

Line Configurations in $\mathbb{P}^3$ Whose Symbolic Square Has Initial Degree $\alpha(I) + 1$ Are Coplanar or Pseudostars: A Negative Answer to a Question of Janssen

Alper Ferudun*

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Abstract

For a finite union L of lines in $\mathbb{P}^3$ with homogeneous ideal I , let $\alpha(I)$ be the least degree of a non-zero form in I , and let $I^{(2)}$ be the second symbolic power of I , the ideal of the forms that vanish to order at least two along every line of L ; one always has $\alpha(I^{(2)}) \geq \alpha(I) + 1$. Janssen proved (J. Pure Appl. Algebra 219 (2015)) that an arithmetically Cohen–Macaulay configuration with $\alpha(I^{(2)}) = \alpha(I) + 1$ is contained in a plane or is a pseudostar, that is, the set of all pairwise intersections of $m \geq 3$ planes no three of which contain a common line. He asked, there and in Oberwolfach Report 9/2015, whether a configuration of lines that is not arithmetically Cohen–Macaulay can satisfy this equality. We show that it cannot. Over an algebraically closed field of arbitrary characteristic, the equality $\alpha(I^{(2)}) = \alpha(I) + 1$ holds if and only if L is contained in a plane or is a pseudostar; both kinds of configurations are arithmetically Cohen–Macaulay, so the hypothesis in Janssen’s theorem is not needed. The same holds for reduced curves of pure dimension one: the equality holds exactly for plane curves and for pseudostars of lines. In the proof we factor a form of degree $\alpha(I) + 1$ that is singular along L ; the case of an irreducible form is excluded by the following statement, which we prove for arbitrary singularities and in every characteristic: the curves along which an irreducible surface of degree d is singular lie on a surface of degree $d - 2$. This statement is classical in spirit (adjoint surfaces). For $\alpha(I) \leq 2$ the classification already follows from a theorem of Haghighi and Mosakhani (2021), which is known to us only through its abstract and its zbMATH review. Computations, among them an exhaustive check of all $2^{35} - 1$ sets of lines of the projective space over the field with two elements, agree with the theorem; they are corroboration only and are not used in the proofs. This is an unrefereed note.

1 Introduction

1.1 The question

Throughout, k is an algebraically closed field of arbitrary characteristic, $S = k[x_0, x_1, x_2, x_3]$ is the homogeneous coordinate ring of $\mathbb{P}^3 = \mathbb{P}_k^3$, a *form* is a homogeneous element of S , and S_t is the space of forms of degree t . For a non-zero homogeneous ideal $J \subset S$ the *initial degree* $\alpha(J)$ is the least degree of a non-zero form in J . A *configuration of lines* is a finite union $L = \ell_1 \cup \cdots \cup \ell_s$ of $s \geq 1$ distinct lines of $\mathbb{P}^3$ with its reduced structure; its ideal is $I = I(L) = I(\ell_1) \cap \cdots \cap I(\ell_s)$. The *second symbolic power* of I is

$$I^{(2)} = S \cap \bigcap_{\mathfrak{p} \in \text{Ass}(I)} I^2 S_{\mathfrak{p}}$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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[Jan15b, Definition 1.1], [Jan15a, p. 513]. For a configuration of lines $I^{(2)} = I(\ell_1)^2 \cap \dots \cap I(\ell_s)^2$ (Lemma 2.1), so that $I^{(2)}$ consists of the forms vanishing to order at least two along every line of L , where “ F vanishes to order at least m along ℓ ” means $F \in I(\ell)^m$. One always has

$$\alpha(I^{(2)}) \geq \alpha(I) + 1$$

(Lemma 2.1(c); compare [Jan15b, p. 1056]). Following [Jan15b, p. 1056], where the terminology of Bocci and Chiantini [BC11] is adopted, one says that L has *type* $(d-1, d)$ if $\alpha(I) = d-1$ and $\alpha(I^{(2)}) = d$; we shall say that L has *gap one*.

The configuration L is *arithmetically Cohen–Macaulay* (ACM) if the ring $S/I(L)$ is Cohen–Macaulay [Jan15b, Definition 2.3]. Not every configuration is ACM; three skew lines are an example [Jan15b, p. 1060]. The configuration is *coplanar* if it is contained in a plane.

Definition 1.1 ([Jan15b, Definition 2.1]). Let $m \geq 3$, and let $H_1, \dots, H_m$ be planes in $\mathbb{P}^3$ no three of which contain a common line. The configuration formed by the lines $\ell_{ij} = H_i \cap H_j$, $1 \leq i < j \leq m$, is the *pseudostar* of $H_1, \dots, H_m$.

A pseudostar has $\binom{m}{2}$ lines (Lemma 3.2(a)). If the planes are in general position, it is a star configuration of lines in the sense of [GHM13]; for $m = 4$ these are the six edges of a tetrahedron. If all planes pass through one point, it is the cone over a star configuration of points in $\mathbb{P}^2$; for $m = 3$ these are three concurrent lines that are not coplanar (Figure 1).

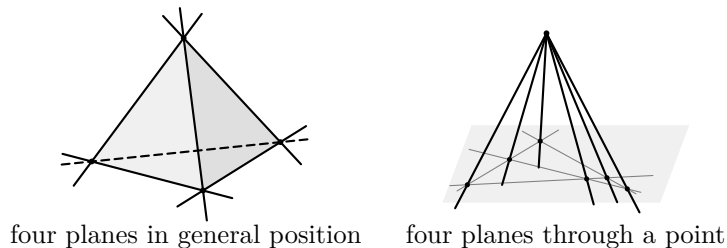


Figure 1: Two pseudostars with four planes: the six edges of a tetrahedron (left), and the six lines joining a point to the intersection points of four lines (grey) of a plane (right).

Janssen proved the following theorem, which extends the theorem of Bocci and Chiantini [BC11] on finite sets of points in $\mathbb{P}^2$ (gap one if and only if the points are collinear or form a star configuration; quoted as [Jan15b, Theorem 1.3]).

Theorem 1.2 (Janssen [Jan15b, Theorem 2.13], [Jan15a, p. 514]). *Let L be a configuration of lines in $\mathbb{P}^3$.*

- (a) *If L is ACM and has type $(d-1, d)$ for some $d > 1$, then L is a pseudostar or coplanar.*
- (b) *If L is a pseudostar or coplanar, then L has type $(d-1, d)$ for some $d > 1$.*

In the proof of (a) the ACM hypothesis is used to pass to a general plane section, which for an ACM configuration has the same initial degree [Jan15b, Corollary 2.8]; then [BC11] applies. Coplanar configurations are ACM [Jan15b, Lemma 2.4], and so are pseudostars ([Jan15b, Proposition 2.6], derived there from [GHM13, Proposition 2.9 and Remark 2.13]). So the only question left by Theorem 1.2 is whether the ACM hypothesis can be dropped. In the extended abstract of his talk in Oberwolfach [Jan15a], Janssen states three questions; the first one, on p. 514, begins “Does there exist a non-ACM configuration of lines L in $\mathbb{P}^3$ ” and continues: with $I = I(L)$ and $\alpha(I^{(2)}) - \alpha(I) = 1$? In the paper this is Question 3.1 [Jan15b, p. 1061]: it asks for a configuration of lines of type $(d-1, d)$

that is not ACM, and it is followed by the remark that a negative answer makes the ACM hypothesis of Theorem 1.2 unnecessary. The question is record OWR-13494-011 of the UnsolvedMath collection [ula26], where it is listed as open.

The statements of [Jan15b] carry the same numbers in the journal version and in the arXiv version 2; our page numbers are those of the journal version. The latter adds that all configurations of lines with gap one known at that time were ACM [Jan15b, p. 1057], and reports computed examples in which a disjoint line added to an ACM configuration of type $(d-1, d)$ gives a configuration of type $(d, d+2)$ that is not ACM [Jan15b, p. 1061]; this is a general fact (Remark 7.5).

Six conventions enter the question, and each of them could change the answer. (i) With the ordinary square I^2 in place of $I^{(2)}$ the difference would be $\alpha(I^2) - \alpha(I) = \alpha(I)$, and the question would be trivial. (ii) The lines carry their reduced structure; multiple structures are not considered. (iii) In [Jan15b] the field is algebraically closed of arbitrary characteristic. In positive characteristic the order of vanishing must be read as membership in $I(\ell)^m$, not through iterated partial derivatives. (iv) The abstract [Jan15a] does not fix the field; over a field that is not algebraically closed we consider lines defined over that field (Corollary 1.4). (v) The question is existential over all configurations; no genericity is involved. (vi) [Jan15b, Definition 2.1] puts no lower bound on the number of planes. Two planes give a single line, which is coplanar, and a pseudostar with $m \geq 3$ planes is not coplanar (Lemma 3.2(b)); so the class “coplanar or pseudostar” is the same with either convention.

1.2 Results

Theorem 1.3 (Classification). *Let k be an algebraically closed field of arbitrary characteristic, let $L \subset \mathbb{P}_k^3$ be a configuration of lines, and let $I = I(L)$. Then $\alpha(I^{(2)}) = \alpha(I) + 1$ if and only if*

- (a) *L is contained in a plane, or*
- (b) *L is a pseudostar.*

In case (a) one has $\alpha(I) = 1$ and $\alpha(I^{(2)}) = 2$. In case (b), if L is the pseudostar of the planes $H_i = \{h_i = 0\}$, $1 \leq i \leq m$, then $\alpha(I) = m - 1$, $\alpha(I^{(2)}) = m$, and every form of degree m in $I^{(2)}$ is a scalar multiple of $h_1 \cdots h_m$.

The “if” part is Theorem 1.2(b), for which we include a short direct proof (Lemmas 3.1 and 3.2).

Corollary 1.4 (Answer to the question). *Let K be an arbitrary field, let L be a finite union of lines of $\mathbb{P}_K^3$ each of which is defined over K , and let $I \subset K[x_0, \dots, x_3]$ be its ideal. If $\alpha(I^{(2)}) = \alpha(I) + 1$, then L is contained in a plane defined over K or is the pseudostar of $m \geq 3$ planes defined over K , and the ring $K[x_0, \dots, x_3]/I$ is Cohen–Macaulay. In particular there is no configuration of lines with $\alpha(I^{(2)}) - \alpha(I) = 1$ that is not ACM, and the ACM hypothesis in Theorem 1.2(a) is not needed.*

The argument does not use that the components are lines. By a *reduced curve* we mean a reduced closed subscheme of $\mathbb{P}^3$ all of whose irreducible components have dimension one.

Theorem 1.5 (Reduced curves). *Let k be as in Theorem 1.3, let $D \subset \mathbb{P}_k^3$ be a reduced curve, and let $I = I(D)$. Then $\alpha(I^{(2)}) = \alpha(I) + 1$ if and only if D is contained in a plane or D is a pseudostar of lines.*

This answers the third question of [Jan15a, p. 515], which is Question 3.2 of [Jan15b] (where it is said to have been suggested by J. Migliore): which reduced, possibly irreducible, curves in $\mathbb{P}^3$ have type $(d-1, d)$? The reading of “reduced curve” as a reduced scheme of pure dimension one is ours. If isolated points were admitted, the list would be longer: the union of the line $x_0 = x_1 = 0$ with the

points $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$ lies in no plane and has type $(2, 3)$: the form x_0x_1 lies in its ideal, and $x_0x_1x_2$, but no quadratic form, in the symbolic square.

The tool is the following statement about the singular curves of a surface. For an irreducible form F of degree d let $X = V(F)$, let $\nu: \bar{X} \rightarrow X$ be the normalisation, and put $\mathcal{O}_{\bar{X}}(n) = \nu^*\mathcal{O}_X(n)$. Let $\chi(n) = \chi(\bar{X}, \mathcal{O}_{\bar{X}}(n))$, and define

$$q = h^1(\bar{X}, \mathcal{O}_{\bar{X}}), \quad p_g = h^2(\bar{X}, \mathcal{O}_{\bar{X}}), \quad \pi = 1 - \chi(0) + \chi(-1). \quad (1)$$

If $\bar{C} \subset \bar{X}$ is the pull-back of a plane section of X , then $\chi(\mathcal{O}_{\bar{C}}) = 1 - \pi$; for a general plane, $\bar{C}$ is a reduced and connected curve and $\pi = h^1(\bar{C}, \mathcal{O}_{\bar{C}}) \geq 0$ is its arithmetic genus (Section 4.4).

Theorem 1.6 (Singular curves lie on adjoint surfaces). *Let k be as in Theorem 1.3, let $F \in S$ be an irreducible form of degree $d \geq 2$, let $X = V(F) \subset \mathbb{P}^3$, and let $\nu: \bar{X} \rightarrow X$ be the normalisation. Let $W \subset S_{d-2}$ be the space of the forms G of degree $d-2$ such that multiplication by G maps $\nu_*\mathcal{O}_{\bar{X}}$ into $\mathcal{O}_X(d-2)$. Let $D \subset X$ be a reduced curve, possibly empty, such that F vanishes to order at least two along every irreducible component of D , that is, $F \in \mathfrak{p}^2S_{\mathfrak{p}}$ for the prime ideal $\mathfrak{p}$ of every component. Then:*

- (a) *every form in W vanishes on D ;*
- (b) $\dim W = h^2(\bar{X}, \mathcal{O}_{\bar{X}}(-2)) \geq \max\{p_g, d-1+2\pi-q+p_g\};$
- (c) $q-p_g \leq \pi$, hence $\dim W \geq d-1+\pi \geq 1$.

In particular D lies on a surface of degree $d-2$.

For $d=2$ the last statement says that D is empty. For a cone over a plane curve Theorem 1.6 is the classical fact that the adjoint curves of degree $d-2$ pass through the singular points (Remark 4.11).

1.3 What was known, and what is new

Theorem 1.2 is due to Janssen, the theorem for points in the plane to Bocci and Chiantini [BC11], and the ACM property of pseudostars to Geramita, Harbourne and Migliore [GHM13] (as cited in [Jan15b, Proposition 2.6]). Fashami, Haghighi and Szemberg [FHS20, Theorem 1.1] proved the analogue of Theorem 1.2(a) for unions of codimension two linear subspaces of $\mathbb{P}^N$ whose ideal is Cohen–Macaulay, in characteristic zero; their Problem 4.4 asks whether a codimension two pseudo-star configuration can fail to be ACM, which is a different question. For points in $\mathbb{P}^3$, the subject of Question 3.3 of [Jan15b], see Bauer and Szemberg [BS15].

Haghighi and Mosakhani [HM21] classified the configurations of codimension two linear subspaces of $\mathbb{P}^N$ whose Waldschmidt constant is less than two, without a Cohen–Macaulay hypothesis. We know this paper only through its abstract and through the review Zbl 1473.14102, which states its Theorem B; the paper is not openly accessible and was not read. Theorem B, as stated there, implies Theorem 1.3 for $\alpha(I) \leq 2$ (Remark 7.1); abstract and review do not mention Janssen’s question.

Theorem 1.6 is classical in spirit. The space W is the part of degree $d-2$ of the conductor of the normalisation, classically the space of adjoint surfaces of degree $d-2$, which contain the multiple curves by definition; what has to be proved is that $W \neq 0$. In characteristic zero this follows from duality for the normalisation and from the vanishing of $H^1(\bar{X}, \mathcal{O}_{\bar{X}}(-1))$, which, as we recall it, is a theorem of Mumford [Mum67] (see Remark 4.10 for a derivation and for the status of this reference). In positive characteristic that vanishing is not available, and we compare the dimensions of two Picard schemes instead; the central step is a covering argument of Mumford [Mum67, p. 99], which we know from Kleiman’s account [Kle05, Remark 5.8]. We found no reference for Theorem 1.6 for arbitrary singularities in arbitrary characteristic; this is the only reason why its proof is written out.

To our knowledge the following are not in print: Theorem 1.3 for $\alpha(I) \geq 3$, and with it the answer to Janssen’s question in general; Theorem 1.5; and Theorem 1.6 in the stated generality. The searches, and their limits, are described under “Scope and priority” at the end of Section 8; in particular [HM21] itself could not be read. No claim of priority is made.

1.4 Idea of the proof

Let L have gap one, put $d = \alpha(I) + 1$, and let $F \neq 0$ be a form of degree d in $I^{(2)}$. We factor F (Figure 2). If F has a multiple factor, a count of degrees shows that L lies in a plane. If F is squarefree and reducible, we remove one factor and apply induction to the lines not lying on it; it follows that every factor of F is linear, and then L consists of all pairwise intersections of the d planes. If F is irreducible, Theorem 1.6 gives a non-zero form of degree $d - 2 = \alpha(I) - 1$ vanishing on L , which is absurd.

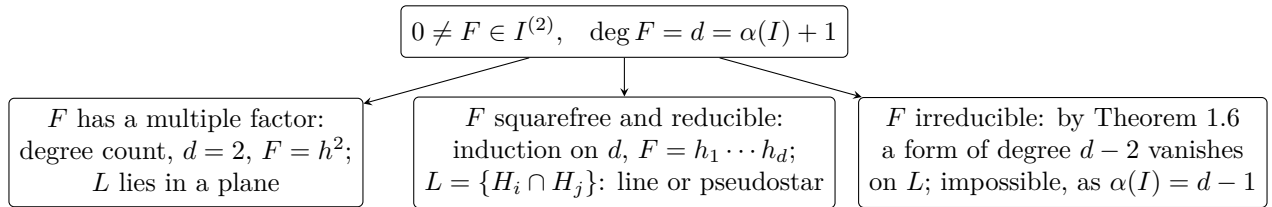


Figure 2: The case distinction in the proof of Theorem 1.3 (Section 5).

The method of [Jan15b, FHS20] needs the ACM hypothesis for the following reason. A configuration that is not ACM imposes more conditions on forms than its general plane section predicts, whereas the irreducible case needs an upper bound for the number of conditions which L imposes on forms of degree $d - 2$. Theorem 1.6 provides such forms on the surface itself.

1.5 What stays open

The analogue of Theorem 1.3 for codimension two linear subspaces of $\mathbb{P}^N$, $N \geq 4$, without the Cohen–Macaulay hypothesis of [FHS20], is not proved here: the third case of our proof would need an analogue of Theorem 1.6 for hypersurfaces in $\mathbb{P}^N$, which we have not carried out. The configurations of lines of type $(d - 2, d)$ (Question 3.4 of [Jan15b]), the configurations of points in $\mathbb{P}^3$ (Question 3.3), and the second question of [Jan15a, p. 514], which configurations of lines are ACM, are not treated.

2 Elementary facts

For a line $\ell \subset \mathbb{P}^3$ and a non-zero form F put

$$\text{mult}_\ell(F) = \max\{m \geq 0 : F \in I(\ell)^m\}.$$

If coordinates x, y, z, w are chosen such that $I(\ell) = (x, y)$, and $F = \sum_j F_j$ with F_j homogeneous of degree j in x, y with coefficients in $k[z, w]$, then $(x, y)^m$ consists of the F with $F_j = 0$ for $j < m$, so that $\text{mult}_\ell(F)$ is the least j with $F_j \neq 0$. For a plane $H = \{h = 0\}$ and a form F , the *restriction* $F|_H$ is the class of F in $S/(h)$, a polynomial ring in three variables.

Lemma 2.1. *Let $\ell \subset \mathbb{P}^3$ be a line and let F, G be non-zero forms.*

- (a) $\text{mult}_\ell(FG) = \text{mult}_\ell(F) + \text{mult}_\ell(G)$. For a non-zero linear form h , $\text{mult}_\ell(h) = 1$ if $\ell \subset \{h = 0\}$ and $\text{mult}_\ell(h) = 0$ otherwise.

- (b) Let $L = \ell_1 \cup \dots \cup \ell_s$ be a configuration of lines with ideal I . Then $I^{(2)} = I(\ell_1)^2 \cap \dots \cap I(\ell_s)^2$. Thus a non-zero form F lies in I if and only if $\text{mult}_{\ell_i}(F) \geq 1$ for all i , and in $I^{(2)}$ if and only if $\text{mult}_{\ell_i}(F) \geq 2$ for all i .
- (c) $\alpha(I^{(2)}) \geq \alpha(I) + 1$ for the ideal I of every configuration of lines. In particular $\alpha(I^{(2)}) \geq 2$.
- (d) Let $H = \{h = 0\}$ be a plane containing ℓ , and let λ be a linear form on H defining ℓ . If $h \nmid F$ and $\text{mult}_\ell(F) \geq m$, then $F|_H$ is a non-zero form divisible by λ^m .

Proof. (a) In the notation above the components of lowest degree multiply: $(FG)_{a+b} = F_a G_b \neq 0$ for $a = \text{mult}_\ell(F)$, $b = \text{mult}_\ell(G)$, because S is a domain. A linear form lies in (x, y) if and only if it vanishes on ℓ , and it does not lie in $(x, y)^2$.

(b) Let $\mathfrak{p} = I(\ell)$. If $fg \in \mathfrak{p}^2$ and $f \notin \mathfrak{p}$, then $\text{mult}_\ell(g) = \text{mult}_\ell(fg) \geq 2$ by (a), whose proof applies to all non-zero polynomials; so $\mathfrak{p}^2$ is $\mathfrak{p}$ -primary, and $\mathfrak{p}^2 S_{\mathfrak{p}} \cap S = \mathfrak{p}^2$. The associated primes of I are the $\mathfrak{p}_i = I(\ell_i)$. Localisation commutes with finite intersections, and $\mathfrak{p}_j S_{\mathfrak{p}_i} = S_{\mathfrak{p}_i}$ for $j \neq i$; hence $I S_{\mathfrak{p}_i} = \mathfrak{p}_i S_{\mathfrak{p}_i}$ and $I^2 S_{\mathfrak{p}_i} \cap S = \mathfrak{p}_i^2 S_{\mathfrak{p}_i} \cap S = \mathfrak{p}_i^2$.

(c) Let $F \neq 0$ be a form in $I^{(2)} = \bigcap \mathfrak{p}_i^2$. By the Leibniz rule every partial derivative of an element of $\mathfrak{p}_i^2$ lies in $\mathfrak{p}_i$, so all $\partial F / \partial x_j$ lie in I . If one of them is not zero, it is a non-zero form of degree $\deg F - 1$ in I . If all of them vanish, then every monomial of F has all its exponents divisible by the characteristic of k ; as F is not constant, the characteristic is a prime p , and, k being perfect, $F = G^p$ for a form G of degree $\deg F/p < \deg F$. Since $G^p \in I$ and I is radical, $G \in I$. In both cases $\alpha(I) \leq \deg F - 1$.

(d) Take coordinates with $h = x$ and $I(\ell) = (x, y)$, so that $\lambda = y$. Setting $x = 0$ maps $(x, y)^m$ into the ideal (y^m) of $k[y, z, w]$, and $F|_H \neq 0$ because $x \nmid F$. $\square$

A noetherian ring R is *Cohen–Macaulay* if $\text{depth } R_{\mathfrak{q}} = \dim R_{\mathfrak{q}}$ for every prime ideal $\mathfrak{q}$ of R . The following criterion suffices for our purposes.

Lemma 2.2. *Let K be a field, $S_K = K[x_0, \dots, x_3]$, and let $J \subset S_K$ be an ideal all of whose minimal primes have height two. If S_K/J has a resolution $0 \rightarrow S_K^a \rightarrow S_K^b \rightarrow S_K \rightarrow S_K/J \rightarrow 0$ by free modules of finite rank, then S_K/J is Cohen–Macaulay.*

Proof. Let $\mathfrak{q} \supset J$ be a prime ideal of S_K . Localising the resolution we see that the $S_{K, \mathfrak{q}}$ -module $(S_K/J)_{\mathfrak{q}}$ has projective dimension at most two. The local ring $S_{K, \mathfrak{q}}$ is regular of dimension $\text{ht } \mathfrak{q}$, so its depth is $\text{ht } \mathfrak{q}$, and the Auslander–Buchsbaum formula [Sta26, Tag 090V] gives $\text{depth}(S_K/J)_{\mathfrak{q}} \geq \text{ht } \mathfrak{q} - 2$. On the other hand $\dim(S_K/J)_{\mathfrak{q}}$ is the largest of the numbers $\text{ht}(\mathfrak{q}/\mathfrak{q}')$ for the minimal primes $\mathfrak{q}' \subset \mathfrak{q}$ of J , and $\text{ht}(\mathfrak{q}/\mathfrak{q}') = \text{ht } \mathfrak{q} - \text{ht } \mathfrak{q}' = \text{ht } \mathfrak{q} - 2$ by the dimension formula in the polynomial ring S_K . As the depth never exceeds the dimension, the two are equal. $\square$

3 The two families

The statements of this section are known or elementary: for the degrees see [Jan15b, Corollary 2.9 and Theorem 2.13(b)], for the ACM property [Jan15b, Lemma 2.4 and Proposition 2.6] and [GHM13]. We give complete proofs, valid in every characteristic, because the precise form of the statements is used in Section 5.

Lemma 3.1 (Coplanar configurations). *Let L consist of $s \geq 1$ distinct lines in a plane $H = \{h = 0\}$, and let $I = I(L)$.*

- (a) $\alpha(I) = 1$, $\alpha(I^{(2)}) = 2$, and S/I is Cohen–Macaulay.
- (b) If $s \geq 2$, the forms of degree two in $I^{(2)}$ are the scalar multiples of h^2 .

- (c) If $s = 1$ and $I = (h_1, h_2)$, the forms of degree two in $I^{(2)}$ are the quadratic forms in h_1, h_2 ; each of them is a product of two linear forms vanishing on the line.

Proof. (a) Let $\lambda_1, \dots, \lambda_s$ be linear forms on H defining the lines; they are pairwise non-proportional. In the factorial ring $S/(h)$ the intersection of the ideals (λ_i) is generated by $\lambda_1 \cdots \lambda_s$; hence $I = (h, g)$ for a form g of degree s restricting to $\lambda_1 \cdots \lambda_s$. Now $h \in I$ gives $\alpha(I) = 1$, and $h^2 \in I^{(2)}$ with Lemma 2.1(c) gives $\alpha(I^{(2)}) = 2$. As h, g is a regular sequence, the Koszul complex $0 \rightarrow S \rightarrow S^2 \rightarrow S$ resolves S/I ; the minimal primes of I are the ideals $I(\ell_i)$, of height two; so S/I is Cohen–Macaulay by Lemma 2.2.

(b) Let $G \in I^{(2)}$ have degree two, and let ℓ, ℓ' be two lines of L , defined in H by λ, λ' . If $h \nmid G$, then by Lemma 2.1(d) the non-zero form $G|_H$ of degree two is divisible by λ^2 and by λ'^2 , hence by $\lambda^2 \lambda'^2$, which has degree four. So $G = hh'$ with a linear form h' . By Lemma 2.1(a), $\text{mult}_\ell(h') = \text{mult}_\ell(G) - 1 \geq 1$, and likewise for ℓ' ; thus h' vanishes on $\ell \cup \ell'$, which spans H , and $h' \in kh$.

(c) The part of degree two of $(h_1, h_2)^2$ is spanned by $h_1^2, h_1 h_2, h_2^2$, and a binary quadratic form over the algebraically closed field k is a product of two linear forms. $\square$

Lemma 3.2 (Pseudostars). *Let $m \geq 3$, let $H_i = \{h_i = 0\}$, $1 \leq i \leq m$, be planes no three of which contain a common line, let L be their pseudostar, with lines $\ell_{ij} = H_i \cap H_j$ and ideal I , and put $\Phi = h_1 \cdots h_m$ and $\Phi_i = \Phi/h_i$.*

- (a) *The planes are distinct, the ℓ_{ij} with $i < j$ are $\binom{m}{2}$ distinct lines, and $\ell_{ij} \not\subset H_r$ for $r \notin \{i, j\}$.*
- (b) *$\alpha(I) = m - 1$ and $\alpha(I^{(2)}) = m$. In particular L is not coplanar.*
- (c) *The forms of degree m in $I^{(2)}$ are the scalar multiples of Φ .*
- (d) *$I = (\Phi_1, \dots, \Phi_m)$, the module S/I has a free resolution $0 \rightarrow S^{m-1} \rightarrow S^m \rightarrow S \rightarrow S/I \rightarrow 0$, and L is ACM.*

Proof. (a) If $H_i = H_j$ for some $i \neq j$, then H_i, H_j and any third plane contain a common line. If $\ell_{ij} \subset H_r$ with $r \notin \{i, j\}$, then H_i, H_j, H_r contain ℓ_{ij} . If $\ell_{ij} = \ell_{i'j'}$ for two different pairs, say $i' \notin \{i, j\}$, then $\ell_{ij} \subset H_{i'}$.

(b) Every line ℓ_{ij} lies on one of the planes H_r with $r \neq 1$, so $\Phi_1 \in I$; and $\text{mult}_{\ell_{ij}}(\Phi) = 2$ by Lemma 2.1(a) and part (a), so $\Phi \in I^{(2)}$. Let $G \neq 0$ be a form in I of degree at most $m - 2$. If $h_1 \nmid G$, then $G|_{H_1}$ is a non-zero form on the plane H_1 which vanishes on the $m - 1$ distinct lines ℓ_{1j} , so its degree is at least $m - 1$, a contradiction. Thus h_1 divides G , and likewise every h_i ; as the h_i are pairwise non-proportional, Φ divides G , which is impossible. Hence $\alpha(I) = m - 1$, and $\alpha(I^{(2)}) = m$ by Lemma 2.1(c). A coplanar configuration has $\alpha(I) = 1 < m - 1$.

(c) Let $G \in I^{(2)}$ have degree m . If $h_1 \nmid G$, then by Lemma 2.1(d), applied to the $m - 1$ lines $\ell_{1j} \subset H_1$, the non-zero form $G|_{H_1}$ is divisible by the product of the squares of their equations, of degree $2(m - 1) > m$; here $m \geq 3$ is used. So h_1 , and likewise every h_i , divides G , and $G \in k\Phi$.

(d) We first show that $J = (\Phi_1, \dots, \Phi_m)$ equals $I = \bigcap_{i < j} (h_i, h_j)$; this holds for any $m \geq 2$ distinct planes no three of which contain a common line, by induction on m . For $m = 2$ both ideals are (h_1, h_2) . Let $m \geq 3$, put $\Phi' = h_1 \cdots h_{m-1}$ and $J' = (\Phi'/h_1, \dots, \Phi'/h_{m-1})$; by induction $J' = \bigcap_{i < j < m} (h_i, h_j)$. We have $J = h_m J' + (\Phi')$ and $\Phi' \in J'$, so $J \subset J' \cap (h_m, \Phi')$. Conversely, let $g = ah_m + b\Phi'$ lie in J' . Then $ah_m \in J'$; by (a), h_m lies in none of the prime ideals (h_i, h_j) with $i < j < m$, so $a \in J'$ and $g \in J$. Thus $J = J' \cap (h_m, \Phi')$. Finally, in the factorial ring $S/(h_m)$ the classes of $h_1, \dots, h_{m-1}$ are pairwise non-proportional linear forms, by (a), so the ideal generated by their product is the intersection of the ideals which they generate: $(h_m, \Phi') = \bigcap_{i < m} (h_i, h_m)$. Together, $J = I$.

Now let $\sum a_i \Phi_i = 0$ with $a_i \in S$. Modulo h_1 this reads $a_1 \Phi_1 \equiv 0$, and $h_1 \nmid \Phi_1$, so $a_1 = b_1 h_1$; in the same way $a_i = b_i h_i$ for all i . Then $(\sum b_i) \Phi = 0$, so $\sum b_i = 0$, and with $c_j = b_1 + \dots + b_j$ we get $(a_1, \dots, a_m) = \sum_{j=1}^{m-1} c_j (h_j e_j - h_{j+1} e_{j+1})$. The $m-1$ vectors $h_j e_j - h_{j+1} e_{j+1}$ are relations between $\Phi_1, \dots, \Phi_m$, and they are linearly independent over S . Hence $0 \rightarrow S^{m-1} \rightarrow S^m \rightarrow S \rightarrow S/I \rightarrow 0$ is exact, and S/I is Cohen–Macaulay by Lemma 2.2. $\square$

Remark 3.3. The proofs of Lemma 2.1(a), (b), (d), of Lemma 2.2, of the Cohen–Macaulay property in Lemma 3.1(a), and of Lemma 3.2(a) and (d) do not use that the field is algebraically closed; they are valid over an arbitrary field K , for lines and planes defined over K .

4 Singular curves lie on adjoint surfaces

We prove Theorem 1.6: part (a) in Section 4.2, part (b) in Section 4.3, and, after the study of a general plane section in Section 4.4, $W \neq 0$ and part (c) in Section 4.5.

4.1 Notation and conventions

Let F be an irreducible form of degree $d \geq 2$. Then $X = V(F)$ is an integral projective surface in $P = \mathbb{P}^3$; let $i: X \rightarrow P$ be the inclusion. The normalisation $\nu: \bar{X} \rightarrow X$ is a finite birational morphism, and $\bar{X}$ is an integral normal projective surface [Sta26, Tag 0BXR]; the sheaf $\mathcal{O}_{\bar{X}}(1) = \nu^* \mathcal{O}_X(1)$ is ample [Sta26, Tag 0892]. We put

$$\mathcal{A} = \nu_* \mathcal{O}_{\bar{X}}, \quad \mathcal{T} = \mathcal{A} / \mathcal{O}_X.$$

Thus $\mathcal{A}$ is a coherent sheaf of $\mathcal{O}_X$ -algebras with $\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{K}$, where $\mathcal{K}$ is the constant sheaf of rational functions on X ; the stalk $\mathcal{A}_x$ is the integral closure of $\mathcal{O}_{X,x}$; and $\mathcal{T}$ is supported on the closed set of dimension at most one where X is not normal. For a sheaf $\mathcal{G}$ on X we write $\mathcal{G}(n) = \mathcal{G} \otimes \mathcal{O}_X(n)$. By the projection formula $\nu_* \mathcal{O}_{\bar{X}}(n) = \mathcal{A}(n)$, and $H^j(\bar{X}, \mathcal{O}_{\bar{X}}(n)) = H^j(X, \mathcal{A}(n))$ because ν is affine. Since $\bar{X}$ is integral and projective and k is algebraically closed, $H^0(\bar{X}, \mathcal{O}_{\bar{X}}) = k$. Since $\bar{X}$ is normal of dimension two, its local rings at closed points have depth two [Sta26, Tag 031S]. The numbers $\chi(n)$, q , p_g , π are those of (1).

Let $D_0 \subset X$ be an integral curve, $\mathfrak{p} \subset S$ its prime ideal, η its generic point, and $\mathfrak{n}$ the maximal ideal of the regular local ring $\mathcal{O}_{P,\eta}$, which has dimension two. If $x_0 \notin \mathfrak{p}$, then $f = F/x_0^d$ is a local equation of X at η , and

$$F \in \mathfrak{p}^2 S_{\mathfrak{p}} \iff f \in \mathfrak{n}^2, \quad (2)$$

because $S_{\mathfrak{p}}$ is a localisation of $\mathcal{O}_{P,\eta}[x_0, x_0^{-1}]$, hence faithfully flat over $\mathcal{O}_{P,\eta}$, with $\mathfrak{p} S_{\mathfrak{p}} = \mathfrak{n} S_{\mathfrak{p}}$. This is the hypothesis of Theorem 1.6. If D_0 is a line ℓ , it is equivalent to $F \in I(\ell)^2$, by Lemma 2.1(b).

We use two standard facts on $P = \mathbb{P}^3$: $H^1(P, \mathcal{O}_P(n)) = 0$ and $\chi(P, \mathcal{O}_P(n)) = \binom{n+3}{3}$ for all n ([Har77, III, Theorem 5.1], [Sta26, Tag 01XT]); and Serre duality: for a coherent sheaf $\mathcal{F}$ on P there are natural isomorphisms $\text{Ext}_P^j(\mathcal{F}, \mathcal{O}_P(-4)) \cong H^{3-j}(P, \mathcal{F})^\vee$ ([Har77, III, Theorem 7.1]; in the language of derived categories, [Sta26, Tag 0A9W]).

4.2 The space W ; proof of part (a)

The sequence $0 \rightarrow \mathcal{O}_P(n-d) \xrightarrow{F} \mathcal{O}_P(n) \rightarrow i_* \mathcal{O}_X(n) \rightarrow 0$ and $H^1(P, \mathcal{O}_P(n-d)) = 0$ show that $S_n \rightarrow H^0(X, \mathcal{O}_X(n))$ is surjective with kernel FS_{n-d} . In particular $S_{d-2} = H^0(X, \mathcal{O}_X(d-2))$. For $G \in S_{d-2}$, multiplication by G is a homomorphism of $\mathcal{O}_X$ -modules $\mu_G: \mathcal{A} \rightarrow \mathcal{K}(d-2)$, and by definition

$$W = \{ G \in S_{d-2} : \mu_G(\mathcal{A}) \subset \mathcal{O}_X(d-2) \}.$$

Lemma 4.1. *The map $G \mapsto \mu_G$ is an isomorphism of W onto $\text{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X(d-2))$.*

Proof. It is injective because $\mu_G(1) = G$. Let $\varphi: \mathcal{A} \rightarrow \mathcal{O}_X(d-2)$ be a homomorphism, and let G be the form with $\varphi(1) = G$. A local section a of $\mathcal{A}$ can be written as $a = b/c$ with local sections b, c of $\mathcal{O}_X$, $c \neq 0$; then $c\varphi(a) = \varphi(b) = bG$, hence $\varphi(a) = aG$ in $\mathcal{K}(d-2)$. So $\varphi = \mu_G$ and $G \in W$. $\square$

Proof of Theorem 1.6(a). Let η be the generic point of a component of D . We may assume that x_0 does not vanish at η , and we keep the notation of (2). The local ring $\mathcal{O}_{X,\eta} = \mathcal{O}_{P,\eta}/(f)$ is a local domain of dimension one with maximal ideal $\mathfrak{n}/(f)$. As $f \in \mathfrak{n}^2$, the cotangent space $\mathfrak{n}/(\mathfrak{n}^2 + (f)) = \mathfrak{n}/\mathfrak{n}^2$ has dimension two, so $\mathcal{O}_{X,\eta}$ is not regular. A normal noetherian local domain of dimension one is a discrete valuation ring, hence regular; therefore $\mathcal{O}_{X,\eta}$ is not normal, and $\mathcal{O}_{X,\eta} \subsetneq \mathcal{A}_\eta$. Let $G \in W$, and put $\gamma = G/x_0^{d-2} \in \mathcal{O}_{X,\eta}$. Then $\gamma\mathcal{A}_\eta \subset \mathcal{O}_{X,\eta}$. If γ were a unit of $\mathcal{O}_{X,\eta}$, we would get $\mathcal{A}_\eta \subset \mathcal{O}_{X,\eta}$. So γ lies in the maximal ideal: G vanishes at η , and therefore on the component. $\square$

4.3 Duality and Euler characteristics; proof of part (b)

Lemma 4.2. *For every coherent sheaf $\mathcal{G}$ on X , $\dim_k \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X(d-4)) = h^2(X, \mathcal{G})$.*

Proof. By Serre duality on P , the dual of $H^2(X, \mathcal{G}) = H^2(P, i_*\mathcal{G})$ is $\text{Ext}_P^1(i_*\mathcal{G}, \mathcal{O}_P(-4))$. Apply $\text{Hom}_P(i_*\mathcal{G}, -)$ to $0 \rightarrow \mathcal{O}_P(-4) \xrightarrow{F} \mathcal{O}_P(d-4) \rightarrow i_*\mathcal{O}_X(d-4) \rightarrow 0$:

$$\text{Hom}_P(i_*\mathcal{G}, \mathcal{O}_P(d-4)) \rightarrow \text{Hom}_P(i_*\mathcal{G}, i_*\mathcal{O}_X(d-4)) \rightarrow \text{Ext}_P^1(i_*\mathcal{G}, \mathcal{O}_P(-4)) \xrightarrow{F} \text{Ext}_P^1(i_*\mathcal{G}, \mathcal{O}_P(d-4)).$$

The first group is zero, because $i_*\mathcal{G}$ is annihilated by F and $\mathcal{O}_P(d-4)$ is torsion-free. The last map is zero: multiplication by F on the second argument of Ext_P^j agrees with multiplication by F on the first argument (the two agree on Hom , hence on its derived functors), and F annihilates $i_*\mathcal{G}$. Hence $\text{Hom}_X(\mathcal{G}, \mathcal{O}_X(d-4)) = \text{Hom}_P(i_*\mathcal{G}, i_*\mathcal{O}_X(d-4)) \cong \text{Ext}_P^1(i_*\mathcal{G}, \mathcal{O}_P(-4))$. $\square$

By Lemmas 4.1 and 4.2, applied to $\mathcal{G} = \mathcal{A}(-2)$,

$$\dim W = \dim \text{Hom}_{\mathcal{O}_X}(\mathcal{A}(-2), \mathcal{O}_X(d-4)) = h^2(X, \mathcal{A}(-2)) = h^2(\overline{X}, \mathcal{O}_{\overline{X}}(-2)). \quad (3)$$

Lemma 4.3. *Let h be a non-zero linear form, and let $\overline{C} \subset \overline{X}$ be the zero scheme of the section ν^*h of $\mathcal{O}_{\overline{X}}(1)$.*

- (i) $\chi(n) - 2\chi(n-1) + \chi(n-2) = d$ for all $n \in \mathbb{Z}$.
- (ii) *The sequences $0 \rightarrow \mathcal{O}_{\overline{X}}(n-1) \xrightarrow{\nu^*h} \mathcal{O}_{\overline{X}}(n) \rightarrow \mathcal{O}_{\overline{C}}(n) \rightarrow 0$ are exact, and $\chi(\mathcal{O}_{\overline{C}}(n)) = nd + 1 - \pi$.*
- (iii) $\chi(-2) = d - 1 + 2\pi - q + p_g$.
- (iv) $H^0(\overline{X}, \mathcal{O}_{\overline{X}}(n)) = 0$ for $n < 0$.
- (v) *The maps $H^2(\overline{X}, \mathcal{O}_{\overline{X}}(-2)) \rightarrow H^2(\overline{X}, \mathcal{O}_{\overline{X}}(-1)) \rightarrow H^2(\overline{X}, \mathcal{O}_{\overline{X}})$ induced by ν^*h are surjective.*

Proof. (i) We have $\chi(n) = \chi(X, \mathcal{O}_X(n)) + \chi(X, \mathcal{T}(n))$. The first summand is $\binom{n+3}{3} - \binom{n+3-d}{3}$; the second difference of $\binom{n+3}{3}$ is $n+1$, so that of the first summand is $(n+1) - (n+1-d) = d$. For the second summand choose a linear form h' that vanishes at none of the finitely many associated points of $\mathcal{T}$. Then $0 \rightarrow \mathcal{T}(n-1) \xrightarrow{h'} \mathcal{T}(n) \rightarrow (\mathcal{T}/h'\mathcal{T})(n) \rightarrow 0$ is exact, and $\mathcal{T}/h'\mathcal{T}$ has finite support, so that $\chi(\mathcal{T}(n)) - \chi(\mathcal{T}(n-1)) = h^0(\mathcal{T}/h'\mathcal{T})$ does not depend on n . Thus the second difference of $\chi(X, \mathcal{T}(n))$ is zero.

(ii) As $d \geq 2$, h does not vanish on X , so ν^*h is a non-zero section of an invertible sheaf on the integral scheme $\overline{X}$; this gives the exact sequences. Put $c(n) = \chi(\mathcal{O}_{\overline{C}}(n)) = \chi(n) - \chi(n-1)$. By (i), $c(n) - c(n-1) = d$, and $c(0) = 1 - \pi$ by the definition of π .

(iii) $\chi(-2) = \chi(0) - c(0) - c(-1) = \chi(0) - (1 - \pi) - (1 - \pi - d)$, and $\chi(0) = 1 - q + p_g$.

(iv) If $s \neq 0$ is a section of $\mathcal{O}_{\overline{X}}(n)$ with $n < 0$, then $s \cdot (\nu^*h)^{-n}$ is a non-zero element of $H^0(\overline{X}, \mathcal{O}_{\overline{X}}) = k$ which vanishes on $\overline{C} \neq \emptyset$, a contradiction.

(v) This follows from (ii), because $H^2(\overline{C}, \mathcal{O}_{\overline{C}}(n)) = 0$ for the scheme $\overline{C}$ of dimension one. $\square$

Proof of Theorem 1.6(b). The equality is (3). By Lemma 4.3(v), $h^2(\overline{X}, \mathcal{O}_{\overline{X}}(-2)) \geq p_g$. By Lemma 4.3(iii) and (iv), $h^2(\overline{X}, \mathcal{O}_{\overline{X}}(-2)) = \chi(-2) + h^1(\overline{X}, \mathcal{O}_{\overline{X}}(-2)) \geq d - 1 + 2\pi - q + p_g$. $\square$

4.4 A general plane section

Lemma 4.4. *For every plane $H = \{h = 0\}$ the scheme $\overline{C}$ of Lemma 4.3 is connected, and for a suitable plane it is reduced. If $\overline{C}$ is reduced, then $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = k$ and $h^1(\overline{C}, \mathcal{O}_{\overline{C}}) = \pi$. In particular $\pi \geq 0$.*

Proof. Connectedness. The scheme $\overline{X}$ is proper and connected, its local rings at closed points have depth two, $\mathcal{O}_{\overline{X}}(1)$ is ample, and ν^*h is a regular section; so its zero scheme is connected by [Sta26, Tag 0FD9].

Reducedness. Let $U \subset X$ be the open set where not all partial derivatives of F vanish. It is not empty: otherwise every $\partial F / \partial x_j$ would be divisible by F , hence zero, and F would be a constant or, in characteristic p , a p -th power (as in the proof of Lemma 2.1(c)), which is not irreducible. By the Jacobian criterion X is smooth at the points of U , so ν is an isomorphism over U , and $X \setminus U$ has dimension at most one. The tangent planes $T_x X$, $x \in U$, are the points of the image of the morphism $U \rightarrow (\mathbb{P}^3)^\vee$, $x \mapsto [\partial F / \partial x_0(x) : \cdots : \partial F / \partial x_3(x)]$; the closure of this image has dimension at most two. The planes containing a given curve form a closed subset of $(\mathbb{P}^3)^\vee$ of dimension at most one. Hence there is a plane H which is not in the closure of the set of tangent planes and contains no curve of $X \setminus U$. For $x \in U \cap H$ the curve $X \cap H \subset H$ is defined by $F|_H$, whose differential at x is not zero because $H \neq T_x X$; so $X \cap H$ is smooth at x . The set $(X \setminus U) \cap H$ is finite. Consequently $\overline{C} = \nu^{-1}(X \cap H)$ is smooth outside a finite set. Every irreducible component of the effective Cartier divisor $\overline{C}$ has dimension one, so $\overline{C}$ is reduced at its generic points; and $\overline{C}$ has no embedded points, because the local rings $\mathcal{O}_{\overline{C}, y} = \mathcal{O}_{\overline{X}, y} / (\nu^*h)$ at closed points have depth one. Hence $\overline{C}$ is reduced.

Cohomology. If $\overline{C}$ is reduced, $H^0(\overline{C}, \mathcal{O}_{\overline{C}})$ is a reduced finite-dimensional k -algebra, hence a product of copies of k , and it has no idempotents other than 0 and 1 because $\overline{C}$ is connected; so it is k . Then $h^1(\mathcal{O}_{\overline{C}}) = 1 - \chi(\mathcal{O}_{\overline{C}}) = \pi$ by Lemma 4.3(ii). $\square$

4.5 Two Picard schemes; proof of part (c)

We use the theory of the Picard scheme in the form of Kleiman's article [Kle05]; the numbers of the statements are those of the arXiv version math/0504020, in which they were read. Let Y be a projective scheme over k with $H^0(Y, \mathcal{O}_Y) = k$; below, Y is $\overline{X}$ or a reduced curve $\overline{C}$ as in Lemma 4.4.

(K1) The Picard scheme $\text{Pic}_{Y/k}$ exists. It is a commutative group scheme, separated and locally of finite type over k , and it represents the functor $T \mapsto \text{Pic}(Y \times T) / \text{Pic}(T)$ on k -schemes. In particular $\text{Pic}_{Y/k}(k) = \text{Pic}(Y)$, and $\text{Pic}_{Y/k}(B) = \text{Pic}(Y_B)$ for every local artinian k -algebra B , where $Y_B = Y \times_k \text{Spec } B$. ([Kle05, Corollary 4.18.3] gives the existence for the fppf sheaf associated with this functor, [Kle05, Proposition 5.3] the separatedness. Since $H^0(Y, \mathcal{O}_Y) = k$ holds universally, by flat base change, and since Y has a k -rational point, the functor is already a sheaf: [Kle05, Theorem 2.5 and Exercise 2.3].)

- (K2) The tangent space of $\mathrm{Pic}_{Y/k}$ at 0 is $H^1(Y, \mathcal{O}_Y)$ [Kle05, Theorem 5.11]. Hence $\dim \mathrm{Pic}_{Y/k} \leq h^1(Y, \mathcal{O}_Y)$, with equality if and only if $\mathrm{Pic}_{Y/k}$ is smooth [Kle05, Corollary 5.13].
- (K3) $\mathrm{Pic}_{Y/k}$ is smooth if $H^2(Y, \mathcal{O}_Y) = 0$ [Kle05, Proposition 5.19], and if k has characteristic zero [Kle05, Corollary 5.14].
- (K4) The connected component $\mathrm{Pic}_{Y/k}^0$ of 0 is an open and closed group subscheme of finite type, and it is irreducible [Kle05, Lemma 5.1 and Proposition 5.3]. If Y is integral and normal, $\mathrm{Pic}_{Y/k}^0$ is projective [Kle05, Theorem 5.4].

We also need two facts on group schemes over the algebraically closed field k .

- (G1) If G is a group scheme of finite type over k , then G_{red} is a closed subgroup scheme, and it is smooth (the argument in the proof of [Kle05, Lemma 5.1], given there for the connected component of the identity; [Sta26, Tags 047N, 047P]). If moreover G is proper and connected, G_{red} is an abelian variety [Sta26, Tag 0H2U].
- (G2) If B is an abelian variety of dimension g over k and n is a positive integer invertible in k , then the group of points of $B(k)$ of order dividing n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$ [Sta26, Tag 03RP].

By (K4) and (G1),

$$A = (\mathrm{Pic}_{\overline{X}/k}^0)_{\mathrm{red}}$$

is an abelian variety, and $\dim A = \dim \mathrm{Pic}_{\overline{X}/k}$.

Proposition 4.5. $\dim \mathrm{Pic}_{\overline{X}/k} \leq \pi$.

Proof. Fix a plane H such that $\overline{C}$ is reduced (Lemma 4.4). Restriction of invertible sheaves from $\overline{X} \times T$ to $\overline{C} \times T$ is a morphism of the functors in (K1), hence a homomorphism of group schemes $\mathrm{Pic}_{\overline{X}/k} \rightarrow \mathrm{Pic}_{\overline{C}/k}$; let $r: A \rightarrow \mathrm{Pic}_{\overline{C}/k}$ be its restriction to A . We claim that the kernel N of r is finite. Then the fibres of r are finite, so $\dim A \leq \dim \mathrm{Pic}_{\overline{C}/k} \leq h^1(\overline{C}, \mathcal{O}_{\overline{C}}) = \pi$ by (K2) and Lemma 4.4, which proves the proposition.

Suppose that N , a closed subgroup scheme of A , has positive dimension. By (G1), $B = (N^0)_{\mathrm{red}}$ is an abelian variety of positive dimension. Let n be a prime different from the characteristic of k . By (G2) there is a point of $B(k)$ of order n . By (K1) it is the class of an invertible sheaf $\mathcal{N}$ on $\overline{X}$ such that

$$\mathcal{N}^{\otimes j} \not\cong \mathcal{O}_{\overline{X}} \quad (0 < j < n), \quad \mathcal{N}^{\otimes n} \cong \mathcal{O}_{\overline{X}}, \quad \mathcal{N}|_{\overline{C}} \cong \mathcal{O}_{\overline{C}}.$$

Step 1. $H^0(\overline{X}, \mathcal{N}^{\otimes j}) = 0$ for $0 < j < n$. Indeed, if $s \neq 0$ is a section of $\mathcal{N}^{\otimes j}$, then $s^{\otimes n}$ is a non-zero section of $\mathcal{N}^{\otimes jn} \cong \mathcal{O}_{\overline{X}}$, hence a non-zero constant; so s vanishes nowhere and $\mathcal{N}^{\otimes j}$ is trivial.

Step 2. Fix an isomorphism $\sigma: \mathcal{N}^{\otimes n} \rightarrow \mathcal{O}_{\overline{X}}$ and let $\mathcal{B} = \bigoplus_{j=0}^{n-1} \mathcal{N}^{\otimes j}$, with the structure of an $\mathcal{O}_{\overline{X}}$ -algebra given by the tensor product and by σ . Let $Y = \mathrm{Spec} \mathcal{B}$, a scheme finite over $\overline{X}$. By Step 1, $H^0(Y, \mathcal{O}_Y) = \bigoplus_j H^0(\overline{X}, \mathcal{N}^{\otimes j}) = k$.

Step 3. For $t \geq 1$ let $D_t \subset Y$ be the zero scheme of the pull-back of $(\nu^* h)^t$. As $\mathcal{B}$ is locally free over the integral scheme $\overline{X}$, there are exact sequences $0 \rightarrow \mathcal{B}(-t) \rightarrow \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{O}_{t\overline{C}} \rightarrow 0$, where $t\overline{C}$ is the zero scheme of $(\nu^* h)^t$, and they give exact sequences

$$k = H^0(Y, \mathcal{O}_Y) \rightarrow H^0(D_t, \mathcal{O}_{D_t}) \rightarrow \bigoplus_{j=0}^{n-1} H^1(\overline{X}, \mathcal{N}^{\otimes j}(-t)).$$

The stalks of the invertible sheaves $\mathcal{N}^{\otimes j}$ at closed points have depth two, so by the lemma of Enriques–Severi–Zariski [Sta26, Tag 0FD8] the group on the right vanishes for $t \gg 0$. For such t the ring

$H^0(D_t, \mathcal{O}_{D_t})$ is a quotient of k ; therefore it has no non-trivial idempotents, and D_t is connected. As D_1 has the same underlying space as D_t , D_1 is connected.

Step 4. On the other hand $D_1 = \text{Spec}(\mathcal{B}|_{\overline{C}})$. Let e be a generator of $\mathcal{N}|_{\overline{C}} \cong \mathcal{O}_{\overline{C}}$. Then $u = \sigma(e^{\otimes n})$ is a unit of $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = k$ (Lemma 4.4), and $\mathcal{B}|_{\overline{C}} \cong \mathcal{O}_{\overline{C}} \otimes_k k[e]/(e^n - u)$. As n is invertible in k and $u \neq 0$, the polynomial $e^n - u$ has n distinct roots in k , so D_1 is the disjoint union of $n \geq 2$ copies of $\overline{C} \neq \emptyset$. This contradicts Step 3. $\square$

Corollary 4.6. *If $p_g = 0$, or if k has characteristic zero, then $q \leq \pi$. In every case $W \neq 0$.*

Proof. Under either hypothesis $\text{Pic}_{\overline{X}/k}$ is smooth by (K3), so its dimension is q by (K2), and $q \leq \pi$ by Proposition 4.5. If $p_g \geq 1$, then $\dim W \geq p_g \geq 1$ by Theorem 1.6(b). If $p_g = 0$, then $\dim W \geq d - 1 + 2\pi - q \geq d - 1 + \pi \geq 1$ by Theorem 1.6(b) and $q \leq \pi$. $\square$

Corollary 4.6 and part (a) show that the curve D of Theorem 1.6 lies on a surface of degree $d - 2$; this is all that is used in Sections 5 and 6. The inequality $q - p_g \leq \pi$ of part (c), without any hypothesis, follows from Proposition 4.5 and the next lemma, the usual estimate of the dimension of a deformation space by its tangent space and its space of obstructions. For $H^2(Y, \mathcal{O}_Y) = 0$ it is the smoothness statement in (K3), and our proof follows that of [Kle05, Proposition 5.19]. (The lemma also follows from a theorem of Mumford on Bockstein operations, recalled in [Kle05, Remark 5.15], which we do not use.)

Lemma 4.7. *Let Y be a projective scheme over k with $H^0(Y, \mathcal{O}_Y) = k$. Then $\dim \text{Pic}_{Y/k} \geq h^1(Y, \mathcal{O}_Y) - h^2(Y, \mathcal{O}_Y)$.*

Proof. Let R be the completion of the local ring of $\text{Pic}_{Y/k}$ at 0, so that $\dim R = \dim \text{Pic}_{Y/k}^0 = \dim \text{Pic}_{Y/k}$. Put $q_Y = h^1(Y, \mathcal{O}_Y)$ and $p_Y = h^2(Y, \mathcal{O}_Y)$. By (K2) the cotangent space of R has dimension q_Y , so $R = Q/J$ with $Q = k[[t_1, \dots, t_{q_Y}]]$ and an ideal $J \subset \mathfrak{n}^2$, where $\mathfrak{n} = (t_1, \dots, t_{q_Y})$. Let $E = J/\mathfrak{n}J$. The ideal J is generated by $\dim_k E$ elements, so $\dim R \geq q_Y - \dim_k E$ by Krull's theorem, and it suffices to show $\dim_k E \leq p_Y$.

By the theorem of Artin and Rees there is an integer $N \geq 2$ with $J \cap \mathfrak{n}^N \subset \mathfrak{n}J$. Put

$$B = Q/(J + \mathfrak{n}^N), \quad B' = Q/(\mathfrak{n}J + \mathfrak{n}^N).$$

These are local artinian k -algebras with residue field k , and the kernel of $B' \rightarrow B$ is $(J + \mathfrak{n}^N)/(\mathfrak{n}J + \mathfrak{n}^N) \cong J/(\mathfrak{n}J + J \cap \mathfrak{n}^N) = E$; it is annihilated by the maximal ideal of B' . For a subspace $E_1 \subset E$ let $B_1 = B'/E_1$. The scheme Y_{B_1} has the same underlying space as Y and contains Y_B as the closed subscheme defined by the ideal $\mathcal{O}_Y \otimes_k (E/E_1)$, whose square is zero. Hence there is an exact sequence of sheaves of abelian groups

$$0 \rightarrow \mathcal{O}_Y \otimes_k (E/E_1) \xrightarrow{a \mapsto 1+a} \mathcal{O}_{Y_{B_1}}^* \rightarrow \mathcal{O}_{Y_B}^* \rightarrow 1,$$

which gives an exact sequence

$$\text{Pic}(Y_{B_1}) \rightarrow \text{Pic}(Y_B) \xrightarrow{\omega_{E_1}} H^2(Y, \mathcal{O}_Y) \otimes_k (E/E_1), \quad (4)$$

compatible with the projections $E/E_1 \rightarrow E/E_2$ for $E_1 \subset E_2$.

The canonical map $R \rightarrow B$ is a B -point of $\text{Pic}_{Y/k}$; by (K1) it is the class of an invertible sheaf $\mathcal{M}$ on Y_B . For the subspace 0 of E write $\omega_0(\mathcal{M}) = \sum_{a=1}^{p_Y} \varepsilon_a \otimes e_a$ with a basis (ε_a) of $H^2(Y, \mathcal{O}_Y)$ and elements $e_a \in E$, and let E_1 be the span of $e_1, \dots, e_{p_Y}$. Then $\omega_{E_1}(\mathcal{M}) = 0$, so by (4) $\mathcal{M}$ extends to an invertible sheaf on Y_{B_1} , that is, the B -point extends to a B_1 -point. It corresponds to a local homomorphism of k -algebras $\psi: R \rightarrow B_1$ which lifts the canonical map $R \rightarrow B$.

Let $J_1 \subset J$ be the preimage of E_1 under $J \rightarrow E$; thus $\mathfrak{n}J \subset J_1$ and $B_1 = Q/(J_1 + \mathfrak{n}^N)$. Choose $u_i \in Q$ such that ψ maps the class of t_i to the class of $t_i + u_i$; as ψ lifts $R \rightarrow B$, we have $u_i \in J + \mathfrak{n}^N$. Let Ψ be the endomorphism of the k -algebra Q with $\Psi(t_i) = t_i + u_i$. Then ψ maps the class of g to the class of $\Psi(g)$, for every $g \in Q$; in particular $\Psi(g) \in J_1 + \mathfrak{n}^N$ for $g \in J$. By Taylor's formula

$$\Psi(g) - g \in \sum_i \frac{\partial g}{\partial t_i} u_i + (u_1, \dots, u_{q_Y})^2 \subset \mathfrak{n}(J + \mathfrak{n}^N) + (J + \mathfrak{n}^N)^2 \subset \mathfrak{n}J + \mathfrak{n}^N$$

for $g \in J$, because $g \in \mathfrak{n}^2$ and $J \subset \mathfrak{n}$. Hence $g \in J \cap (J_1 + \mathfrak{n}^N) = J_1 + J \cap \mathfrak{n}^N = J_1$ for all $g \in J$. So $J = J_1$, $E = E_1$, and $\dim_k E \leq p_Y$. $\square$

Proof of Theorem 1.6(c). By Lemma 4.7 for $Y = \overline{X}$ and by Proposition 4.5, $q - p_g \leq \dim \text{Pic}_{\overline{X}/k} \leq \pi$. With part (b), $\dim W \geq d - 1 + 2\pi - (q - p_g) \geq d - 1 + \pi$, and $d - 1 + \pi \geq 1$ because $d \geq 2$ and $\pi \geq 0$ (Lemma 4.4). $\square$

4.6 Remarks

Remark 4.8 (The case $d = 2$). For $d = 2$ we have $W \subset S_0 = k$, and $W \neq 0$ means $W = k$. By part (a) the non-zero constants vanish on D , so $D = \emptyset$. In this form Theorem 1.6 is used for $d = 2$ in Section 5.

Remark 4.9 (Mumford's argument). The proof of Proposition 4.5 adapts an argument of Mumford [Mum67, p. 99], which we know from [Kle05, Remark 5.8]: in characteristic zero the restriction map $\text{Pic}_X^0 \rightarrow \text{Pic}_Y^0$ is injective for a general hyperplane section Y of a normal projective variety X of dimension at least two, because an unramified cyclic covering of X defined by a point of the kernel would split over the ample divisor Y . The same remark states that the kernel is finite in every characteristic, with a reference to an article of Kleiman which we did not consult. We wrote the argument out because $\overline{C}$ is the zero scheme of a section of an ample and globally generated sheaf, not necessarily a hyperplane section of $\overline{X}$, and because in characteristic $p > 0$ it has to be applied to a point of prime order $n \neq p$ of an abelian variety in the kernel (a finite kernel may be non-reduced).

Remark 4.10 (Characteristic zero). Let k have characteristic zero. Then group schemes over k are reduced [Sta26, Tag 047N], so the finite kernel of r in the proof of Proposition 4.5 is étale, and the differential of r at 0 is injective. By [Kle05, Remark 5.8 and Theorem 5.11] this differential is the restriction map $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \rightarrow H^1(\overline{C}, \mathcal{O}_{\overline{C}})$, whose kernel is $H^1(\overline{X}, \mathcal{O}_{\overline{X}}(-1))$ because $H^0(\mathcal{O}_{\overline{X}}) \rightarrow H^0(\mathcal{O}_{\overline{C}})$ is surjective. Hence $H^1(\overline{X}, \mathcal{O}_{\overline{X}}(-1)) = 0$. For smooth surfaces this is the classical theorem of Picard and Severi on the regularity of the adjoint system, for which [Kle05, Remark 5.8] refers to pp. 181 and 204–206 of [Zar71] and to [Mum67, pp. 94–97]; as we recall it, [Mum67] proves the vanishing for normal surfaces in characteristic zero and shows by an example that it fails in positive characteristic. (We did not open [Zar71] and [Mum67].) A section of $\mathcal{O}_{\overline{C}}(-1)$ vanishes on every component of the reduced curve $\overline{C}$, so $H^0(\overline{C}, \mathcal{O}_{\overline{C}}(-1)) = 0$, and Lemma 4.3(ii) for $n = -1$ gives $H^1(\overline{X}, \mathcal{O}_{\overline{X}}(-2)) = 0$. Therefore, in characteristic zero, $\dim W = \chi(-2) = d - 1 + 2\pi - q + p_g$, and the bound $\dim W \geq d - 1 + \pi$ is attained if and only if $q - p_g = \pi$. This remark is not used elsewhere.

Remark 4.11 (Adjoints). By duality for the finite morphism ν , which we do not use, $\nu_* \omega_{\overline{X}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X(d - 4))$, so that $W \cong H^0(\overline{X}, \omega_{\overline{X}}(2))$: the elements of W are the adjoint surfaces of degree $d - 2$ of the classical theory. For the cone with vertex $(1 : 0 : 0 : 0)$ over an integral plane curve Γ of degree d and geometric genus g one finds $W = \bigoplus_{j=0}^{d-2} x_0^j A_{d-2-j}$, where A_e is the space of the adjoint curves of degree e of Γ (the forms of degree e in x_1, x_2, x_3 that multiply the normalisation of the coordinate ring of Γ into this ring), and $\dim A_{d-2} = g + d - 1$ by the argument of Section 4.3, one dimension lower.

5 Proof of the classification and of the corollary

Proof of Theorem 1.3. The “if” part and the additional statements are contained in Lemma 3.1(a) and in Lemma 3.2(b) and (c).

For the “only if” part we use induction on $d = \alpha(I^{(2)})$. Let L be a configuration with $\alpha(I) = d - 1$; then $d \geq 2$. Let $F \neq 0$ be a form of degree d in $I^{(2)}$. By Lemma 2.1(b), $\text{mult}_\ell(F) \geq 2$ for every line ℓ of L , and no non-zero form of degree at most $d - 2$ vanishes on L . We assume that the theorem holds for all configurations L' with gap one and $\alpha(I(L')^{(2)}) < d$.

Case 1: F is not squarefree. Write $F = E^2G$ with an irreducible form E of degree $e \geq 1$, and let M be the set of the lines of L not contained in $V(E)$. If $M = \emptyset$, then $E \in I$, so $e \geq d - 1$; together with $2e \leq d$ this gives $d = 2$ and $e = 1$, and L lies in the plane $V(E)$. If $M \neq \emptyset$, then for $\ell \in M$ we have $\text{mult}_\ell(E) = 0$, hence $\text{mult}_\ell(G) = \text{mult}_\ell(F) \geq 2$ by Lemma 2.1(a). So G is a non-zero element of $I(M)^{(2)}$, and by Lemma 2.1(c) there is a non-zero form G' of degree at most $\deg G - 1 = d - 2e - 1$ vanishing on M . Then EG' is a non-zero form vanishing on L , of degree at most $d - e - 1 \leq d - 2$, a contradiction.

Case 2a: F is irreducible. Then F is an irreducible form of degree $d \geq 2$ with $F \in I(\ell)^2$ for every line ℓ of L . By Theorem 1.6, with $D = L$ (see (2) for the hypothesis), there is a non-zero form of degree $d - 2$ vanishing on L , a contradiction. So this case does not occur. (For this conclusion parts (a) and (b) of Theorem 1.6 and Corollary 4.6 suffice; for $d = 2$ see Remark 4.8.)

Case 2b: F is squarefree and reducible. Write $F = F_1 \cdots F_r$ with $r \geq 2$ and pairwise non-proportional irreducible forms F_i of degree d_i . Fix i , put $G_i = F/F_i$, of degree $e_i = d - d_i \geq 1$, and let M_i be the set of the lines of L not contained in $V(F_i)$. We claim:

$$G_i \text{ is a product of pairwise non-proportional linear forms.} \quad (5)$$

If $M_i = \emptyset$, then $F_i \in I$, so $d_i \geq d - 1$ and $e_i = 1$; the claim holds. Let $M_i \neq \emptyset$, and let J_i be the ideal of the configuration M_i . For $\ell \in M_i$ we have $\text{mult}_\ell(G_i) = \text{mult}_\ell(F) \geq 2$, so G_i is a non-zero form of degree e_i in $J_i^{(2)}$. Put $d' = \alpha(J_i^{(2)})$ and $a' = \alpha(J_i)$. Then $d' \leq e_i$, and $a' \leq d' - 1$ by Lemma 2.1(c). If $A \in J_i$ is a non-zero form of degree a' , then $F_i A$ is a non-zero form vanishing on L , so $d_i + a' \geq d - 1$, that is, $a' \geq e_i - 1$. Hence

$$e_i - 1 \leq a' \leq d' - 1 \leq e_i - 1,$$

so that M_i has gap one, $\alpha(J_i^{(2)}) = e_i < d$, and G_i is a non-zero form of the least degree e_i in $J_i^{(2)}$; moreover $e_i \geq 2$. By the induction hypothesis, M_i is coplanar or a pseudostar. If M_i is coplanar with at least two lines, then $e_i = 2$ and G_i is a scalar multiple of the square of a linear form (Lemma 3.1(b)); this is impossible, because G_i divides the squarefree form F . If M_i is a single line, then $e_i = 2$ and G_i is a product of two linear forms (Lemma 3.1(c)), which are not proportional because F is squarefree. If M_i is the pseudostar of m planes, then $e_i = m$ by Lemma 3.2(b), and G_i is a scalar multiple of the product of the equations of these planes (Lemma 3.2(c)). This proves (5).

By (5) for $i = 1$, the forms $F_2, \dots, F_r$ are linear, and by (5) for $i = 2$, F_1 is linear. Thus $F = h_1 \cdots h_d$ with pairwise non-proportional linear forms h_i ; let $H_i = \{h_i = 0\}$. By Lemma 2.1(a), $\text{mult}_\ell(F)$ is the number of the planes H_r containing ℓ ; so every line of L lies on at least two of the planes, and is therefore equal to $H_i \cap H_j$ for some $i < j$. Conversely, let $i < j$. The non-zero form $F/(h_i h_j)$ has degree $d - 2$, so it does not vanish on L : there is a line ℓ of L that lies on no plane H_r with $r \notin \{i, j\}$. As ℓ lies on two of the planes, $\ell = H_i \cap H_j$. Hence L consists exactly of the lines $\ell_{ij} = H_i \cap H_j$, $i < j$, and $\ell_{ij} \not\subset H_r$ for $r \notin \{i, j\}$; in particular no three of the planes contain a common line. If $d = 2$, L is a single line, which is coplanar; if $d \geq 3$, L is the pseudostar of $H_1, \dots, H_d$.

The induction is well founded: the induction hypothesis was applied only to the configurations M_i , for which $2 \leq \alpha(J_i^{(2)}) = e_i < d$. For $d = 2$ it is not applied at all, since then $e_i = 1$. $\square$

Proof of Corollary 1.4. Let $\overline{K}$ be an algebraic closure of K , and put $S_K = K[x_0, \dots, x_3]$. Let $L = \ell_1 \cup \dots \cup \ell_s$, where $I(\ell_i) \subset S_K$ is generated by two linear forms with coefficients in K , and let $I = \bigcap I(\ell_i)$. By Lemma 2.1(b), whose proof is valid over K , $I^{(2)} = \bigcap I(\ell_i)^2$. Since $\overline{K}$ is flat over K , forming finite intersections of ideals commutes with the extension of scalars from S_K to $S_{\overline{K}}$. Hence $I \otimes_K \overline{K}$ and $I^{(2)} \otimes_K \overline{K}$ are the ideal of the configuration $L_{\overline{K}} \subset \mathbb{P}_{\overline{K}}^3$ and its second symbolic power, and $\alpha(I)$ and $\alpha(I^{(2)})$ do not change under the extension. By Theorem 1.3, $L_{\overline{K}}$ lies in a plane or is the pseudostar of $m \geq 3$ planes $H_1, \dots, H_m$ of $\mathbb{P}_{\overline{K}}^3$. In the first case, if $s \geq 2$ the plane is spanned by two lines of L and is therefore defined over K ; if $s = 1$ the line lies in a plane defined over K . In the second case every H_i contains the $m - 1 \geq 2$ lines ℓ_{ij} , which are defined over K , so H_i is defined over K . By Lemma 3.1(a) and Lemma 3.2(d), whose proofs are valid over K (Remark 3.3), S_K/I is Cohen–Macaulay. $\square$

6 Reduced curves

Let $D \subset \mathbb{P}^3$ be a reduced curve with irreducible components $D_1, \dots, D_s$, let $\mathfrak{p}_i \subset S$ be the prime ideal of D_i , of height two, and $I = \bigcap \mathfrak{p}_i$. The local ring $S_{\mathfrak{p}_i}$ is regular of dimension two. For a non-zero form F let $\text{ord}_{D_i}(F)$ be the order of F in $S_{\mathfrak{p}_i}$, the largest m with $F \in \mathfrak{p}_i^m S_{\mathfrak{p}_i}$. For a line this is the multiplicity of Section 2: the proof of Lemma 2.1(b) shows that every power of $I(\ell)$ is $I(\ell)$ -primary. The following lemma replaces Lemmas 2.1 and 3.1.

Lemma 6.1. *Let D_0 be an integral curve in $\mathbb{P}^3$ with prime ideal $\mathfrak{p}$, and let F, G be non-zero forms.*

- (a) $\text{ord}_{D_0}(FG) = \text{ord}_{D_0}(F) + \text{ord}_{D_0}(G)$.
- (b) *For a reduced curve $D = D_1 \cup \dots \cup D_s$ with ideal I , a non-zero form F lies in $I^{(2)}$ if and only if $\text{ord}_{D_i}(F) \geq 2$ for all i .*
- (c) $\alpha(I^{(2)}) \geq \alpha(I) + 1$ for the ideal I of every reduced curve.
- (d) *Let h be a non-zero linear form and $H = \{h = 0\}$. Then $\text{ord}_{D_0}(h) = 1$ if $D_0 \subset H$, and $\text{ord}_{D_0}(h) = 0$ otherwise. If $D_0 \subset H$, if f_0 is an equation of D_0 in H , and if $h \nmid F$ and $\text{ord}_{D_0}(F) \geq m$, then $F|_H$ is a non-zero form divisible by f_0^m .*
- (e) *Let D be a reduced curve of degree $\delta \geq 2$ contained in a plane $H = \{h = 0\}$, with ideal I . Then $\alpha(I) = 1$, $\alpha(I^{(2)}) = 2$, and the forms of degree two in $I^{(2)}$ are the scalar multiples of h^2 .*

Proof. (a) The associated graded ring of the regular local ring $S_{\mathfrak{p}}$ is a polynomial ring over its residue field [Sta26, Tag 00NO], hence a domain; so the order function of $S_{\mathfrak{p}}$ is additive.

(b) The associated primes of I are the $\mathfrak{p}_i$, and $IS_{\mathfrak{p}_i} = \mathfrak{p}_i S_{\mathfrak{p}_i}$. So, by the definition of the symbolic power, $I^{(2)} = \bigcap_i (\mathfrak{p}_i^2 S_{\mathfrak{p}_i} \cap S)$. (Here $\mathfrak{p}_i^2 S_{\mathfrak{p}_i} \cap S$ can be larger than $\mathfrak{p}_i^2$.)

(c) Let $F \neq 0$ be a form in $I^{(2)}$, and fix i . There is $c \in S \setminus \mathfrak{p}_i$ with $cF \in \mathfrak{p}_i^2$. For a partial derivative ∂ we get $c\partial F + F\partial c = \partial(cF) \in \mathfrak{p}_i$ by the Leibniz rule, and $F \in \mathfrak{p}_i$; hence $\partial F \in \mathfrak{p}_i$. So all partial derivatives of F lie in I , and the proof of Lemma 2.1(c) applies.

(d) If $D_0 \not\subset H$, then h is a unit of $S_{\mathfrak{p}}$. Let $D_0 \subset H$. The ring $S_{\mathfrak{p}}/(h)$ is the localisation of the polynomial ring $S/(h)$ at the prime ideal generated by the irreducible polynomial f_0 ; this is a discrete valuation ring with uniformiser f_0 , and its valuation, restricted to $S/(h)$, counts the multiplicity of the factor f_0 . As $S_{\mathfrak{p}}/(h)$ is regular of dimension one less than $S_{\mathfrak{p}}$, the element h does not lie in $\mathfrak{p}^2 S_{\mathfrak{p}}$, so $\text{ord}_{D_0}(h) = 1$. If $F \in \mathfrak{p}^m S_{\mathfrak{p}}$, the class of F in $S_{\mathfrak{p}}/(h)$ lies in the m -th power of the maximal ideal, so f_0^m divides $F|_H$; and $F|_H \neq 0$ if $h \nmid F$.

(e) $h \in I$ and $h^2 \in I^{(2)}$ give the degrees, with (c). Let $G \in I^{(2)}$ have degree two. If $h \nmid G$, then by (d) the non-zero form $G|_H$ is divisible by the squares of the equations of the components of D , which

are pairwise non-associated irreducible polynomials; their product has degree $2\delta \geq 4$, a contradiction. So $G = hh'$ with a linear form h' , and $\text{ord}_{D_i}(h') = \text{ord}_{D_i}(G) - 1 \geq 1$ for all i by (a) and (d): h' vanishes on D . If h' were not proportional to h , then D would be contained in the line $H \cap \{h' = 0\}$, and $\delta = 1$. So $h' \in kh$. $\square$

Proof of Theorem 1.5. “If”. A reduced curve contained in a plane has $\alpha(I) = 1$ and $\alpha(I^{(2)}) = 2$, by Lemma 6.1(e) if its degree is at least two and by Lemma 3.1(a) if it is a line. Pseudostars have gap one by Lemma 3.2(b).

“Only if”. We repeat the proof of Theorem 1.3, by induction on $d = \alpha(I^{(2)})$ for reduced curves D with $\alpha(I) = d - 1$, with “component of D ” in place of “line of L ”, with ord_{D_0} in place of mult_ℓ , and with Lemma 6.1(a), (b), (c) in place of Lemma 2.1(a), (b), (c). Let $F \neq 0$ be a form of degree d in $I^{(2)}$.

Case 1 ($F = E^2G$ with E irreducible). Let M be the union of the components of D not contained in $V(E)$. If $M = \emptyset$, then $E \in I$, and as before $d = 2$, $\deg E = 1$, and D lies in the plane $V(E)$. If $M \neq \emptyset$, then $\text{ord}_{D_0}(E) = 0$ and $\text{ord}_{D_0}(G) \geq 2$ for the components D_0 of M , and the argument of Case 1 in Section 5 gives a non-zero form of degree at most $d - 2$ vanishing on D , a contradiction.

Case 2a (F irreducible). Theorem 1.6 applies to F and D , and gives a non-zero form of degree $d - 2$ vanishing on D , a contradiction.

Case 2b ($F = F_1 \cdots F_r$ squarefree, $r \geq 2$). With $G_i = F/F_i$, $e_i = \deg G_i$, and M_i the union of the components of D not contained in $V(F_i)$, the chain of inequalities of Section 5 shows, without change, that either $M_i = \emptyset$ and $e_i = 1$, or M_i is a reduced curve with gap one, $\alpha(I(M_i)^{(2)}) = e_i < d$, and G_i is a non-zero form of degree e_i in $I(M_i)^{(2)}$. In the second case the induction hypothesis says that M_i is contained in a plane or is a pseudostar of lines. If M_i is a plane curve of degree at least two, then G_i is a scalar multiple of the square of a linear form by Lemma 6.1(e), which is impossible because F is squarefree. If M_i is a line or a pseudostar, G_i is a product of pairwise non-proportional linear forms by Lemma 3.1(c) and Lemma 3.2(c), as in Section 5. Hence, again, $F = h_1 \cdots h_d$ with pairwise non-proportional linear forms, defining planes $H_1, \dots, H_d$. By Lemma 6.1(a) and (d), for every component D_0 of D the number $\text{ord}_{D_0}(F) \geq 2$ is the number of the planes H_r that contain D_0 . So D_0 is contained in two different planes H_i, H_j , hence in the line $H_i \cap H_j$; being an irreducible curve, D_0 is equal to this line. Therefore D is a configuration of lines with gap one, and Theorem 1.3 shows that it is coplanar or a pseudostar. $\square$

Pseudostars are ACM (Lemma 3.2(d)), and reduced plane curves are complete intersections; so reduced curves with gap one are ACM as well.

7 Remarks

Remark 7.1 (The case $\alpha(I) \leq 2$ and the theorem of Haghighi and Mosakhani). For a homogeneous ideal I the *Waldschmidt constant* is $\hat{\alpha}(I) = \lim_m \alpha(I^{(m)})/m = \inf_m \alpha(I^{(m)})/m$. According to the review Zbl 1473.14102, Theorem B of [HM21] states: if Z is a configuration of codimension two linear subspaces of $\mathbb{P}^N$, $N \geq 3$, over an algebraically closed field, and $1 < \hat{\alpha}(I(Z)) < 2$, then $\hat{\alpha}(I(Z)) = 2 - 1/n$ for an integer $n \geq 2$, and Z consists of n subspaces contained in a hyperplane and of one further subspace that is coplanar with each of them (the configuration $Z_{1,n}$ of the review). No Cohen–Macaulay hypothesis is made; the review calls the field algebraically closed and does not mention its characteristic. We have not read [HM21] and have not checked this theorem.

Granting it, Theorem 1.3 for $\alpha(I) \leq 2$ follows. If $\alpha(I) = 1$, then L is coplanar. Let L have gap one with $\alpha(I) = 2$; then L is not coplanar, and $\hat{\alpha}(I) \leq \alpha(I^{(2)})/2 = \frac{3}{2}$. The Waldschmidt constant does not decrease when lines are added, and $I^{(m)} = \bigcap I(\ell_i)^m$ as in Lemma 2.1(b). For two skew lines, with ideals (x, y) and (z, w) , every monomial in $(x, y)^m \cap (z, w)^m$ has degree at least $2m$, so the constant is

2; hence any two lines of L meet. As L is not coplanar, all its lines pass through one point, and three of them are not coplanar. For three such lines, with ideals (y, z) , (x, z) , (x, y) , a monomial $x^a y^b z^c w^e$ lies in the m -th symbolic power if and only if $b + c$, $a + c$ and $a + b$ are at least m , so its degree is at least $\frac{3}{2}m$. Therefore $\hat{\alpha}(I) = \frac{3}{2}$, and Theorem B applies with $2 - 1/n = \frac{3}{2}$: $n = 2$, and L consists of three lines. Three concurrent lines that are not coplanar form the pseudostar of the three planes which they span in pairs.

Remark 7.2 (The elementary range). For $d = \alpha(I^{(2)}) \leq 4$, Theorem 1.6 can be replaced in Case 2a of Section 5 by a count of parameters. Let F be an irreducible form of degree d that is singular along the s lines of L . A general plane section of $V(F)$ is an integral plane curve of degree d , by Bertini's theorem, and it is singular at the s points in which the plane meets L ; hence $s \leq \frac{1}{2}(d-1)(d-2)$. A line imposes at most $d-1$ conditions on the forms of degree $d-2$. So the space of the forms of degree $d-2$ vanishing on L has dimension at least $\binom{d+1}{3} - \frac{1}{2}(d-1)^2(d-2)$, which is 1, 2, 1 for $d = 2, 3, 4$. Thus Theorem 1.3 for $\alpha(I) \leq 3$ does not need Theorem 1.6. For $d = 5$ the number is -4 , and the count no longer suffices: five general lines impose 20 independent conditions on the 20-dimensional space of cubic forms (computed for random lines over the field with $2^{31} - 1$ elements, see Section 8). We do not assert that Theorem 1.6 is necessary for $d = 5$; Proposition 7.3 is a partial result in the other direction. We did not find an elementary proof of the irreducible case for all d . (This remark is not used in the proofs.)

Proposition 7.3. *Let $X = V(F)$ be an irreducible quintic surface, and let $\ell_1, \dots, \ell_5$ be distinct lines with $F \in I(\ell_j)^2$ for all j . If ℓ_1 is skew to $\ell_2, \dots, \ell_5$, then the five lines lie on a quadric; in particular a non-zero cubic form vanishes on them.*

Proof. Let H be a plane through ℓ_1 containing none of the finitely many points in which two of the lines $\ell_2, \dots, \ell_5$ meet. By Lemma 2.1(d), $F|_H = \lambda^2 R$, where λ defines ℓ_1 in H and R is a non-zero cubic form. The points $p_j = \ell_j \cap H$, $2 \leq j \leq 5$, are distinct and not on ℓ_1 , and $F|_H$ has multiplicity at least two at each of them; so R is singular at four points. As a reduced plane cubic has at most three singular points, $R = \mu^2 \mu'$ with linear forms μ, μ' , and the p_j lie on the line $m_H = \{\mu = 0\}$. Thus m_H meets all five lines, and $m_H \neq m_{H'}$ for $H \neq H'$. If two of $\ell_2, \dots, \ell_5$ are skew, say ℓ_2 and ℓ_3 , then ℓ_1, ℓ_2, ℓ_3 lie on a quadric Q ; every m_H meets Q in three points and so lies on Q ; and ℓ_4, ℓ_5 contain the infinitely many points $\ell_j \cap m_H$ of Q , so they lie on Q . If $\ell_2, \dots, \ell_5$ meet pairwise, they are coplanar or concurrent. If they are coplanar, their plane and a plane through ℓ_1 form the quadric. If they pass through a point p and are not coplanar, then $p \notin H$, and they lie in the plane spanned by p and m_H , a contradiction. $\square$

Proposition 7.4 (The bound of Theorem 1.6(c) is attained). *Let $a, b \geq 2$, and let F be a bihomogeneous form of bidegree (a, b) in the two pairs of variables (x, y) and (z, w) such that the curve $E = \{F = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is smooth and irreducible; its genus is $g = (a-1)(b-1)$. Then $X = V(F) \subset \mathbb{P}^3$ is an irreducible surface of degree $d = a + b$, the lines $\{x = y = 0\}$ and $\{z = w = 0\}$ are a -fold and b -fold on X , and*

$$\pi = q = g, \quad p_g = 0, \quad W = k[x, y]_{a-1} \otimes k[z, w]_{b-1}, \quad \dim W = ab = d - 1 + \pi.$$

Proof. Put $S_{i,j} = k[x, y]_i \otimes k[z, w]_j$ and $B_{i,j} = H^0(E, \mathcal{O}_E(i, j))$. By the sequences $0 \rightarrow \mathcal{O}(i-a, j-b) \xrightarrow{F} \mathcal{O}(i, j) \rightarrow \mathcal{O}_E(i, j) \rightarrow 0$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and the Künneth formula, for $i, j \geq 0$ the map $S_{i,j} \rightarrow B_{i,j}$ has kernel $FS_{i-a, j-b}$, and it is surjective unless $(i \geq a, j \leq b-2)$ or $(i \leq a-2, j \geq b)$. Let $p: Y = \mathbb{P}(\mathcal{O}_E(1, 0) \oplus \mathcal{O}_E(0, 1)) \rightarrow E$; then $H^0(Y, \mathcal{O}_Y(n)) = \bigoplus_{i+j=n} B_{i,j}$ for $n \geq 0$, and x, y, z, w define a morphism $\varphi: Y \rightarrow \mathbb{P}^3$, which maps the point $(s:t)$ of the fibre over $((x_0:y_0), (z_0:w_0)) \in E$ to $(sx_0: sy_0: tz_0: tw_0)$. Its image is X , it is an isomorphism over the complement of the two lines, and its fibres are finite; so, Y being smooth, φ is the normalisation of X . Hence X and F are

irreducible (E is reduced), $\mathcal{O}_{\overline{X}}(n) = \mathcal{O}_Y(n)$, and $p_*\mathcal{O}_Y = \mathcal{O}_E$, $R^1p_*\mathcal{O}_Y = 0$, $Rp_*\mathcal{O}_Y(-1) = 0$ give $q = g$, $p_g = 0$, $\chi(0) = 1 - g$, $\chi(-1) = 0$, $\pi = g$. The multiplicities are read off from the bidegree of F .

As $\mathcal{A}(n)$ is generated by its global sections for $n \gg 0$, a form $G \in S_{d-2}$ lies in W if and only if $GB_{i,j}$ is contained in the image of S for all $i, j \geq 0$, and this can be tested on the bihomogeneous components of G . It holds for $G \in S_{a-1,b-1}$, because $S_{i+a-1,j+b-1} \rightarrow B_{i+a-1,j+b-1}$ is surjective. Let $0 \neq G \in S_{u,v}$ with $u+v = d-2$ and, by symmetry, $v \leq b-2$. As $v < b$, G does not vanish identically on the integral curve E , so multiplication by G is injective on $B_{i,0}$; the image of $S_{i+u,v}$ has dimension $(i+u+1)(v+1)$, and $\dim B_{i,0} \geq ib+1-g$ by the theorem of Riemann and Roch, $\mathcal{O}_E(1,0)$ being of degree b . As $b > v+1$, $GB_{i,0}$ is not contained in the image of S for large i . Hence $W = S_{a-1,b-1}$, and $ab = (a+b) - 1 + (a-1)(b-1)$. $\square$

By Bertini's theorem such curves E exist for all $a, b \geq 2$ in every characteristic. The proposition is not used elsewhere.

Remark 7.5 (Adding a disjoint line). Let L have gap one, $d = \alpha(I(L)^{(2)})$, and let ℓ be a line meeting no line of L . Then $L' = L \cup \ell$ has type $(d, d+2)$ and is not ACM (the observation quoted in Section 1.1). Indeed, for a linear form h vanishing on ℓ we have $hG \in I(L')$ for $G \in I(L)$ and $h^2F \in I(L')^{(2)}$ for $F \in I(L)^{(2)}$, so $\alpha(I(L')) \leq d$ and $\alpha(I(L')^{(2)}) \leq d+2$. No non-zero form of degree $d-1$ vanishes on L' : if L is coplanar, no plane contains L' , since two lines in a plane meet; if L is the pseudostar of $H_1, \dots, H_d$, such a form is $\sum c_i \Phi_i$ (Lemma 3.2(d)), and ℓ , which meets no line of L , meets H_i in a single point, which lies on no H_j with $j \neq i$, so that evaluation at this point gives $c_i = 0$. As ℓ meets no other line of L' , L' is neither coplanar nor a pseudostar, so $\alpha(I(L')^{(2)}) \geq d+2$ by Theorem 1.3; and L' is not connected, whereas the curve of an ACM configuration is connected (a standard fact, quoted from memory).

8 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation. The computations of this section are corroboration only: they test the statements on explicit configurations and surfaces, and they are not part of any proof. Every number below is computed, and is given with the program that produced it. The reproducibility package has four parts with their recorded outputs: `original/` (the programs with which the results were first obtained and tested), `verification_run_A/` and `verification_run_B/` (the programs of the first two independent verification runs), and `independent_run_2/` (the programs of the third run).

Method. For lines through points with coordinates in $\mathbb{Q}$ or in a prime field, $\alpha(I)$ and $\alpha(I^{(2)})$ are found by exact linear algebra: for a form F of degree t the condition $F \in I(\ell)$ gives $t+1$ linear equations in the coefficients, and $F \in I(\ell)^2$ gives $3t+1$ equations (F and two transversal first derivatives vanish on ℓ ; this is valid in every characteristic). For lines defined over a prime field the dimensions are those over its algebraic closure. The first three parts use three independent implementations (`linetools.py`, with python-flint; `vlines.py` and `olib.py`, with numpy). Coplanar configurations and pseudostars are recognised combinatorially. The ACM property is decided through a plane section (the first difference of the Hilbert function of L is compared with the Hilbert function of $L \cap H$, for a plane H meeting L in distinct points), or, in run B, through the Hartshorne–Rao module $\bigoplus_t H^1(\mathbb{P}^3, \mathcal{I}_L(t))$, which vanishes if and only if L is ACM. The plane-section test of `linetools.py` uses the regularity bound of [DS02] and a plane defined over the ground field; it is not available over $\mathbb{F}_2$. The third run (`r2lib.py`, with python-flint) forms $(I(\ell)^2)_t$ from the generators of the ideal, without derivatives, and decides the ACM property without plane sections: if g_1, g_2 are forms of degrees e_1, e_2 without common zero on L , and g_1 vanishes identically on no line of L , then the length of $S/(I, g_1, g_2)$ is at

least e_1e_2s , with equality if and only if L is ACM (proved in `independent_run_2/README.md`); over $\mathbb{F}_2$ it takes cubic forms.

The projective space over $\mathbb{F}_2$. The space $\mathbb{P}^3(\mathbb{F}_2)$ has 35 lines, hence $2^{35} - 1 = 34,359,738,367$ non-empty sets of lines. The program `exhaust_F2_full.py` (`original/`) enumerates them up to the action of $\mathrm{GL}_4(\mathbb{F}_2)$: the orbits of sets of at most 17 lines by orderly generation, the others as complements; at every level the sum of the orbit sizes is the binomial coefficient. It found 1,771,439 orbits. Exactly 17 of them have gap one: 9 orbits of coplanar sets (the sets of 1 to 7 lines in a plane; 1,835 sets) and 8 orbits of pseudostars (one, two, two, one, one, one orbits with $m = 3, \dots, 8$ planes, consisting of 420, 945, 1008, 420, 120, 15 sets; 2,928 sets in all). There is no violation of Theorem 1.3 in either direction. Run B repeated the complete enumeration with its own program `f2_full.py`, which compares the number of orbits at every level with Burnside’s lemma (`f2_group.py`: 1,771,439 in all); the numbers of orbits, the distribution of $\alpha(I)$ and the orbits with gap one agree level by level with those of `exhaust_F2_full.py` (`compare_claim_f2.py`). The numbers 1,835 and 2,928 were also obtained by direct counts (`f2_count_families.py` of run B, `capcount_F2.py` of run A). Run A checked, by a different method, 19,554,065 sets which represent all sets of at most ten lines up to projective equivalence (`exhaust_small_F2.py`). The program of `original/` did not test the ACM property over $\mathbb{F}_2$; for the 17 orbits with gap one it was computed by run B (the Hartshorne–Rao module is zero, `f2_validate.py`) and by run A (`crosscheck_claim_reps.py`, with a plane defined over $\mathbb{F}_{16}$). Of the 6,440 orbits of sets of at most nine lines, 5,612 are not ACM, and their least gap is two (`f2_small.py` of run B). The third run enumerated the 16,672 orbits of sets of at most ten lines with its own program and its own test (`r2_f2_orbits.py`): 14 have gap one, the nine orbits of coplanar sets and the five orbits of pseudostars with at most five planes; 14,804 are not ACM, with least gap two (5,612 of the 6,440 orbits of sets of at most nine lines, as above). All this concerns only configurations whose lines are defined over $\mathbb{F}_2$.

Characteristic zero and odd characteristic. The program `survey_Q.py` (`original/`) treats 16,376 configurations of 1 to 28 lines through points with integer coordinates, exactly over $\mathbb{Q}$ (sub-configurations of pseudostars, grids on a quadric, cones, subsets of the 27 lines of a cubic surface, random sets, unions). Gap one occurs exactly for the 919 coplanar configurations and the 358 pseudostars; among the configurations with at most ten lines, for which the ACM property was tested, 9,647 are not ACM (4,452 of them connected), with gaps from 2 to 5. The program `survey_Fp.py` treats 71,882, 14,398 and 4,800 sets of lines over $\mathbb{F}_3$, $\mathbb{F}_5$ and $\mathbb{F}_7$, with the same result. Run A computed 88 selected configurations over $\mathbb{Q}$ with their reductions (`candidates.py`) and 391,173 configurations over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$ (`survey.py`), of which the 90,850 with gap one are coplanar or pseudostars, and ACM. Run B computed all 4,791,322 sets of at most eight of the 28 lines joining two vertices of a cube, in 102,817 orbits under its symmetries (`survey.py cube`; modulo $2^{31} - 1$, with $\alpha(I)$ certified over $\mathbb{Q}$): the 65 orbits with gap one are coplanar or pseudostars, and none of the 90,214 orbits that are not ACM has gap one. It also computed the 101,510 sub-configurations of ten pseudostars (`survey.py pstar`) and classical configurations (`survey2.py`). The third run computed 541 configurations exactly over $\mathbb{Q}$ (`r2_survey_Q.py`; among them the 27 lines of a cubic surface, a double six, and pseudostars with a line removed or added): gap one occurs exactly for the 56 coplanar ones and the 42 pseudostars; 246 are not ACM, among them 42 examples of Remark 7.5. No violation of Theorem 1.3 or of Corollary 1.4 was found.

Surfaces and curves. For Theorem 1.6 the space W was computed from its definition on surfaces with a known normalisation: by run B on 125 parametrised rational surfaces of degrees 3 to 6 over seven prime fields, where $\dim W = d - 1 + 2\pi$ in every case (`lemmaA_toric.py`), and on twenty

random surfaces as in Proposition 7.4, where $\dim W = ab$ (`lemmaA_bigraded.py`; the smoothness of the curves E was not certified); by run A on 98 surfaces in six characteristics (`conductor.py`); and by the third run in 33 cases over $\mathbb{Q}$ and over $\mathbb{F}_p$, $p \leq 11$ (`r2_conductor.py`: cones over singular rational plane curves, ruled surfaces with a $(d-1)$ -fold line, the surfaces $x^p w = z^p y$ in characteristic $p = 2, 3, 5$, seven surfaces as in Proposition 7.4 with $a = 2$, $b \leq 4$ and certified smooth E , and others), where W vanishes on the singular lines and $\dim W \geq d - 1 + \pi$. The program `lemmaA_examples.py` (`original/`) computes, for six surfaces over $\mathbb{Q}$ singular along lines, the dimension of the space of all forms of degree $d - 2$ vanishing on these lines, which is an upper bound for $\dim W$, not $\dim W$ itself. For Theorem 1.5, run A computed 30 reduced curves over $\mathbb{F}_{32003}$ (`curves.py`); gap one occurs exactly for the plane curves and the two pseudostars among them. The statement on five general lines in Remark 7.2 was computed by `elementary_range.py` of run B.

Reproduction. The script `run_quick.sh` of the package re-runs, in a single process, every program that finishes within a few minutes, and compares the results with the recorded outputs (about 25 minutes). The longer runs (the two complete enumerations over $\mathbb{F}_2$, the enumeration of run A, the survey of the cube) are documented by their commands and recorded logs. They were run once more when the package was assembled, and the third run ran `run_quick.sh` again from the archive: all recorded results were reproduced. Details are in the file `README.md` of the package.

Verification. The results were first obtained, with proofs and with the programs in `original/`, in a first written version. Three independent verification runs, all AI-assisted, followed, each with its own programs. All corrections which they required were made; the verification report of the package lists them.

Runs A and B examined the first written version. Run A compared the question, the conventions and Theorem 1.2 with [Jan15a] and with the arXiv version of [Jan15b], re-derived every step of the proofs line by line (Lemmas 2.1, 3.1 and 3.2, the proof of Theorem 1.6 up to $W \neq 0$ in the form of Corollary 4.6, Sections 5 and 6), and read the statements of [Kle05] that are used. Run B repeated and extended the computations with its own programs, computed W on explicit surfaces, re-derived the proof of Theorem 1.6 to the same extent, examined the elementary range, and searched the literature; Proposition 7.3, the statement of Proposition 7.4 and the credit to [HM21] are due to it. Neither run found a gap or an error; their corrections concerned the exposition, citations and credits.

The third run examined the final text. It began with what had been changed or added when this note was written, after runs A and B: Lemma 4.7, and with it the inequality $q - p_g \leq \pi$ of Theorem 1.6(c) without the hypothesis of Corollary 4.6 (Theorems 1.3 and 1.5 and Corollary 1.4 do not depend on this lemma); in the proof of Proposition 4.5, the use of the lemma of Enriques–Severi–Zariski on $\overline{X}$ and of the reduced identity component A ; the proof of $I = (\Phi_1, \dots, \Phi_m)$ in Lemma 3.2(d) by induction, and Lemma 2.2; the last steps of the proofs of Theorem 1.5 and of Corollary 1.4; and citations of [Sta26] in place of citations of textbooks. It re-derived these parts step by step, opened the thirteen cited tags of [Sta26], read every cited statement of [Kle05] in the source of the arXiv version, and compared statements and hypotheses with the use made of them. Then it re-derived all other proofs line by line, read [Jan15a] and both versions of [Jan15b] (the journal version for the first time), verified the bibliography and repeated the literature search. It found no wrong theorem, proposition or lemma and no gap. Of the eleven corrections which it required, two are additions: the statement of Proposition 7.4, quoted before from run B without proof, is now proved in the text, in a form checked by this run, and Remark 7.5 was added. The others concern sources, attributions and wording. Its programs were written before it opened the other programs of the package and use other methods; no proof depends on them.

Scope and priority. The note answers the question stated in Section 1.1 completely: the answer is no (Corollary 1.4). This is to be read with the following qualifications. The case $\alpha(I) \leq 2$ of Theorem 1.3 already follows from Theorem B of Haghighi and Mosakhani [HM21] (Remark 7.1); we know that paper only through its abstract and through the review Zbl 1473.14102, because it is not openly accessible, and we have not verified its theorem. Theorem 1.2, in particular the fact that coplanar configurations and pseudostars have gap one, is due to Janssen [Jan15b]; the ACM property of pseudostars to Geramita, Harbourne and Migliore [GHM13], as cited in [Jan15b]; the theorem for points in the plane to Bocci and Chiantini [BC11]; and the theorem for codimension two linear subspaces of $\mathbb{P}^N$ with Cohen–Macaulay ideal, in characteristic zero, to Fashami, Haghighi and Szemberg [FHS20]. Theorem 1.6 is classical in spirit (Section 1.3), and the covering argument in the proof of Proposition 4.5 is Mumford’s; we claim only that we found no reference for the statement in this generality. Theorem 1.5 answers Question 3.2 of [Jan15b] under our reading of “reduced curve”. The questions of Section 1.5 are not treated.

What was read: [Jan15a] in the publisher’s file (Janssen’s abstract is on pp. 513–515); [Jan15b] completely, in the journal version and in the arXiv version 2; [FHS20] completely, in the arXiv version 1; of [GHM13], the definitions and Proposition 2.9 in the arXiv version 1 (the Remark 2.13 cited in [Jan15b] is not in that version); of [Kle05], in the arXiv version, the statements cited in Section 4.5, Lemma 5.1 and Proposition 5.19 with their proofs, and Remarks 5.8 and 5.15; of [Sta26], the statements of the tags cited; of [HM21], the abstract and the review only; of [BS15], the abstract only. The proofs of the cited statements were not checked. Not read were: the journal versions of [FHS20] and [GHM13]; [BC11] (what we say about [BC11] and about the ACM property in [GHM13] is taken from [Jan15b, Jan15a]); [Mum67] and [Zar71] (Remark 4.10); [Har77] and [DS02]. Quoted from memory are: the cohomology of the sheaves $\mathcal{O}(n)$ and Serre duality on $\mathbb{P}^3$ [Har77], for which we also cite tags of [Sta26] that were read; the bound of [DS02] and the Castelnuovo–Mumford theorem, used only in programs; Bertini’s theorem and the bound for the number of singular points of a plane curve (Remark 7.2); the Künneth formula, projective bundles over a curve, the theorem of Riemann and Roch and Bertini’s theorem (Proposition 7.4); the connectedness of ACM curves (Remark 7.5); and standard facts of commutative algebra and projective geometry (Krull’s height theorem, the theorem of Artin and Rees, the dimension formula in polynomial rings, discrete valuation rings, the quadric through three skew lines).

The searches were made on 10 October 2026: when the results were first obtained, by the three verification runs, and when this note was written. They comprise about fifty-five queries to the arXiv (their subjects are listed in the verification report of the package); the lists of the works citing [Jan15b], [FHS20], [HM21] and [BS15] in OpenAlex (eight works cite the article of Janssen and two its preprint); queries to zbMATH, whose lists of citing works could not be obtained, to Crossref, DataCite and Zenodo; and nine web searches. We found no publication that answers Janssen’s question in general, or that states Theorem 1.3 for $\alpha(I) \geq 3$, Theorem 1.5, or Theorem 1.6 in this generality. Of the citing works only the titles and the available abstracts were examined, apart from [FHS20]; in particular [HM21], which cites [Jan15b], could not be read, and its text may contain more than its abstract and its review say. Three repository pages found by the web searches refused automated access. The classical literature on adjoint surfaces cannot be searched in this way, so for Theorem 1.6 the negative result is weak. A search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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