear (and not collocated), or they span the plane. The first two cases are known.

Proposition 3.1 (cf. [SHA15, Lemma 6], [PSAA18, Lemma 3]). *Let P be an incorrect equilibrium whose agents lie on a line. Then $\lambda_{\min}(\nabla^2 V(P)) \leq \lambda_{\min}(L(e)) < 0$.*

Proof. Choose a unit vector $a \in \mathbb{R}^2$ with $\langle p_{ij}, a \rangle = 0$ for all i, j and put $Q = va^\top$ for $v \in \mathbb{R}^4$, so $q_{ij} = (v_i - v_j)a$. Then $\langle p_{ij}, q_{ij} \rangle = 0$, and (3) gives $H_P[va^\top] = v^\top L(e)v$, while $\|va^\top\| = \|v\|$. By Lemma 2.2, $L(e)$ is not positive semidefinite. Taking for v a unit eigenvector for $\lambda_{\min}(L(e))$ gives the claim. $\square$

Lemma 3.2 (self-stress). *Let P be an equilibrium whose agents span the plane, with affine dependency λ . Then there is $\kappa \in \mathbb{R}$ with $e_{ij} = \kappa \lambda_i \lambda_j$ for all $i < j$, and $L(e) = -\kappa \lambda \lambda^\top$. If P is incorrect, then $\kappa > 0$. Conversely, if the agents of P span the plane and $e_{ij} = \kappa \lambda_i \lambda_j$ for all $i < j$, then P is an equilibrium.*

Proof. We may assume $P^\top \mathbf{1} = 0$. Since P has rank 2 and its columns are orthogonal to $\mathbf{1}$, the columns of P and $\mathbf{1}$ span a 3-dimensional subspace, whose orthogonal complement is spanned by λ . This subspace lies in the kernel of the symmetric matrix $L(e)$, because $L(e)P = 0$ and $L(e)\mathbf{1} = 0$. Hence $L(e) = c\lambda\lambda^\top$ for some $c \in \mathbb{R}$, and the off-diagonal entries give $e_{ij} = \kappa \lambda_i \lambda_j$ with $\kappa = -c$. If $\kappa \leq 0$, then $L(e) = -\kappa \lambda \lambda^\top \succeq 0$, and Lemma 2.2 gives $e = 0$. Conversely, $e_{ij} = \kappa \lambda_i \lambda_j$ implies $L(e)_{ii} = \kappa \lambda_i \sum_{j \neq i} \lambda_j = -\kappa \lambda_i^2$, so $L(e) = -\kappa \lambda \lambda^\top$ and $L(e)P = -\kappa \lambda (\lambda^\top P) = 0$. $\square$

The weights $\omega_{ij} = \lambda_i \lambda_j$ form the classical self-stress of four points spanning the plane, which spans the space of self-stresses of this framework (for stresses and stress matrices see [Con82]).

4 An inequality for the affine dependency

In this section $\lambda \in \mathbb{R}^4$ is any nonzero vector with $\sum_i \lambda_i = 0$. Let $s_k = \sum_i \lambda_i^k$, so $s_2 = \|\lambda\|^2$, let $D = \text{diag}(\lambda_1^2, \dots, \lambda_4^2)$, and let η_2, η_3, η_4 be the elementary symmetric functions of $\lambda_1, \dots, \lambda_4$ (and $\eta_1 = \sum_i \lambda_i = 0$). Put

$$\sigma = -\eta_2 = \frac{s_2}{2} > 0, \quad \Delta = \sigma^2 + 2\eta_4 = \sum_{i < j} \lambda_i^2 \lambda_j^2 = \frac{s_2^2 - s_4}{2} > 0.$$

($\Delta > 0$ because λ has at least two nonzero entries.) Let $W = \{x \in \mathbb{R}^4 : \mathbf{1}^\top x = \lambda^\top x = 0\}$, let $U \in \mathbb{R}^{4 \times 2}$ have orthonormal columns spanning W , and let $\Pi = UU^\top = I - \frac{1}{4}\mathbf{1}\mathbf{1}^\top - \lambda\lambda^\top/s_2$ be the orthogonal projection onto W . Define the 2×2 matrices and the vector

$$M = U^\top DU, \quad b = U^\top D\lambda, \quad K = s_2 I + 4M, \quad N = M + \frac{bb^\top}{2\Delta}. \quad (4)$$

Then $M \succeq 0$, $N \succeq 0$ and $K \succ 0$. Replacing U by UR with R orthogonal replaces M, N, K by $R^\top MR, R^\top NR, R^\top KR$, so the spectrum of NK depends only on λ .

Proposition 4.1. $\lambda_{\max}(NK) \leq s_2^2/2$, with equality if and only if $\eta_3 = 0$, that is, if and only if $\lambda = (a, -a, c, -c)$ up to the order of the entries.

The proof rests on closed forms for the trace and determinant of NK .

Lemma 4.2. *With the notation above,*

$$\begin{aligned} \text{tr } M &= \frac{\sigma^2 + 4\eta_4}{2\sigma}, & \text{tr } M^2 &= \frac{\sigma^2}{4} - \frac{3\eta_3^2}{4\sigma} + \frac{4\eta_4^2}{\sigma^2}, & \det M &= \eta_4 + \frac{3\eta_3^2}{8\sigma}, \\ \|b\|^2 &= 2\sigma\eta_4 + \frac{3\eta_3^2}{4} - \frac{8\eta_4^2}{\sigma}, & b^\top Mb &= \frac{4\eta_4^2(\sigma^2 - 4\eta_4)}{\sigma^2} + \frac{\eta_3^2(\sigma^2 + 24\eta_4)}{8\sigma}, \end{aligned} \quad (5)$$

and

$$\text{tr}(NK) = \frac{2(\sigma^4 + 5\sigma^2\eta_4 + 12\eta_4^2 - \sigma\eta_3^2)}{\Delta}, \quad \det(NK) = \frac{(3\sigma\eta_4 + \eta_3^2)(4\sigma^3 + 16\sigma\eta_4 + 3\eta_3^2)}{\Delta}. \quad (6)$$

Proof. The quantities in (5) do not depend on U : $\text{tr } M = \text{tr}(\Pi D)$, $\text{tr } M^2 = \text{tr}(\Pi D \Pi D)$, $\|b\|^2 = \lambda^\top D \Pi D \lambda$ and $b^\top Mb = \lambda^\top D \Pi D \Pi D \lambda$. Write $\Pi = I - A - B$ with $A = \frac{1}{4}\mathbf{1}\mathbf{1}^\top$ and $B = \lambda\lambda^\top/s_2$. Since $\mathbf{1}^\top D \mathbf{1} = s_2$, $\lambda^\top D \lambda = s_4$ and $\mathbf{1}^\top D \lambda = s_3$, expanding gives

$$\text{tr}(\Pi D) = \frac{3}{4}s_2 - \frac{s_4}{s_2}, \quad \text{tr}(\Pi D \Pi D) = \frac{s_4}{2} - \frac{2s_6}{s_2} + \frac{s_2^2}{16} + \frac{s_4^2}{s_2^2} + \frac{s_3^2}{2s_2}, \quad \lambda^\top D \Pi D \lambda = s_6 - \frac{s_3^2}{4} - \frac{s_4^2}{s_2},$$

and, since $\Pi D \lambda = (\lambda_i^3)_i - \frac{s_3}{4}\mathbf{1} - \frac{s_4}{s_2}\lambda$,

$$\lambda^\top D \Pi D \Pi D \lambda = \sum_i \lambda_i^2 \left(\lambda_i^3 - \frac{s_3}{4} - \frac{s_4}{s_2} \lambda_i \right)^2 = s_8 + \frac{s_2 s_3^2}{16} + \frac{s_4^3}{s_2^2} - \frac{s_3 s_5}{2} - \frac{2s_4 s_6}{s_2} + \frac{s_3^2 s_4}{2s_2}.$$

Newton's identities with $\eta_1 = 0$ give $s_2 = 2\sigma$, $s_3 = 3\eta_3$, $s_4 = 2\sigma^2 - 4\eta_4$, $s_5 = 5\sigma\eta_3$, $s_6 = 2\sigma^3 + 3\eta_3^2 - 6\sigma\eta_4$ and $s_8 = 2\sigma^4 - 8\sigma^2\eta_4 + 8\sigma\eta_3^2 + 4\eta_4^2$. Substituting and simplifying yields (5), with $\det M = \frac{1}{2}((\text{tr } M)^2 - \text{tr } M^2)$. For instance, $\text{tr } M = \frac{3}{2}\sigma - (2\sigma^2 - 4\eta_4)/(2\sigma) = (\sigma^2 + 4\eta_4)/(2\sigma)$.

For (6) we use three identities for 2×2 matrices: $\text{tr}(NK) = s_2 \text{tr} M + 4 \text{tr} M^2 + (s_2 \|b\|^2 + 4b^\top Mb)/(2\Delta)$; $\det K = s_2^2 + 4s_2 \text{tr} M + 16 \det M$; and, by the matrix determinant lemma with the adjugate $(\text{tr} M)I - M$ of M , $\det N = \det M + ((\text{tr} M)\|b\|^2 - b^\top Mb)/(2\Delta)$. Substituting (5) gives

$$\det K = \frac{2(4\sigma^3 + 16\sigma\eta_4 + 3\eta_3^2)}{\sigma}, \quad \det N = \frac{\sigma(3\sigma\eta_4 + \eta_3^2)}{2\Delta},$$

the formula for $\text{tr}(NK)$, and $\det(NK) = \det N \det K$. All identities in this proof are identities between rational functions of $\lambda_1, \lambda_2, \lambda_3$ (with $\lambda_4 = -\lambda_1 - \lambda_2 - \lambda_3$); they were also verified by computer algebra (see the Verification paragraph). $\square$

Proof of Proposition 4.1. The matrix NK is similar to $K^{1/2}NK^{1/2} \succeq 0$, so its eigenvalues $\nu_1 \geq \nu_2$ are real. Let $\mu = s_2^2/2 = 2\sigma^2$. From (6) one obtains the identities

$$(2\mu - \text{tr}(NK)) \Delta = 2[(\sigma^2 - 4\eta_4)(\sigma^2 + 3\eta_4) + \sigma\eta_3^2], \quad (7)$$

$$(\mu^2 - \mu \text{tr}(NK) + \det(NK)) \Delta = \eta_3^2(8\sigma^3 + 25\sigma\eta_4 + 3\eta_3^2). \quad (8)$$

By the inequality of arithmetic and geometric means applied to $\lambda_1^2, \dots, \lambda_4^2$, $|\eta_4| = |\lambda_1\lambda_2\lambda_3\lambda_4| \leq (s_2/4)^2 = \sigma^2/4$. Hence $\sigma^2 - 4\eta_4 \geq 0$, $\sigma^2 + 3\eta_4 \geq \sigma^2/4 > 0$ and $8\sigma^2 + 25\eta_4 \geq \frac{7}{4}\sigma^2 > 0$, so the right sides of (7) and (8) are nonnegative. As $\Delta > 0$, we get $(\mu - \nu_1) + (\mu - \nu_2) \geq 0$ and $(\mu - \nu_1)(\mu - \nu_2) \geq 0$, which together force $\nu_1 \leq \mu$.

If $\eta_3 \neq 0$, the right side of (8) is positive, so both factors $\mu - \nu_i$ are positive and $\nu_1 < \mu$. If $\eta_3 = 0$, then $(\mu - \nu_1)(\mu - \nu_2) = 0$ and $\nu_2 \leq \nu_1 \leq \mu$, so $\nu_1 = \mu$. Finally, $\eta_1 = \eta_3 = 0$ means that $\prod_i(t - \lambda_i) = t^4 + \eta_2 t^2 + \eta_4$ is even, that is, that the multiset $\{\lambda_i\}$ is symmetric under $t \mapsto -t$. $\square$

Remark 4.3. The certificate is a sum of squares: with $\sigma = s_2/2$, $A = \lambda_1^2 + \lambda_2^2$ and $B = \lambda_3^2 + \lambda_4^2$, for all real λ

$$\sigma^2 \mp 4\eta_4 = \frac{1}{4}(A - B)^2 + (\lambda_1\lambda_3 \mp \lambda_2\lambda_4)^2 + (\lambda_1\lambda_4 \mp \lambda_2\lambda_3)^2,$$

and $\sigma^2 + 3\eta_4 = \frac{3}{4}(\sigma^2 + 4\eta_4) + \frac{1}{4}\sigma^2$, $8\sigma^2 + 25\eta_4 = \frac{25}{4}(\sigma^2 + 4\eta_4) + \frac{7}{4}\sigma^2$. So the right sides of (7) and (8) are explicit sums of squares of polynomials in λ .

5 Proof of Theorem 1.1

Part (a) is Proposition 3.1. For (b), let P be an incorrect equilibrium whose agents span the plane. We may assume $P^\top \mathbf{1} = 0$. By Lemma 3.2, $e_{ij} = \kappa \lambda_i \lambda_j$ with $\kappa > 0$. Since $P^\top \mathbf{1} = P^\top \lambda = 0$, the columns of P lie in W , so $P = UB$ with $B = U^\top P$ invertible. We use the notation of Section 4.

Step 1: the test direction. For $z \in \mathbb{R}^2$ let $Q = \lambda z^\top$, so agent i moves by $\lambda_i z$. We claim

$$H_P[\lambda z^\top] = 2z^\top B^\top K B z - \kappa s_2^2 \|z\|^2. \quad (9)$$

Indeed $q_{ij} = (\lambda_i - \lambda_j)z$. With $u = Pz$ and $w_{ij} = (\lambda_i - \lambda_j)^2$, the first part of (3) is $2 \sum_{i < j} (\lambda_i - \lambda_j)^2 \langle p_{ij}, z \rangle^2 = 2u^\top L(w)u$. Using $\sum_i \lambda_i = 0$ one finds $L(w) = \text{diag}(4\lambda_i^2 + s_2) - \delta \mathbf{1}^\top - \mathbf{1} \delta^\top + 2\lambda \lambda^\top$ with $\delta = (\lambda_1^2, \dots, \lambda_4^2)^\top$. Since $\mathbf{1}^\top u = \lambda^\top u = 0$, this gives $u^\top L(w)u = s_2 \|u\|^2 + 4u^\top D u = z^\top B^\top (s_2 I + 4U^\top D U) B z = z^\top B^\top K B z$. The second part of (3) is $\kappa \|z\|^2 \sum_{i < j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2$, and

$$\sum_{i < j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 = \sum_{i \neq j} \lambda_i^3 \lambda_j - 2 \sum_{i < j} \lambda_i^2 \lambda_j^2 = -s_4 - (s_2^2 - s_4) = -s_2^2.$$

Step 2: the target Gram matrix. For $i \neq j$ we have $\bar{d}_{ij}^2 = \|p_{ij}\|^2 - \kappa\lambda_i\lambda_j$, so the matrix of squared target distances is $\bar{\mathcal{D}} = \mathcal{D}(P) - \kappa(\lambda\lambda^\top - D)$, where $\mathcal{D}(P)$ is the matrix of the $\|p_{ij}\|^2$. Since $-\frac{1}{2}J\mathcal{D}(P)J = PP^\top$ for centred P , and $J\lambda = \lambda$,

$$X^* = -\frac{1}{2}J\bar{\mathcal{D}}J = PP^\top + \frac{\kappa}{2}(\lambda\lambda^\top - JDJ). \quad (10)$$

Let $\hat{\lambda} = \lambda/\|\lambda\|$, so that $(U, \hat{\lambda})$ is an orthonormal basis of $\mathbf{1}^\perp$. Using $U^\top\lambda = 0$ and $JU = U$ we get

$$U^\top X^* U = BB^\top - \frac{\kappa}{2}M, \quad U^\top X^* \hat{\lambda} = -\frac{\kappa}{2\sqrt{s_2}}b, \quad \hat{\lambda}^\top X^* \hat{\lambda} = \frac{\kappa}{2}\left(s_2 - \frac{s_4}{s_2}\right) = \frac{\kappa\Delta}{s_2} > 0. \quad (11)$$

Step 3: the Schur complement. Since $X^* \succeq 0$ and $X^*\mathbf{1} = 0$, the 3×3 matrix of X^* in the basis $(U, \hat{\lambda})$ of $\mathbf{1}^\perp$ is positive semidefinite and has rank $\text{rank } X^* \leq 2$. Its last diagonal entry is positive, so its Schur complement

$$S = U^\top X^* U - \frac{(U^\top X^* \hat{\lambda})(U^\top X^* \hat{\lambda})^\top}{\hat{\lambda}^\top X^* \hat{\lambda}} = BB^\top - \frac{\kappa}{2}M - \frac{\kappa}{4\Delta}bb^\top = BB^\top - \frac{\kappa}{2}N \quad (12)$$

is positive semidefinite, and $\text{rank } X^* = 1 + \text{rank } S$. (The 3×3 matrix is congruent to $\text{diag}(S, \hat{\lambda}^\top X^* \hat{\lambda})$.) As the target is planar, $\text{rank } S \leq 1$, so there is $y_0 \neq 0$ with $Sy_0 = 0$, that is, $y_0^\top BB^\top y_0 = \frac{\kappa}{2}y_0^\top N y_0$. This is the only place where planarity of the target is used.

Step 4: conclusion. The matrices $B^\top KB$ and $K^{1/2}BB^\top K^{1/2}$ have the same eigenvalues, so

$$\lambda_{\min}(B^\top KB) = \min_{y \neq 0} \frac{y^\top BB^\top y}{y^\top K^{-1}y} \leq \frac{y_0^\top BB^\top y_0}{y_0^\top K^{-1}y_0} = \frac{\kappa}{2} \cdot \frac{y_0^\top N y_0}{y_0^\top K^{-1}y_0} \leq \frac{\kappa}{2} \lambda_{\max}(NK),$$

because the maximum of $y^\top N y / y^\top K^{-1}y$ over $y \neq 0$ is $\lambda_{\max}(K^{1/2}NK^{1/2})$, which equals $\lambda_{\max}(NK)$. So Proposition 4.1 gives $\lambda_{\min}(B^\top KB) \leq \kappa s_2^2/4$. Let z be a unit eigenvector of $B^\top KB$ for its smallest eigenvalue. Then (9) gives

$$H_P[\lambda z^\top] \leq \frac{1}{2}\kappa s_2^2 - \kappa s_2^2 = -\frac{1}{2}\kappa s_2^2,$$

and $\|\lambda z^\top\|^2 = s_2$, so $\lambda_{\min}(\nabla^2 V(P)) \leq -\frac{1}{2}\kappa s_2$. Finally, since $s_4 \geq s_2^2/4$ by the Cauchy–Schwarz inequality,

$$V(P) = \frac{\kappa^2}{4} \sum_{i < j} \lambda_i^2 \lambda_j^2 = \frac{\kappa^2}{8}(s_2^2 - s_4) \leq \frac{3}{32}\kappa^2 s_2^2,$$

that is, $\frac{1}{2}\kappa s_2 \geq \sqrt{8V(P)/3}$. This proves (b).

Part (c). Taking $Q = P$ in (3) gives $H_P[P] = 2\sum_{i < j} \|p_{ij}\|^4 + \text{tr}(P^\top L(e)P)$, and the last term vanishes at an equilibrium. So $H_P[P] > 0$ unless the agents are collocated. If they are collocated, then $e_{ij} = -\bar{d}_{ij}^2$ and (3) gives $H_P[Q] = -\sum_{i < j} \bar{d}_{ij}^2 \|q_{ij}\|^2 = -\text{tr}(Q^\top L(\bar{d}^2)Q)$. If the target is not a single point, the graph on $\{1, 2, 3, 4\}$ with an edge ij whenever $\bar{d}_{ij} > 0$ is complete multipartite with at least two parts, hence connected, so the kernel of $L(\bar{d}^2)$ is spanned by $\mathbf{1}$ and H_P is negative definite on $\{Q : Q^\top \mathbf{1} = 0\}$. (For a one-point target, $L(e) \succeq 0$ for every P , so by Lemma 2.2 there are no incorrect equilibria.) The sharpness claim is Proposition 6.1 below. $\square$

Remark 5.1 (All spanning incorrect equilibria). Steps 2 and 3 can be reversed. Given $\lambda \neq 0$ with $\sum_i \lambda_i = 0$, $\kappa > 0$ and $r \in \mathbb{R}^2$ with $\frac{\kappa}{2}N + rr^\top \succ 0$, choose B with $BB^\top = \frac{\kappa}{2}N + rr^\top$, put $P = UB$, and define X^* by the right side of (10). By (11)–(12), which hold for every such P , $S = rr^\top$, so $X^* \succeq 0$ has rank at most 2: it is the Gram matrix of a planar target with squared distances $\|p_{ij}\|^2 - \kappa\lambda_i\lambda_j$, for which P is an incorrect equilibrium (Lemma 3.2). Every centred spanning incorrect equilibrium arises in this way, with $B = U^\top P$ and $rr^\top = S$.

Remark 5.2 (More agents). Consider $n = \ell + 2$ agents in $\mathbb{R}^\ell$ with all distances prescribed by a target in $\mathbb{R}^\ell$. The proofs of Lemmas 2.2 and 3.2 and Proposition 3.1 (with lines replaced by proper affine subspaces) and of Steps 1–4 of Section 5 carry over, with $W = \{\mathbf{1}, \lambda\}^\perp$ now of dimension ℓ and $K = s_2 I + nM$. They give $H_P[\lambda z^\top] \leq \kappa(\lambda_{\max}(NK) - s_2^2)\|z\|^2$ for a suitable unit vector z , so every incorrect equilibrium is unstable provided $\lambda_{\max}(NK) < s_2^2$ for all $\lambda \neq 0$ with $\sum_i \lambda_i = 0$. For $\lambda = (0, \dots, 0, 1, -1)$ one computes $\lambda_{\max}(NK)/s_2^2 = (n-2)/4$, so the method cannot work for $n \geq 6$. There the obstruction is genuine: for $\ell = 4$ (respectively $\ell \geq 5$) [CMRB26, Prop. 6.3] gives a non-strict (respectively strict) spurious second-order critical point with two coincident points, that is, with $\lambda = (0, \dots, 0, 1, -1)$. For five agents in $\mathbb{R}^3$ a numerical maximization gives $\sup_\lambda \lambda_{\max}(NK)/s_2^2 \approx 0.75$, which would suffice; we have not proved this inequality in five variables, so this case remains open here.

6 Sharpness and examples

Proposition 6.1. *Let the target be the rectangle $\bar{q}_1 = (0, 0)$, $\bar{q}_2 = (4, 0)$, $\bar{q}_3 = (4, \sqrt{6})$, $\bar{q}_4 = (0, \sqrt{6})$. The configuration $p_1 = (2, -1)$, $p_2 = (-2, 1)$, $p_3 = (-1, 2)$, $p_4 = (1, -2)$ is an incorrect equilibrium with $\lambda = (1, 1, -1, -1)$, $\kappa = 4$, $\|\lambda\|^2 = 4$ and $V(P) = 24$. The characteristic polynomial of $\nabla^2 V(P)$ is $x^3(x+8)(x-56)(x-80)(x^2-160x+1728)$, so*

$$\lambda_{\min}(\nabla^2 V(P)) = -8 = -\frac{\kappa}{2}\|\lambda\|^2 = -\sqrt{8V(P)/3}.$$

Hence both inequalities in Theorem 1.1(b) can be equalities.

Proof. The squared target distances in the order 12, 13, 14, 23, 24, 34 are (16, 22, 6, 6, 22, 16), and the errors are $e = (4, -4, -4, -4, -4, 4)$, so $e_{ij} = 4\lambda_i\lambda_j$, and $p_1 + p_2 - p_3 - p_4 = 0$; by Lemma 3.2, P is an equilibrium. The Hessian is an explicit 8×8 integer matrix, by (3), and its characteristic polynomial is computed directly. $\square$

Here P is itself a rectangle, with other side lengths and another order of the agents around its boundary, as in the incorrect rectangles described in [And15, p. 648]: its diagonals join agents 1, 2 and 3, 4, those of the target join 1, 3 and 2, 4. By Proposition 4.1, $\lambda_{\max}(NK) = s_2^2/2$ exactly when $\lambda = (a, -a, c, -c)$ up to order, that is, when two disjoint pairs of agents are joined by parallel segments or two agents coincide. The next examples show spanning incorrect equilibria for both target types of the question; all claims are checked in exact arithmetic (Verification, items 1 and 4).

Example 6.2 (Triangle with an interior agent). Let the squared target distances $\bar{d}_{ij}^2$, in the order 12, 13, 14, 23, 24, 34, be (21, 45, 336, 96, 309, 621). Here $X^* + \mathbf{11}^\top$ has characteristic polynomial $x(x-4)(x^2-357x+6480)$, so X^* is positive semidefinite of rank 2: the target is planar, for instance $\bar{q}_1 = (0, 0)$, $\bar{q}_2 \approx (4.583, 0)$, $\bar{q}_3 \approx (-3.273, 5.855)$, $\bar{q}_4 \approx (5.237, -17.566)$. Its affine dependency is $(-5, 1, 3, 1)$, so $\bar{q}_1 = \frac{1}{5}(\bar{q}_2 + 3\bar{q}_3 + \bar{q}_4)$ lies inside the triangle $\bar{q}_2\bar{q}_3\bar{q}_4$. The configuration P with rows (3, -2), (2, 0), (10, 0), (-15, 2) has $e = (-16, 8, 4, -32, -16, 8) = 4\lambda_i\lambda_j$ for $\lambda = (-1, 4, -2, -1)$, so it is an incorrect equilibrium with $\kappa = 4$ and $\|\lambda\|^2 = 22$. The characteristic polynomial of $\nabla^2 V(P)$ is

$$x^3(x^5 - 5360x^4 + 7214832x^3 - 2571180544x^2 - 137562323968x + 8413501194240);$$

it has exactly one negative root, $\lambda_{\min} \approx -77.69$, below the bound $-\frac{\kappa}{2}\|\lambda\|^2 = -44$ of Theorem 1.1, and four positive roots. For $z = (0, 1)$, $H_P[\lambda z^\top] = -1520 \leq -\frac{\kappa}{2}s_2^2\|z\|^2 = -968$.

Example 6.3 (Convex quadrilateral). Let the squared target distances, in the same order, be (417, 33, 48, 672, 297, 105). Now $X^* + \mathbf{11}^\top$ has characteristic polynomial $x(x-4)(x^2-393x+7200)$, and

the target's affine dependency is $(3, -1, -3, 1)$. So the segments $\bar{q}_1\bar{q}_4$ and $\bar{q}_2\bar{q}_3$ cross, and the target is a convex quadrilateral with these diagonals; since $3(\bar{q}_1 - \bar{q}_3) = \bar{q}_2 - \bar{q}_4$ it is a trapezoid, not a parallelogram. The configuration with rows $(-5, 0), (15, 5), (-9, -5), (-1, 0)$ has $e = (8, 8, -32, 4, -16, -16) = 4\lambda_i\lambda_j$ for $\lambda = (-2, -1, -1, 4)$, an incorrect equilibrium with $\kappa = 4$ and $\|\lambda\|^2 = 22$. The characteristic polynomial of $\nabla^2 V(P)$ is

$$x^3(x^5 - 5936x^4 + 9010800x^3 - 3736566784x^2 - 130423352320x + 13331005440000),$$

with exactly one negative root, $\lambda_{\min} \approx -71.69 < -44$, and four positive roots. For $z = (1, -2)$, $H_P[\lambda z^\top] = -6256 \leq -4840$.

7 Proof of Corollary 1.2

Existence and convergence. The right side of (1) is polynomial, so solutions exist locally and are unique. Summing (1) over i gives $\sum_i \dot{p}_i = 0$, so the centroid is constant. Along a solution $\frac{d}{dt}V = -\|\nabla V\|^2 \leq 0$, hence $e_{ij}^2 \leq 4V(p(0))$ and $\|p_{ij}\|^2 \leq \bar{d}_{ij}^2 + 2\sqrt{V(p(0))}$; with the centroid fixed, the solution stays in a compact set and exists for all $t \geq 0$. Since V is real-analytic, the Łojasiewicz gradient inequality implies that every bounded trajectory of its gradient flow has finite length and converges to a single critical point; see [AMA05] and the references there.

The time-one map. Let $\varphi : \mathbb{R}^8 \rightarrow \mathbb{R}^8$ be the time-one map of (1), defined on all of $\mathbb{R}^8$ by the first step. It is C^∞ , it is injective by uniqueness of solutions, and $\det D\varphi(x) = \exp(-\int_0^1 \text{tr} \nabla^2 V(\varphi_s(x)) ds) > 0$ by Liouville's formula, where φ_s is the time- s map. So φ is a C^∞ diffeomorphism onto an open subset of $\mathbb{R}^8$. At an equilibrium x we have $D\varphi(x) = \exp(-\nabla^2 V(x))$. If x is incorrect, Theorem 1.1 shows that $D\varphi(x)$ has an eigenvalue greater than 1.

Centre-stable manifolds. By the centre-stable manifold theorem [Shu87, Thm. III.7], in the form used in [LPP⁺19], every incorrect equilibrium x has an open neighbourhood B_x and an embedded C^1 disc $\mathcal{W}_x$ of dimension at most 7 such that every y with $\varphi^k(y) \in B_x$ for all $k \geq 0$ lies in $\mathcal{W}_x$. The set $\mathcal{A}$ of incorrect equilibria is covered by the sets B_x , $x \in \mathcal{A}$; since $\mathbb{R}^8$ is second countable, countably many of them, $B_{x_1}, B_{x_2}, \dots$, cover $\mathcal{A}$.

Measure zero. Suppose the solution from y converges to $x^* \in \mathcal{A}$. Then $\varphi^k(y) \rightarrow x^*$. Choose m with $x^* \in B_{x_m}$; since B_{x_m} is open, there is K with $\varphi^k(y) \in B_{x_m}$ for all $k \geq K$, so $\varphi^K(y) \in \mathcal{W}_{x_m}$. Hence the set of such y lies in $\bigcup_{m,K} \varphi^{-K}(\mathcal{W}_{x_m})$. Each $\mathcal{W}_{x_m}$ is a Lebesgue null set, and so is each $\varphi^{-K}(\mathcal{W}_{x_m})$, the image of a null set under the C^1 map $(\varphi^K)^{-1}$ on the open set $\varphi^K(\mathbb{R}^8)$. A countable union of null sets is null. All other solutions converge to an equilibrium with $e = 0$, that is, to a configuration congruent to the target.

The exceptional set. Every equilibrium is its own limit, and if the target is not a single point, the collocated configurations are incorrect equilibria. If all agents lie on a line $\{c + sa : s \in \mathbb{R}\}$, then every term $e_{ij}(p_j - p_i)$ in (1) is parallel to a , so the vector field is tangent to the affine subspace of configurations on this line, and by uniqueness the solution stays on it. Its limit is a collinear or collocated equilibrium, which is incorrect if the target is not collinear. (For a one-point target there are no incorrect equilibria, by the proof of Theorem 1.1(c), and the exceptional set is empty.) $\square$

Verification. The claims were checked as follows.

1. A program written for this note, `check_paper_formulas.py` (sympy, about three minutes), verifies in exact arithmetic Lemmas 2.1, 3.2 and 4.2 (the latter as identities of rational functions, and once more from the 4×4 projector form), (9)–(11), (7)–(8), Remark 4.3, and (12) at 25 random rational points. It also certifies Proposition 6.1, Examples 6.2 and 6.3 and the value $(n-2)/4$ of Remark 5.2 for $4 \leq n \leq 8$.

2. The programs with which the result was found verify Lemmas 2.1 and 3.2, (9)–(12) and (7)–(8) by separate code. Their numerical tests, which are evidence only, found 2,397 incorrect critical points for 320 targets in eight families, all with a negative Hessian eigenvalue, and $\max \lambda_{\min}(\nabla^2 V)/(\kappa s_2) = -0.500000$ over the 725 spanning ones; over 97,325 constructed spanning incorrect equilibria and an adversarial maximization the supremum of this ratio was $-\frac{1}{2}$, at $\eta_3 = 0$, and over 19,992 others $\max \lambda_{\min}(\nabla^2 V)/\sqrt{2V} = -1.1547 \approx -2/\sqrt{3}$. All 6,000 integrations of (1), for 150 targets, reached the target shape (45 after continued integration).
3. A first independent verification run, which was AI-assisted, checked every step of the proofs, re-ran the programs of item 2 (some with smaller samples) with consistent results, and wrote its own code. This code re-derives Proposition 4.1 with a non-orthonormal basis of W and checks a second certificate, $\mu K - K N K \succeq 0$, at 400 exact rational points; a damped Newton search on 67 targets found 426 incorrect critical points (134 spanning, 225 collinear, 67 collocated), all with a negative Hessian eigenvalue; and for 40,000 spanning incorrect equilibria built as in Remark 5.1, with a hill climb, the supremum of $\lambda_{\min}(\nabla^2 V)/(\kappa s_2)$ was -0.5 to ten decimal places.
4. A second independent verification run, also AI-assisted, re-read the question on the rendered pages of the report, checked the proofs line by line, including the derivation of Lemma 4.2, and wrote its own code from the text of the paper. This code obtains (6) without a basis of W , from the characteristic polynomial of the 4×4 matrix $\Pi(D + D\lambda\lambda^\top D/(2\Delta))\Pi(s_2\Pi + 4\Pi D\Pi) = U(NK)U^\top$, and checks (5), (7)–(8), Remark 4.3, (9)–(12), Proposition 6.1 and Examples 6.2 and 6.3 in exact arithmetic, with the Hessian obtained by differentiating V itself. Its numerical tests found 369 incorrect critical points for 56 targets of eleven types (116 spanning, 197 collinear, 56 collocated), all with a negative Hessian eigenvalue and, except the collocated ones, a positive one. Over 40,000 spanning incorrect equilibria built as in Remark 5.1 and an adversarial search, the suprema of $\lambda_{\min}(\nabla^2 V)/(\kappa s_2)$ and $\lambda_{\min}(\nabla^2 V)/\sqrt{2V}$ were -0.5000000000 and $-1.1547005384 \approx -2/\sqrt{3}$, and all 420 of its integrations of (1) reached the target shape. It also re-ran the programs of items 1–3 from the released source files, with consistent results.

Scope and priority. Theorem 1.1 answers the question of record OWR-13497-006 [ula26], as posed in [And15], affirmatively for every planar target, with “saddle point” read as in [And15] (where collocated equilibria count as saddle points); parts (a)–(c) are stronger. Corollary 1.2 answers records OWR-13497-001, -004 and -005 in the almost-global sense, and the literal global statement fails. The instability in part (a) is known [SHA15, PSAA18, CMRB26], Lemma 3.2 is classical, and for targets that satisfy the acuteness condition of [SYAD09] the instability of all incorrect equilibria is stated there. What is new, as far as we know, is the instability of the spanning incorrect equilibria for all targets that are not collinear, the bound in part (b) with its sharp constants, and Proposition 4.1. We read [And15], the authors’ version of [PSAA18], the arXiv versions of [SYDA13, SGJ⁺25, CMRB26, Cri26], and the abstracts of [SYAD09, AYDS10, DAYS11, SHA15, PST⁺16, SdQ24]. We did not see the full texts of these six works; what we say about them is taken from [And15, PSAA18, SYDA13] and from their abstracts. We searched arXiv (up to October 1, 2026), Crossref, zbMATH and Semantic Scholar, and the lists of works citing [PSAA18, PST⁺16, AYDS10, SHA15, SYDA13, SYAD09, SdQ24] in Semantic Scholar and OpenCitations, and ran four web searches in all; OpenAlex was rate-limited and could not be used for citation searches. We found no proof for the standard law with a general non-rectangular target. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments and the examples, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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