

Containment of the Symbolic Cube in the Square for the Roulleau–Urzúa Configurations of Cubics

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Abstract

In an Oberwolfach report of 2015 Harbourne considered the configurations C_n of $\frac{4}{3}(n^2 - 3)$ plane cubics due to Roulleau and Urzúa, defined when 3 divides n , and asked whether $I^{(3)} \not\subseteq I^2$ for the ideal I of the singular points of C_n . He reported that for C_3 the containment $I^{(3)} \subseteq I^2$ holds, with I the ideal of the 12 proper singular points taken with multiplicity 2 and of the 24 infinitely near triple points taken with multiplicity 1. For $n = 3m$ we take I to be the ideal of the cluster of all singular points of C_n , proper and infinitely near, with weight $3m^2 - 1$ at each of the 12 points of multiplicity $n^2 - 3$ and weight 1 at each infinitely near triple point and at each quadruple point; for $m = 1$ this is Harbourne’s ideal, and for $m \geq 2$ it is our extension of his description. We prove that $I^{(3)} \subseteq I^2$ for every $m \geq 1$, so that the answer to the question is negative for every n divisible by 3. From the description of C_n as the union of the Halphen cubics of order m , due to Bauer, Harbourne, Roé and Szemberg, Bézout’s theorem gives $\alpha(I^{(3)}) = 36m^2 - 12$, and for $m \geq 2$ the Kawamata–Viehweg vanishing theorem on the blow-up of the cluster gives $\text{reg}(I) \leq 18m^2 - 8$; for $m = 1$ an exact computation, made with a computer, gives $\text{reg}(I) = 12$, the value reported by Harbourne. The containment then follows from a criterion of Bocci and Harbourne. The question, the case $n = 3$ and its numbers are Harbourne’s; for $m \geq 2$ we found no answer in the literature accessible to us. The second part of the question, on interesting analogous configurations of conics, is not addressed. This is an unrefereed note.

1 Introduction

1.1 The question

Let $R = \mathbb{C}[x, y, z]$. Huneke asked whether $I^{(3)} \subseteq I^2$ for the ideal $I \subseteq R$ of every finite set of points of $\mathbb{P}^2$; the first negative answer, the ideal of the 12 triple points of the dual Hesse arrangement of 9 lines, is due to Dumnicki, Szemberg and Tutaj-Gasińska [DSTG13] (see [Har15, p. 511]). In his extended abstract “ H -constants and the containment problem” in Oberwolfach Report 9/2015 [Har15], Harbourne relates such examples to H -constants, and in his Example 3 [Har15, p. 512] he turns to the configurations of cubics of Roulleau and Urzúa [Rou17, RU15]; the ground field is $\mathbb{C}$ [Har15, p. 510]. When 3 divides n , C_n is the union of $\frac{4}{3}(n^2 - 3)$ cubics, and its singular points are the 12 vertices of the dual Hesse arrangement, of multiplicity $n^2 - 3$; $4(n^2 - 3)$ triple points, $\frac{1}{3}(n^2 - 3)$ infinitely near each vertex; and $\frac{1}{3}(n^2 - 3)(n^2 - 9)$ quadruple points. Harbourne asks, for the ideal I of the singular points of C_n , whether $I^{(3)} \not\subseteq I^2$.

An interpolated comment in the source settles C_3 . There the singular points are the 12 vertices, of multiplicity 6, and the 24 infinitely near triple points; I is the ideal of the 12 vertices taken with

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multiplicity 2 and of the 24 infinitely near points taken with multiplicity 1; a direct calculation gives $\alpha(I^{(3)}) = 24$ and $\text{reg}(I) = 12$; and a criterion of Bocci and Harbourne [BH10] gives $I^{(3)} \subseteq I^2$. Here α is the least degree of a nonzero element and reg is the Castelnuovo–Mumford regularity. The source then records a second question, due to Szemberg: “Are there interesting somehow analogous configurations of conics?”

The problem is record OWR-13494-009 of the UnsolvedMath collection [ula26]. The record asks whether $I^{(3)} \subseteq I^2$ holds for all n , and whether there are interesting analogous configurations of conics. It turns the source’s “does non-containment hold?” into “does containment hold for all n ?”; both ask to decide, for each n , whether $I^{(3)} \subseteq I^2$. A proof of containment for all n , or one n with an element of $I^{(3)} \setminus I^2$, answers this first clause. The second clause asks for examples, and “interesting” is not defined; it has no yes-or-no answer, and we do not address it (Remark 7.1).

1.2 The configurations and the range of n

Bauer, Harbourne, Ro   and Szemberg [BHRS17] describe C_n explicitly: for $n = 3m$ it is the union of the Halphen cubics of order m , which are $3m^2 - 1$ members of each of four pencils of cubics with base points among the 12 vertices [BHRS17, Theorem 2]; Section 2 recalls the definitions and the facts we use. Throughout,

$$n = 3m, \quad m \geq 1, \quad k := 3m^2 - 1 = \frac{1}{3}(n^2 - 3),$$

so that C_n is the union of $4k$ cubics and has degree $12k$.

The source introduces C_n only when 3 divides n [Har15, p. 512]; Bauer, Harbourne, Ro   and Szemberg work with $n = 3m$ [BHRS17, Introduction, Theorem 1], Roulleau with n a positive multiple of 3 [Rou17, §1], and Roulleau and Urz  a, who consider only this case, with Eisenstein integers n divisible by $1 - \zeta^2$, $\zeta = e^{2\pi i/6}$ [RU15, §3], which for a rational integer means $3 \mid n$. The source’s C_n , whose data are given in terms of n^2 , and this note concern rational integers n . So “for all n ” means for all $n = 3m$ with $m \geq 1$.

1.3 The ideal

Let V be the set of the 12 vertices, let Q be the set of the $12k$ infinitely near triple points, k over each vertex v , and let S be the set of the $9k(m^2 - 1) = \frac{1}{3}(n^2 - 3)(n^2 - 9)$ quadruple points. The k points of Q over v are k distinct tangent directions at v ; let $l_{v,1}, \dots, l_{v,k}$ be linear forms defining the corresponding lines through v (Lemma 2.3).

Definition 1.1. For $n = 3m$, let $I = I_n \subseteq R$ be the ideal of the cluster of singular points of C_n with weight $k = 3m^2 - 1$ at each point of V , weight 1 at each point of Q , and weight 1 at each point of S . Explicitly, a form F lies in I if and only if F vanishes at every point of S and, at every $v \in V$, either F has multiplicity at least $k + 1$ at v , or F has multiplicity exactly k at v and its tangent cone at v contains the k lines $l_{v,1}, \dots, l_{v,k}$ (that is, the strict transform of the curve $F = 0$ passes through the k points of Q over v).

In Section 3 the same ideal is written as $I = \{F : \text{div}(\pi^*F) \geq Z\}$ for a divisor Z on a blow-up π of $\mathbb{P}^2$, and its local structure is described (Lemma 3.1). The following remarks explain the choice.

Remark 1.2. (a) For $m = 1$ we have $k = 2$, and Definition 1.1 is literally the ideal of the source’s comment: the 12 vertices with multiplicity 2 and the 24 infinitely near points with multiplicity 1. It reproduces the source’s numbers $\alpha(I^{(3)}) = 24$ and $\text{reg}(I) = 12$ (Propositions 4.1 and 5.4).

(b) For $m \geq 2$ the source does not say which weights are meant; the weight k at the vertices is our extension. Locally, I is spanned by the germs of curves that pass through all singular points of C_n , proper and infinitely near: at a vertex, by the germs of multiplicity at least k , the least multiplicity

with which a curve can pass through k distinct infinitely near points of a point, whose strict transforms pass through the k points of Q over it (Lemma 3.1(c)), and at a point of S by the germs vanishing there. So I is the ideal of the consistent cluster with the smallest weights that contains all singular points.

(c) The reading in which I is the ideal of the proper singular points, $V \cup S$ with weight 1, is excluded by the source: for $m = 1$ it is the ideal of the 12 vertices, for which $I^{(3)} \not\subseteq I^2$ [DSTG13], whereas the source reports containment for C_3 .

(d) The reading with weight 2 at the vertices for every m gives, for $k \geq 3$, the ideal of the vertices with multiplicity 3 and of S , since a curve with multiplicity 2 at v has at most two tangent directions there; the infinitely near points then play no role. This reading is not studied in this note.

1.4 Results

Symbolic powers are taken as in [BH10, §1.1]; for the ideal I this means $I^{(r)} = \text{sat}(I^r)$ (Lemma 3.2).

Theorem 1.3. *Let $m \geq 1$, $n = 3m$, and let I be the ideal of Definition 1.1. Then $I^{(3)} \subseteq I^2$.*

So the source's question "does $I^{(3)} \not\subseteq I^2$ hold?" has the answer no for every n divisible by 3, and the first clause of the record has the answer yes.

Theorem 1.4. *For every $m \geq 1$, $\alpha(I^{(3)}) = 36m^2 - 12$. Moreover $\text{reg}(I) = 12$ for $m = 1$, and $\text{reg}(I) \leq 18m^2 - 8$ for $m \geq 2$.*

Theorem 1.3 follows from Theorem 1.4 by the criterion of Bocci and Harbourne [BH10, Lemma 2.3.4]: if $2\text{reg}(I) \leq \alpha(I^{(3)})$, then $I^{(3)} \subseteq I^2$ (Section 6). Indeed $2 \cdot 12 = 24$ for $m = 1$, and $2(18m^2 - 8) = 36m^2 - 16 < 36m^2 - 12$ for $m \geq 2$. The equality $\alpha(I^{(3)}) = 36m^2 - 12$ is proved by Bézout's theorem (Proposition 4.1). The bound $\text{reg}(I) \leq 18m^2 - 8$ for $m \geq 2$ is proved by the Kawamata–Viehweg vanishing theorem on the blow-up of $\mathbb{P}^2$ along the cluster (Proposition 5.2). For $m = 1$ the vanishing argument does not apply, and $\text{reg}(I) = 12$ is an exact computation over $\mathbb{Q}(\varepsilon)$, $\varepsilon^2 + \varepsilon + 1 = 0$, made with a computer (Proposition 5.4); this is the only computer-assisted step of the proof.

1.5 What is new, what is not, and what stays open

The case $m = 1$ is the source's result, with the same numbers and criterion; we reproduce it. The geometry of C_n is that of [BHRS17, Theorems 1, 1.5 and 2], cited, and corroborated by computation for $m = 2$ and $m = 3$ only. The criterion is that of Bocci and Harbourne. To our knowledge, and relative to the literature we found, the containment for $m \geq 2$ under the reading of Definition 1.1 is new: no answer for $m \geq 2$ was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and we claim no priority. Harbourne's survey chapter [Har21] could not be read (closed access). Not addressed: the conics clause, the reading of Remark 1.2(d) for $m \geq 2$, and the exact value of $\text{reg}(I)$ for $m \geq 2$. On the way we note two misprints in the arXiv version of [BHRS17], which do not affect its theorems (Remark 7.4).

2 The configurations

2.1 The Hesse singular point cubic pencils

We recall the construction of [BHRS17, §1]. Let ε be a primitive cube root of unity. The *dual Hesse arrangement* consists of the nine lines given by the linear factors of $(x^3 - y^3)(y^3 - z^3)(z^3 - x^3)$. They

meet by threes in the twelve *vertices*

$$\begin{aligned} v_0 &= (1 : 0 : 0), \quad v_1 = (0 : 1 : 0), \quad v_\infty = (0 : 0 : 1), \quad h_0 = (1 : 1 : 1), \quad h_1 = (1 : \varepsilon : \varepsilon^2), \quad h_\infty = (1 : \varepsilon^2 : \varepsilon), \\ \delta_0 &= (\varepsilon : 1 : 1), \quad \delta_1 = (1 : \varepsilon : 1), \quad \delta_\infty = (1 : 1 : \varepsilon), \quad \gamma_0 = (\varepsilon^2 : 1 : 1), \quad \gamma_1 = (1 : \varepsilon^2 : 1), \quad \gamma_\infty = (1 : 1 : \varepsilon^2) \end{aligned}$$

[BHRS17, p. 4]; each vertex lies on exactly three of the lines, and each line contains exactly four vertices. Let V be the set of vertices, $\mathbf{T} = \{v, h, \delta, \gamma\}$, and $V_t = \{t_0, t_1, t_\infty\}$ for $t \in \mathbf{T}$. The four sets V_t partition V , and each of the nine lines contains exactly one point of each V_t ; these are the four ways of choosing three vertices with exactly one on each line [Har15, p. 512]. Put $\Lambda_t = V \setminus V_t$.

For $t \in \mathbf{T}$ the cubics through the nine points of Λ_t form a pencil $\mathcal{C}_t$, a *Hesse singular point cubic pencil*. It has exactly three singular members C_{t_u} , $u \in \{0, 1, \infty\}$, where C_{t_u} is the union of the three lines of the arrangement through t_u [BHRS17, p. 4], [Har15, p. 512]. The equations are scaled so that $C_{t_0} + C_{t_\infty} = C_{t_1}$, and the member of $\mathcal{C}_t$ with parameter $u \in \mathbb{C}$ is $C_{t_0} + uC_{t_\infty}$ [BHRS17, pp. 4–5]; the singular members have the parameters 0, 1, ∞ .

Let $\mathcal{E} = \mathbb{C}/\mathbb{Z}[\zeta]$ and let $f: \mathcal{E} \rightarrow \mathbb{P}^1$ be a morphism of degree 3 with triple ramification over 0, 1, ∞ ; it is unique up to translation and inversion on $\mathcal{E}$ and up to the automorphisms of $\mathbb{P}^1$ that preserve $\{0, 1, \infty\}$ [BHRS17, Lemma 1.1 and p. 6]. Take the origin of the group law of $\mathcal{E}$ at one of the three ramification points. For $m \geq 1$ the set $\mathbb{P}^1[3m] = f(\mathcal{E}[3m]) \setminus \{0, 1, \infty\}$ of *equianharmonic $3m$ -torsion parameters* does not depend on these choices and has $3m^2 - 1$ elements [BHRS17, p. 6 and Remark 1.2]; $\mathbb{P}^1[3] = \{-\varepsilon, -\varepsilon^2\}$ [BHRS17, p. 6]. The *Halphen cubics of order m* in $\mathcal{C}_t$ are its $k = 3m^2 - 1$ members with parameter in $\mathbb{P}^1[3m]$ [BHRS17, p. 7]. We write $\mathcal{H}_t$ for the set of these k curves and $\mathcal{C} = \bigcup_{t \in \mathbf{T}} \mathcal{H}_t$.

Theorem 2.1 ([BHRS17, Theorem 2]). *For $n = 3m$, the curves of the Rouleau–Urzúa configuration are the Halphen cubics of order m . Thus C_n is the union of the $4k$ curves of $\mathcal{C}$.*

Theorem 2.2 ([BHRS17, Theorem 1.5]). *Let $X \rightarrow \mathbb{P}^2$ be the blow-up of the twelve vertices, with exceptional curves E_v . The union H_m of the strict transforms of the curves of $\mathcal{C}$ has the following singularities:*

1. k ordinary triple points on each E_v , $12k$ in all;
2. $9k(m^2 - 1)$ ordinary quadruple points off $\bigcup_v E_v$.

Moreover, each component of H_m passes through 9 of the triple points and through $9(m^2 - 1)$ of the quadruple points.

The statement in [BHRS17] also gives the linear system of H_m ; see Remark 7.4(a) for a misprint there. We read the list as the complete list of singular points of H_m , as in its proof [BHRS17, pp. 8–9]; [BHRS17, Theorem 1] states the same numbers for C_n on $\mathbb{P}^2$, and so does the source [Har15, p. 512], with the explicit statement that there are no other singular points.

2.2 Consequences

Lemma 2.3. (a) *The $4k$ curves of $\mathcal{C}$ are pairwise distinct smooth cubics. A curve of $\mathcal{H}_t$ passes through each point of Λ_t and through no point of V_t . Two distinct curves of $\mathcal{H}_t$ meet only at the nine points of Λ_t , and transversally.*

- (b) *Every vertex v lies on exactly $3k$ curves of $\mathcal{C}$, namely the curves of $\mathcal{H}_t$ for the three t with $v \in \Lambda_t$. These curves have exactly k distinct tangent lines at v ; each of them is tangent to exactly three of the curves, one from each of the three pencils. The points $q_{v,1}, \dots, q_{v,k}$ of E_v that correspond to these lines are the k triple points of H_m on E_v .*

- (c) Let $S \subset \mathbb{P}^2 \setminus V$ be the set of the quadruple points of Theorem 2.2(2). Each $s \in S$ lies on exactly four curves of $\mathcal{C}$, one from each pencil, with four distinct tangent lines at s . Each curve of $\mathcal{C}$ passes through exactly $9(m^2 - 1)$ points of S .
- (d) The singular points of C_n are the twelve vertices, of multiplicity $3k = n^2 - 3$, and the $9k(m^2 - 1)$ points of S , which are ordinary quadruple points. Over a vertex v , the strict transform of C_n on X is singular exactly at $q_{v,1}, \dots, q_{v,k}$, where it has ordinary triple points. So the singular points of C_n , proper and infinitely near, are the points of V , of $Q = \{q_{v,j} : v \in V, 1 \leq j \leq k\}$ and of S , and after these points are blown up the strict transforms of the $4k$ curves are pairwise disjoint.

Proof. (a) By Theorem 2.1 and the definition, the curves of $\mathcal{H}_t$ are the members of $\mathcal{C}_t$ with the k distinct parameters in $\mathbb{P}^1[3m]$, a set that does not contain $0, 1, \infty$. So they are k distinct members, and none of them is singular; hence they are smooth, and in particular irreducible. A member C of $\mathcal{C}_t$ contains Λ_t . If it also passed through $t_u \in V_t$, then C and the singular member C_{t_u} , which passes through t_u , would have the ten points $\Lambda_t \cup \{t_u\}$ in common; this contradicts Bézout's theorem, since C is irreducible and different from C_{t_u} . If $t' \neq t$, then $V_t \subseteq \Lambda_{t'}$, so a curve of $\mathcal{H}_{t'}$ passes through V_t and a curve of $\mathcal{H}_t$ does not; hence curves of different pencils are distinct. Two distinct members of $\mathcal{C}_t$ have the nine points of Λ_t in common; by Bézout's theorem each of these has intersection multiplicity 1, so the two curves meet nowhere else and have distinct tangent lines there.

(b) Since the sets V_t partition V , a vertex v lies in Λ_t for exactly three t , and by (a) exactly the $3k$ curves of these three sets $\mathcal{H}_t$ pass through v , each of them smooth at v . The strict transform on X of a curve that is smooth at v meets E_v in exactly one point, the point of its tangent direction. By Theorem 2.2(1) the strict transforms of these $3k$ curves meet E_v in k ordinary triple points. Each of these points lies on exactly three strict transforms (a strict transform is smooth, so it has one branch at each point), and each strict transform meets E_v only once; so the k triple points lie on $3k$ distinct strict transforms, that is, on all of them. Hence the $3k$ curves through v have exactly k distinct tangent directions, each shared by exactly three curves. Two curves of the same pencil have distinct tangent lines at v by (a), so the three curves with a common tangent line come from the three different pencils. Since each of the three pencils has k curves through v and there are k directions, each pencil has exactly one curve in each direction.

(c) By Theorem 2.2(2) a point $s \in S$ is a point of $\mathbb{P}^2 \setminus V$ where four smooth branches of curves of $\mathcal{C}$ with distinct tangent lines meet. The curves are smooth, so the four branches belong to four distinct curves, and these lie in four different pencils because two curves of one pencil meet only in $\Lambda_t \subseteq V$, by (a). The last statement is the last sentence of Theorem 2.2.

(d) A point of $\mathbb{P}^2 \setminus V$ on two curves of $\mathcal{C}$ is a singular point of H_m off $\bigcup_v E_v$, hence a point of S by Theorem 2.2; so off V the curve C_n is singular exactly at the points of S , which are ordinary quadruple points by (c). At a vertex, C_n has $3k$ smooth branches by (b). On X , the strict transform of C_n meets E_v only at $q_{v,1}, \dots, q_{v,k}$, where it has ordinary triple points. Blowing up an ordinary singular point separates its branches, since they have distinct tangent directions; so after the points of V , Q and S are blown up, the strict transforms of the curves of $\mathcal{C}$ are pairwise disjoint, and there are no further infinitely near singular points. $\square$

The numbers agree with the source: $3k = n^2 - 3$, $|Q| = 12k = 4(n^2 - 3)$ and $|S| = 9k(m^2 - 1) = \frac{1}{3}(n^2 - 3)(n^2 - 9)$.

3 The cluster and its ideal

3.1 The blow-up

Let $\pi: Y \rightarrow \mathbb{P}^2$ be the composite of the blow-up of the twelve vertices, the blow-up of the $12k$ points of Q , which lie on the exceptional curves of the first blow-up, and the blow-up of the points of S . For $x \in V \cup Q \cup S$ let E_x be the total transform on Y of the exceptional curve of the blow-up at x , and let L be the pull-back of a line. Then

$$\text{Pic } Y = \mathbb{Z}L \oplus \bigoplus_x \mathbb{Z}E_x, \quad L^2 = 1, \quad E_x^2 = -1, \quad L \cdot E_x = E_x \cdot E_y = 0 \ (x \neq y), \quad K_Y = -3L + \sum_x E_x.$$

The irreducible exceptional curves are the curves E_q ($q \in Q$) and E_s ($s \in S$), smooth rational curves with self-intersection -1 , and the strict transforms

$$\tilde{E}_v = E_v - \sum_{j=1}^k E_{q_{v,j}} \quad (v \in V)$$

of the first exceptional curves, smooth rational curves with $\tilde{E}_v^2 = -1 - k$. The curves $\tilde{E}_v$ are pairwise disjoint, $\tilde{E}_v \cdot E_{q_{v,j}} = 1$, and $\tilde{E}_v$ does not meet the other curves E_q nor the curves E_s . Sums $\sum_v$, $\sum_q$, $\sum_s$ are over V , Q and S .

For $C \in \mathcal{H}_t$ let $Q(C) \subset Q$ be the set of the nine points of Q on the strict transform of C on X , one over each vertex of Λ_t (Lemma 2.3(b)), and let $S(C) = S \cap C$, a set of $9(m^2 - 1)$ points. Since C is smooth, its strict transform $\tilde{C}$ on Y is an irreducible curve of class

$$\tilde{C} = 3L - \sum_{v \in \Lambda_t} E_v - \sum_{q \in Q(C)} E_q - \sum_{s \in S(C)} E_s. \quad (1)$$

So $\tilde{C} \cdot L = 3$, $\tilde{C} \cdot E_v = 1$ for $v \in \Lambda_t$, $\tilde{C} \cdot E_q = 1$ for $q \in Q(C)$, $\tilde{C} \cdot E_s = 1$ for $s \in S(C)$, all other $\tilde{C} \cdot E_x$ vanish, and $\tilde{C} \cdot \tilde{E}_v = 0$ for every v . By Lemma 2.3, $3k$ of the curves pass through each vertex, three through each point of Q and four through each point of S ; hence

$$\sum_{C \in \mathcal{C}} \tilde{C} = 12k L - 3k \sum_v E_v - 3 \sum_q E_q - 4 \sum_s E_s. \quad (2)$$

3.2 The ideal

Let

$$Z = k \sum_v E_v + \sum_q E_q + \sum_s E_s = \sum_v \left(k \tilde{E}_v + (k+1) \sum_{j=1}^k E_{q_{v,j}} \right) + \sum_s E_s,$$

an effective divisor supported on exceptional curves, let $\mathcal{I}_Z = \pi_* \mathcal{O}_Y(-Z) \subseteq \pi_* \mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^2}$, an ideal sheaf, and let

$$I = \bigoplus_{t \geq 0} H^0(\mathbb{P}^2, \mathcal{I}_Z(t)) = \{F \in R \text{ homogeneous} : \text{div}(\pi^* F) \geq Z\}.$$

For a point x let $\mathcal{O}_x$ be its local ring and $\mathfrak{m}_x$ the maximal ideal. For $v \in V$ we regard $l_{v,j}$ as a local equation at v of the line through v in the direction $q_{v,j}$.

Lemma 3.1. (a) $\mathcal{I}_{Z,x} = \mathcal{O}_x$ for $x \notin V \cup S$, $\mathcal{I}_{Z,s} = \mathfrak{m}_s$ for $s \in S$, and for $v \in V$

$$\mathcal{I}_{Z,v} = (l_{v,1} \cdots l_{v,k}) + \mathfrak{m}_v^{k+1} = \{f \in \mathfrak{m}_v^k : \text{the degree-}k \text{ part of } f \text{ is a multiple of } l_{v,1} \cdots l_{v,k}\}.$$

- (b) $\mathcal{I}_{Z,v}$ has colength $\frac{1}{2}(k^2 + 3k)$ in $\mathcal{O}_v$. The subscheme of $\mathbb{P}^2$ defined by $\mathcal{I}_Z$ has length $\ell = 6k(k + 3) + 9k(m^2 - 1)$; thus $\ell = 60$ for $m = 1$ and $\ell = 1221$ for $m = 2$.
- (c) $\mathcal{I}_{Z,v}$ is spanned, as a vector space, by the germs f with $\text{mult}_v(f) \geq k$ whose strict transform passes through $q_{v,1}, \dots, q_{v,k}$.
- (d) I is the ideal of Definition 1.1.

Proof. (a) Over a point $x \notin V \cup S$ the map π is an isomorphism and Z is zero. Over $s \in S$ it is the blow-up of s , $Z = E_s$ there, and $\text{ord}_{E_s}(\pi^*f) = \text{mult}_s(f)$; so $f \in \mathcal{I}_{Z,s}$ if and only if $f(s) = 0$. Let $v \in V$ and let f be a germ at v with $\mu = \text{mult}_v(f)$; let μ_j be the multiplicity at $q_{v,j}$ of the strict transform of $f = 0$ on the blow-up of v . Then $\text{ord}_{\tilde{E}_v}(\pi^*f) = \mu$ and $\text{ord}_{E_{q_{v,j}}}(\pi^*f) = \mu + \mu_j$. By the second expression for Z , $f \in \mathcal{I}_{Z,v}$ if and only if $\mu \geq k$ and $\mu + \mu_j \geq k + 1$ for all j . This holds if $\mu \geq k + 1$. If $\mu = k$, it holds if and only if $\mu_j \geq 1$ for all j , that is, if and only if the tangent cone f_k , a binary form of degree k in local coordinates, vanishes in the k distinct directions $q_{v,j}$; this means that f_k is a scalar multiple of $l_{v,1} \cdots l_{v,k}$. The set of germs in $\mathfrak{m}_v^k$ whose degree- k part is a multiple of $g = l_{v,1} \cdots l_{v,k}$ is the ideal $(g) + \mathfrak{m}_v^{k+1}$.

(b) $\dim \mathcal{O}_v / \mathfrak{m}_v^{k+1} = \frac{1}{2}(k+1)(k+2)$, and $(g) + \mathfrak{m}_v^{k+1}$ contains $\mathfrak{m}_v^{k+1}$ with quotient spanned by g ; so the colength is $\frac{1}{2}(k+1)(k+2) - 1 = \frac{1}{2}(k^2 + 3k)$. Summing over the twelve vertices and the $9k(m^2 - 1)$ points of S gives ℓ .

(c) A germ with $\mu \geq k$ whose strict transform passes through all $q_{v,j}$ lies in $\mathcal{I}_{Z,v}$ by (a). Conversely, g has these properties, and so has $g + h$ for every $h \in \mathfrak{m}_v^{k+1}$, since it has multiplicity k and tangent cone g ; hence $h = (g + h) - g$ and every $ag = a(v)g + (a - a(v))g$ ($a \in \mathcal{O}_v$) lie in the span.

(d) A form F of degree t lies in I if and only if, for every point x , the germ at x of F/λ^t , for a linear form λ with $\lambda(x) \neq 0$, lies in $\mathcal{I}_{Z,x}$; by (a) these are the conditions of Definition 1.1. $\square$

Lemma 3.1(c) is the precise form of Remark 1.2(b). For $m = 1$ the conditions at a vertex v are: F and its first derivatives vanish at v , and the quadratic part of F at v vanishes in the two directions $q_{v,1}, q_{v,2}$; these are five linear conditions.

3.3 Symbolic powers

Following [BH10, §1.1], for a homogeneous ideal $0 \neq J \subsetneq R$ we let $J^{(r)} = J^r R_A \cap R$, where A is the complement of the union of the associated primes of J ; $\text{sat}(J)$ denotes the saturation and $P_x \subset R$ the homogeneous prime ideal of a point $x \in \mathbb{P}^2$.

Lemma 3.2. (a) I is saturated, it defines the 0-dimensional subscheme of $\mathbb{P}^2$ with ideal sheaf $\mathcal{I}_Z$, of length ℓ and supported on $V \cup S$, and its associated primes are the ideals P_x , $x \in V \cup S$.

(b) Let $r \geq 1$. A form F of degree t lies in $I^{(r)}$ if and only if, for every $x \in V \cup S$, the germ at x of F/λ^t lies in $(\mathcal{I}_{Z,x})^r$, where λ is a linear form with $\lambda(x) \neq 0$. Moreover $I^{(r)} = \text{sat}(I^r)$.

(c) If $F \in I^{(r)}$, then $\text{div}(\pi^*F) \geq rZ$.

Proof. (a) Let F be a form with $x_i^N F \in I$ for $i = 0, 1, 2$. Near every point of Y one of the sections π^*x_i of $\pi^*\mathcal{O}(1)$ does not vanish, so $\text{div}(\pi^*F) \geq Z$ there; hence $F \in I$, and I is saturated. The sheaf associated with $I = \bigoplus_t H^0(\mathcal{I}_Z(t))$ is $\mathcal{I}_Z$, so I defines the subscheme with ideal sheaf $\mathcal{I}_Z$; by Lemma 3.1 it is supported on $V \cup S$ and has length ℓ . The associated primes of a saturated homogeneous ideal defining a 0-dimensional subscheme are the ideals of the points of its support.

(b) The homogeneous primes containing I^r are the P_x ($x \in V \cup S$) and the irrelevant ideal M ; so in a primary decomposition of I^r the P_x -primary component is $I^r R_{P_x} \cap R$, and any other component is M -primary. The set $A = R \setminus \bigcup_x P_x$ meets M (prime avoidance) and no P_x , so $I^{(r)} = I^r R_A \cap R =$

$\bigcap_x (I^r R_{P_x} \cap R) = \text{sat}(I^r)$. For a form F of degree t , $F \in I^r R_{P_x}$ if and only if F/λ^t lies in the degree-zero localization $(I^r)_{(P_x)} = (\mathcal{I}_{Z,x})^r \subseteq \mathcal{O}_x$.

(c) By (b), near each $x \in V \cup S$ the germ of F/λ^t is a finite sum of products $g_1 \cdots g_r$ with $g_i \in \mathcal{I}_{Z,x}$, and $\text{div}(\pi^* g_i) \geq Z$ near $\pi^{-1}(x)$. Orders along a prime divisor are additive on products and satisfy $\text{ord}(a+b) \geq \min(\text{ord } a, \text{ord } b)$; hence $\text{div}(\pi^* F) \geq rZ$ near $\pi^{-1}(x)$. Since Z is supported over $V \cup S$, $\text{div}(\pi^* F) \geq rZ$. $\square$

4 The initial degree of the symbolic cube

Proposition 4.1. *Let $m \geq 1$.*

- (a) *Every nonzero $F \in I^{(3)}$ has degree at least $36m^2 - 12$.*
- (b) *The product Φ of the equations of the $4k$ curves of $\mathcal{C}$, an equation of C_n of degree $12k = 36m^2 - 12$, lies in $I^{(3)}$.*

Hence $\alpha(I^{(3)}) = 36m^2 - 12$.

Proof. (a) Let $0 \neq F \in I^{(3)}$ have degree d . By Lemma 3.2(c), $D := \text{div}(\pi^* F) - 3Z$ is effective, and it is linearly equivalent to $dL - 3Z$. The components of $\text{div}(\pi^* F)$ that are not exceptional are the strict transforms of the components of the curve $F = 0$. Let $C \in \mathcal{C}$ be a curve that is not a component of $F = 0$. Then $\tilde{C}$ is not a component of D , so $\tilde{C} \cdot D \geq 0$. By (1), $\tilde{C}$ meets E_v once for each of the nine vertices $v \in \Lambda_t$, E_q once for each of the nine points $q \in Q(C)$, and E_s once for each of the $9(m^2 - 1)$ points $s \in S(C)$; hence

$$\tilde{C} \cdot Z = 9k + 9 + 9(m^2 - 1) = (27m^2 - 9) + 9m^2 = 36m^2 - 9,$$

and $0 \leq \tilde{C} \cdot D = 3d - 3(36m^2 - 9)$, that is, $d \geq 36m^2 - 9$. If, on the other hand, every curve of $\mathcal{C}$ is a component of $F = 0$, then, since these $4k$ curves are irreducible and pairwise distinct (Lemma 2.3(a)), Φ divides F and $d \geq 12k = 36m^2 - 12$. In both cases $d \geq 36m^2 - 12$.

(b) We use Lemma 3.2(b). Let $s \in S$. Four of the curves pass through s , so the germ of Φ at s lies in $\mathfrak{m}_s^4 \subseteq \mathfrak{m}_s^3 = (\mathcal{I}_{Z,s})^3$. Let $v \in V$, let t_1, t_2, t_3 be the three indices with $v \in \Lambda_{t_i}$, and let t_4 be the fourth. The curves of $\mathcal{H}_{t_4}$ do not pass through v (Lemma 2.3(a)), so their product is a unit at v . For $i \leq 3$ let G_i be the product of the equations of the k curves of $\mathcal{H}_{t_i}$. By Lemma 2.3(b) these k curves are smooth at v and their tangent lines are $l_{v,1}, \dots, l_{v,k}$, one each; so the germ of G_i is $\prod_{j=1}^k (c_j l_{v,j} + h_j)$ with $c_j \in \mathbb{C}^*$ and $h_j \in \mathfrak{m}_v^2$, which equals $(\prod_j c_j) l_{v,1} \cdots l_{v,k}$ plus an element of $\mathfrak{m}_v^{k+1}$. By Lemma 3.1(a), $G_i \in \mathcal{I}_{Z,v}$, so the germ of $\Phi = G_1 G_2 G_3 \cdot (\text{unit})$ lies in $(\mathcal{I}_{Z,v})^3$. At all other points $\mathcal{I}_{Z,x} = \mathcal{O}_x$. Hence $\Phi \in I^{(3)}$.

So $\alpha(I^{(3)}) \leq 36m^2 - 12$, and equality holds by (a). $\square$

For $m = 1$ this gives $\alpha(I^{(3)}) = 24$, the value reported in the source. Part (b) shows directly that Φ lies in $I^{(3)} = \text{sat}(I^3)$, not only in the ideal of forms F with $\text{div}(\pi^* F) \geq 3Z$; we do not need the equality of these two ideals.

5 The regularity

5.1 Regularity and the vanishing of H^1

For a homogeneous ideal J let $\tau(J)$ be the least degree from which the Hilbert function of R/J agrees with its Hilbert polynomial, and $\sigma(J) = \tau(J) + 1$. If J is saturated and defines a 0-dimensional subscheme of $\mathbb{P}^N$, then $\text{reg}(J) = \sigma(J)$ [BH10, §1.2, p. 3], where Bocci and Harbourne refer to [GGP95].

Lemma 5.1. *For $t \geq 0$ we have $\dim_{\mathbb{C}}(R/I)_t = \ell - h^1(\mathbb{P}^2, \mathcal{I}_Z(t))$. If $h^1(\mathcal{I}_Z(t_0)) = 0$ for some $t_0 \geq 0$, then $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \geq t_0$. Hence $\tau(I)$ is the least $t \geq 0$ with $h^1(\mathcal{I}_Z(t)) = 0$, and $\text{reg}(I) = \tau(I) + 1$.*

Proof. Let $\mathcal{O}_Z = \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_Z$, a skyscraper sheaf of length ℓ . The exact sequence $0 \rightarrow \mathcal{I}_Z(t) \rightarrow \mathcal{O}_{\mathbb{P}^2}(t) \rightarrow \mathcal{O}_Z \rightarrow 0$, $H^0(\mathcal{I}_Z(t)) = I_t$ and $H^1(\mathcal{O}_{\mathbb{P}^2}(t)) = 0$ give $\dim R_t - \dim I_t = \ell - h^1(\mathcal{I}_Z(t))$. A linear form that vanishes at no point of $V \cup S$ lies in no associated prime of I (Lemma 3.2(a)), so multiplication by it is injective from $(R/I)_t$ to $(R/I)_{t+1}$. Hence $\dim(R/I)_t$ is non-decreasing in t ; it is at most ℓ , and once it equals ℓ it stays equal to ℓ . The Hilbert polynomial of R/I is the constant ℓ , and the last statement is the one quoted above. $\square$

5.2 The case $m \geq 2$

Proposition 5.2. *Let $m \geq 2$. Then $H^1(\mathbb{P}^2, \mathcal{I}_Z(t)) = 0$ for every $t \geq 18m^2 - 9$. Consequently $\text{reg}(I) \leq 18m^2 - 8$.*

Proof. Fix $t \geq 18m^2 - 9 = 6k - 3$ and put

$$N = tL - Z - K_Y - \sum_v \tilde{E}_v.$$

Since $-Z - K_Y = 3L - (k+1)\sum_v E_v - 2\sum_q E_q - 2\sum_s E_s$ and $\sum_v \tilde{E}_v = \sum_v E_v - \sum_q E_q$,

$$N = (t+3)L - (k+2)\sum_v E_v - \sum_q E_q - 2\sum_s E_s, \quad K_Y + N = tL - Z - \sum_v \tilde{E}_v. \quad (3)$$

Step 1: a decomposition. Put $a = t + 3 - 6k \geq 0$ and $b = \frac{k}{2} - 2$; since $m \geq 2$ we have $k \geq 11$ and $b \geq \frac{7}{2}$. By (2),

$$N = \frac{1}{2} \sum_{C \in \mathcal{C}} \tilde{C} + aL + b \sum_v E_v + \frac{1}{2} \sum_q E_q; \quad (4)$$

indeed the coefficients are $6k + a = t + 3$ on L , $-\frac{3}{2}k + b = -(k+2)$ on each E_v , $-\frac{3}{2} + \frac{1}{2} = -1$ on each E_q and -2 on each E_s . With $E_v = \tilde{E}_v + \sum_j E_{q_{v,j}}$ this becomes

$$N = \frac{1}{2} \sum_{C \in \mathcal{C}} \tilde{C} + aL + b \sum_v \tilde{E}_v + (b + \frac{1}{2}) \sum_q E_q, \quad (5)$$

a combination with non-negative rational coefficients of the nef class L and of the irreducible curves $\tilde{C}$ ($C \in \mathcal{C}$), $\tilde{E}_v$ ($v \in V$) and E_q ($q \in Q$).

Step 2: N is nef. Let $\Gamma \subset Y$ be an irreducible curve. If Γ is not one of the curves $\tilde{C}$, $\tilde{E}_v$, E_q , every term of (5) meets Γ non-negatively, so $N \cdot \Gamma \geq 0$. For the curves of the list, (3) and (1) give

$$\begin{aligned} N \cdot \tilde{C} &= 3(t+3) - 9(k+2) - 9 \cdot 1 - 9(m^2 - 1) \cdot 2 = 3t - 9k - 18m^2 \\ &= 3t - 45m^2 + 9 \geq 9m^2 - 18 > 0, \end{aligned}$$

$$N \cdot \tilde{E}_v = N \cdot E_v - \sum_{j=1}^k N \cdot E_{q_{v,j}} = (k+2) - k = 2,$$

$$N \cdot E_q = 1.$$

(Also $N \cdot E_s = 2$.) So N is nef.

Step 3: N is big. From (3), $N^2 = (t+3)^2 - 12(k+2)^2 - 12k - 4|S|$, and $4|S| = 36k(m^2 - 1) = 12k(k-2)$ because $3(m^2 - 1) = k - 2$. Since $t+3 \geq 6k$,

$$N^2 \geq 36k^2 - 12(k+2)^2 - 12k - 12k(k-2) = 12k^2 - 36k - 48 = 12(k-4)(k+1) > 0.$$

A nef divisor with positive self-intersection on a smooth projective surface is big [Laz04, Theorem 2.2.16].

Step 4: vanishing on Y . By the Kawamata–Viehweg vanishing theorem [Laz04, Theorem 4.3.1], $H^1(Y, \mathcal{O}_Y(K_Y + N)) = 0$, that is, $H^1(Y, \mathcal{O}_Y(D - \sum_v \tilde{E}_v)) = 0$ for $D = tL - Z$. The curves $\tilde{E}_v$ are pairwise disjoint, so we have the exact sequence

$$0 \rightarrow \mathcal{O}_Y\left(D - \sum_v \tilde{E}_v\right) \rightarrow \mathcal{O}_Y(D) \rightarrow \bigoplus_v \mathcal{O}_{\tilde{E}_v}(D) \rightarrow 0.$$

Here $\tilde{E}_v \cong \mathbb{P}^1$, and

$$D \cdot \tilde{E}_v = tL \cdot \tilde{E}_v - Z \cdot \tilde{E}_v = 0 - \left(kE_v + \sum_j E_{q_{v,j}}\right) \cdot \left(E_v - \sum_j E_{q_{v,j}}\right) = -(-k + k) = 0,$$

so $\mathcal{O}_{\tilde{E}_v}(D) \cong \mathcal{O}_{\mathbb{P}^1}$, whose H^1 vanishes. Hence $H^1(Y, \mathcal{O}_Y(D)) = 0$.

Step 5: descent to $\mathbb{P}^2$. By the projection formula, $\pi_*\mathcal{O}_Y(D) = \mathcal{O}_{\mathbb{P}^2}(t) \otimes \pi_*\mathcal{O}_Y(-Z) = \mathcal{I}_Z(t)$. The Leray spectral sequence of π gives an injection $H^1(\mathbb{P}^2, \pi_*\mathcal{O}_Y(D)) \hookrightarrow H^1(Y, \mathcal{O}_Y(D))$, the first map of its five-term exact sequence. So $H^1(\mathbb{P}^2, \mathcal{I}_Z(t)) = 0$.

By Lemma 5.1, $\tau(I) \leq 18m^2 - 9$ and $\text{reg}(I) \leq 18m^2 - 8$. $\square$

Remark 5.3. The curves $\tilde{E}_v$ are subtracted because $(tL - Z - K_Y) \cdot \tilde{E}_v = 1 - k < 0$. For $m = 1$ the argument fails: $b = -1$, and $N \cdot \tilde{C} = 3t - 36 < 0$ at $t = 11$, the degree needed for $\text{reg}(I) \leq 12$.

5.3 The case $m = 1$

Proposition 5.4. *Let $m = 1$, so $k = 2$, $S = \emptyset$ and $\ell = 60$. For $t = 7, 8, \dots, 13$,*

$$\dim_{\mathbb{C}}(R/I)_t = 36, 45, 52, 57, 60, 60, 60,$$

so that $\dim I_t = 0, 0, 3, 9, 18, 31, 45$ and $h^1(\mathcal{I}_Z(t)) = 24, 15, 8, 3, 0, 0, 0$. Hence $\alpha(I) = 9$, $\tau(I) = 11$ and $\text{reg}(I) = 12$.

Proof (computer-assisted). The vertices are listed in Section 2.1, and the eight Halphen cubics of order 1 are the members $C_{t_0} + uC_{t_\infty}$, $u \in \{-\varepsilon, -\varepsilon^2\}$, of the four pencils, listed in [BHRS17, p. 7]; all are defined over $\mathbb{Q}(\varepsilon)$. By Lemma 3.1(a) and the remark after it, $\mathcal{I}_{Z,v}$ is the common kernel of five linear forms on $\mathcal{O}_v$: in local coordinates at v , the value of f at v , its two first derivatives, and its quadratic part evaluated at the two tangent directions at v of the Halphen cubics through v . Since I_t is the common kernel of these 60 forms on R_t (Lemma 3.1(d)), $\dim(R/I)_t$ is the rank of the $60 \times \dim R_t$ matrix of these forms on the monomials of degree t . The program `check_note.py` of the package (Section 8) computes these ranks exactly over $\mathbb{Q}(\varepsilon)$, with the values stated. It also checks exactly the facts used to set up the conditions: the eight cubics agree up to scalars with the list of [BHRS17, p. 7]; they are smooth; six of them pass through each vertex, with exactly two tangent lines, each tangent to three of the six, from three pencils; and two cubics from different pencils have six common vertices and the same tangent at exactly three of them, so that, by Bézout's theorem ($3 \cdot 2 + 3 \cdot 1 = 9$), they meet nowhere else and the 24 infinitely near triple points are ordinary. Since $\dim(R/I)_{10} = 57 < 60 = \dim(R/I)_{11}$, Lemma 5.1 gives $\tau(I) = 11$ and $\text{reg}(I) = 12$. $\square$

Only $h^1(\mathcal{I}_Z(11)) = 0$, that is $\text{reg}(I) \leq 12$, is used in the proof of Theorem 1.3. The same ranks were obtained independently by four other programs in exact arithmetic (Section 8).

6 Proof of the main theorem

We use the second postulational criterion of Bocci and Harbourne, with the symbolic powers of Section 3.3; we write s for the exponent that is called m in [BH10].

Theorem 6.1 ([BH10, Lemma 2.3.4]). *Let J be a homogeneous ideal, not necessarily saturated, of the polynomial ring of $\mathbb{P}^N$ over an algebraically closed field, defining a 0-dimensional subscheme. If $r \operatorname{reg}(J) \leq \alpha(J^{(s)})$, then $J^{(s)} \subseteq J^r$.*

In [BH10] the proof runs as follows: $r \operatorname{reg}(J) \leq \alpha(J^{(s)}) \leq s \alpha(J) \leq s \operatorname{reg}(J)$ gives $r \leq s$ [BH10, Lemma 2.3.3(b)], hence $J^{(s)} \subseteq J^{(r)}$; if J is not saturated, then $J^{(s)} = J^s$; if J is saturated, then $(J^r)_t = (J^{(r)})_t$ for $t \geq r \operatorname{reg}(J)$ [BH10, Lemma 2.3.3(c)], by the bound $\operatorname{reg}(J^r) \leq r \operatorname{reg}(J)$ of [GGP95, Theorem 1.1], while $(J^{(s)})_t = 0$ for $t < \alpha(J^{(s)})$.

Proof of Theorems 1.3 and 1.4. By Proposition 4.1, $\alpha(I^{(3)}) = 36m^2 - 12$ for every $m \geq 1$; by Proposition 5.4, $\operatorname{reg}(I) = 12$ for $m = 1$; by Proposition 5.2, $\operatorname{reg}(I) \leq 18m^2 - 8$ for $m \geq 2$. This is Theorem 1.4.

By Lemma 3.2(a), I is a homogeneous ideal of $\mathbb{C}[x, y, z]$ defining a 0-dimensional subscheme of $\mathbb{P}^2$. This subscheme is not reduced, and the conditions at the infinitely near points are encoded in the local ideals $\mathcal{I}_{Z,v}$; Theorem 6.1 requires nothing more, and the symbolic powers in it are those of Lemma 3.2(b). We apply it with $r = 2$ and $s = 3$. For $m = 1$, $2 \operatorname{reg}(I) = 24 = \alpha(I^{(3)})$. For $m \geq 2$, $2 \operatorname{reg}(I) \leq 36m^2 - 16 < 36m^2 - 12 = \alpha(I^{(3)})$. In both cases $I^{(3)} \subseteq I^2$. $\square$

7 Remarks

Remark 7.1 (The second clause). Szemberg's question, whether there are interesting analogous configurations of conics, is not addressed in this note. For orientation: H -constants of configurations of conics with transversal intersection points are estimated in [PTG16], Harbourne's survey shows four real conics with three of them through each of their intersection points [Har18, Fig. 3], and conics attached to the 6-division points of the Fermat cubic are studied in [SS24], a note that also recalls the containment problem for the twelve vertices; we found no containment result for a conic analogue of C_n .

7.1 What the bounds give

For $m = 1$ the criterion is used with equality, $2 \operatorname{reg}(I) = \alpha(I^{(3)}) = 24$; for $m \geq 2$ there is a margin of at least 4. The bound $\operatorname{reg}(I) \leq 18m^2 - 8$ is probably not sharp: for $m = 2$ it gives 64, while the computations modulo primes in Section 8 would give $\operatorname{reg}(I) \leq 52$ if the points computed modulo p are reductions of the points in characteristic 0; we do not prove this. The divisor of Proposition 5.2 also bounds all symbolic powers from below.

Proposition 7.2. *Let $m \geq 2$. Then $\alpha(I^{(s)}) \geq (3k + 4)s$ for every $s \geq 1$. Consequently $I^{(s)} \subseteq I^r$ whenever $s/r \geq (6k - 2)/(3k + 4)$, and the resurgence $\rho(I)$, the supremum of the ratios s/r with $I^{(s)} \not\subseteq I^r$ [BH10, §1.1], satisfies $\rho(I) \leq (6k - 2)/(3k + 4) < 2$.*

Proof. Let N be the divisor (3) for $t = 6k - 3 = 18m^2 - 9$; it is nef by Steps 1 and 2 of the proof of Proposition 5.2. Let $0 \neq F \in I^{(s)}$ have degree d . By Lemma 3.2(c), $\operatorname{div}(\pi^*F) - sZ$ is effective and linearly equivalent to $dL - sZ$, so $N \cdot (dL - sZ) \geq 0$. Here $N \cdot L = t + 3 = 6k$ and

$$N \cdot Z = 12k(k + 2) + 12k \cdot 1 + 2|S| = 12k(k + 2) + 12k + 6k(k - 2) = 18k^2 + 24k,$$

so $d \geq s(18k^2 + 24k)/(6k) = (3k + 4)s$. By [BH10, Corollary 2.3.5], for a homogeneous ideal defining a 0-dimensional subscheme and a real number $c > 0$ with $sc \leq \alpha(I^{(s)})$ for all $s \geq 1$, one has $I^{(s)} \subseteq I^r$ whenever $s/r \geq \text{reg}(I)/c$, and $\rho(I) \leq \text{reg}(I)/c$. We take $c = 3k + 4$ and use $\text{reg}(I) \leq 6k - 2$ (Proposition 5.2); finally $2(3k + 4) - (6k - 2) = 10 > 0$. $\square$

For $m = 2$ this gives $\rho(I) \leq 64/37$. We have not tried to optimise the constant.

7.2 Comparison with known results

Remark 7.3. (a) For $m = 1$, I is strictly contained in the ideal J of the twelve vertices, and $J^{(3)} \not\subseteq J^2$: the product f of the nine lines lies in $J^{(3)}$ and not in J^2 [DSTG13, Theorem 2.2]. This f has multiplicity 3 at each vertex, so it lies in I (Definition 1.1 with $k = 2$), but not in $I^{(3)}$, whose elements have multiplicity at least $3k = 6$ at the vertices (Lemma 3.2(c)). There is no contradiction with Theorem 1.3: containment of symbolic in ordinary powers is not monotone in the ideal.

(b) Harbourne observes [Har15, p. 511] that every known nontrivial configuration of lines with no point on exactly two lines has H -constant less than -2 and non-containment for the reduced ideal of its singular points. No point lies on exactly two components of C_n , and $H(C_n) = -4(n^4 - 3n^2)/(n^4 + 27) < -2$ for $n \geq 6$ [Har15, p. 512]; so for the cluster ideal the analogous pattern fails for these cubics. This does not contradict the observation, which concerns lines and reduced points.

7.3 Two misprints in the arXiv version of [BHRS17]

Remark 7.4. We read [BHRS17] in its arXiv version 1; the journal version was not compared.

(a) In [BHRS17, Theorem 1.5] the linear system of H_m is printed as $|12(3m^2 - 1)L - 9(3m^2 - 1)E|$, where E is the sum of the twelve exceptional curves. Each curve of $\mathcal{H}_t$ lies in $|3L - E_t|$ with $E_t = \sum_{v \in \Lambda_t} E_v$ [BHRS17, p. 7 and the first line of the proof of Theorem 1.5], and each vertex lies in Λ_t for exactly three t , so $\sum_t E_t = 3E$; hence the class is $12(3m^2 - 1)L - 3(3m^2 - 1)E = 12kL - 3kE$, in accordance with the multiplicity $3k$ of C_n at the vertices.

(b) On p. 10 of [BHRS17] the six parameters of $\mathbb{P}^1[6] \setminus (\mathbb{P}^1[3] \cup \{-1, \frac{1}{2}, 2\})$ are said to be the roots of $P(x) = x^6 - 3x^5 + 5x^3 - 3x + 1 = (x^2 - x - 1)^3 + 2 = (x^2 - x + 1)^3 - 6x^2(x - 1)^2$, on which the function $J(x) = (x^2 - x + 1)^3/(x^2(x - 1)^2)$, invariant under the automorphisms of $\mathbb{P}^1$ preserving $\{0, 1, \infty\}$, equals 6. The parameter $\tau = -\varepsilon(b\varepsilon^2 - 1)/(b\varepsilon - 1)$, $b^3 = 2$, given on the same page has the minimal polynomial $(x^2 - x + 1)^3 + 54x^2(x - 1)^2 = x^6 - 3x^5 + 60x^4 - 115x^3 + 60x^2 - 3x + 1$ (so $J = -54$), and the six parameters, computed from the definition with division polynomials of $y^2 = x^3 + 1$, are its roots (Section 8). The explicit cubics on that page are written in terms of τ and agree with it; for instance $C_{v_\tau} = (b - \varepsilon)x^3 + \varepsilon^2(b - 1)y^3 + (b\varepsilon - 1)z^3$ is proportional to $C_{v_0} + \tau C_{v_\infty} = -\tau x^3 + (\tau - 1)y^3 + z^3$.

Neither misprint affects the theorems of [BHRS17]: the class in (a) is the sum of the classes $(3m^2 - 1)(3L - E_t)$ stated at the start of the proof, and the singularities do not depend on it; and the cubics in (b) are given in terms of τ .

8 Computations and reproducibility

Everything in Sections 2 to 7 is proved by reading, given the cited theorems, with two exceptions: Proposition 5.4, the regularity for $m = 1$, rests on an exact finite computation, and so do the statements about the sextics in Remark 7.4(b), which are not used in any proof. The cited theorems are [BHRS17, Theorems 2 and 1.5] for the geometry of C_n ; [BH10, Lemma 2.3.4], which uses [GGP95, Theorem 1.1], and the equality $\text{reg} = \sigma$ of [BH10, §1.2]; the Kawamata–Viehweg vanishing theorem and the bigness criterion for nef divisors [Laz04]; and [BH10, Corollary 2.3.5] for Proposition 7.2. The other computations reproduce numbers or corroborate cited facts; none is used in a proof. The

package contains the program of this note, the programs with which the results were first obtained (`original/`), those of the two verification runs (`verification_run_A/`, `verification_run_B/`), and those of the verification run on the final text (`independent_run_2/`), each with its recorded outputs. The programs run with Python 3 (tested with 3.13.5, and with numpy 2.5.2, sympy 1.14.0 and mpmath 1.3.0 where needed) inside their folders; each writes into a folder of its own or to the standard output, never over a recorded output.

The certificate for $m = 1$. Five programs, each with its own construction of the configuration, computed the ranks of the 60 conditions of Z exactly over $\mathbb{Q}(\varepsilon)$, and they agree: `check_note.py` of this note (36, 45, 52, 57, 60, 60, 60 for $t = 7, \dots, 13$), `m1_exact.py` of `original/` ($t = 9, \dots, 12$), `m1_verify.py` of run A ($t = 8, \dots, 12$), `b2_m1.py` of run B ($t = 7, \dots, 13$) and `r2_m1_exact.py` of the run on the final text ($t = 7, \dots, 13$). The program `check_note.py` (standard library, about 5 seconds) also checks exactly the facts about C_3 listed in the proof of Proposition 5.4, Remark 7.4(b) in $\mathbb{Q}(\varepsilon, b)$, and the intersection numbers of Sections 2 to 5 and of Proposition 7.2 for $m = 1, \dots, 60$; all checks pass.

Corroboration of Theorem 2.2 for $m = 2$ and $m = 3$. Modulo primes (`m2_modp.py` at $p = 1000039$ and 1000303 and `m3_modp.py` at $p = 2000521$ in `original/`; `conf_driver.py` of run A at $p = 1000171$, 1000291 and, for $m = 3$, 1500283; `b4_m2_modp.py` of run B at $p = 1000171$ and 1000291), and in characteristic 0 by complex numerics with 50 digits (`b3_numeric.py` of run B) and with 80 digits (`r2_numeric.py` of the run on the final text), the programs find for $m = 2$: 44 smooth cubics; at each vertex 33 cubics with 11 tangent directions, three cubics from three pencils on each; 297 further intersection points, each on exactly four cubics, one from each pencil, with distinct tangents; 27 of them on each cubic; and no other intersections. For $m = 3$: 104 cubics, 26 directions at each vertex, 1872 quadruple points, 72 on each cubic. With the sextic printed in [BHRS17] instead of the correct one, both numerical programs find 1593 off-vertex points (only 9 of them on four cubics, by `b3_numeric.py`); so the test detects a wrong input.

Regularity for $m = 2$ modulo primes. At each of the six primes 1000039, 1000303, 1000171, 1000291, 1002073, 1002091 (the last two in `independent_run_2/`) the programs find $h^1(\mathcal{I}_Z(t)) = 18, 3, 0$ for $t = 49, 50, 51$ modulo p , and 0 at $t = 52$ and $t = 63$ where computed; this agrees with Proposition 5.2, which gives vanishing from $t = 63$ on.

Other checks. For $m = 1$, four programs find modulo primes that no nonzero form F of degree $t \leq 23$ has $\text{div}(\pi^*F) \geq 3Z$ and that C_3 spans those of degree 24; `b2_m1.py` (degrees 24, 25, 26) and `r2_m1_containment_modp.py` (degree 24) also check $I^{(3)} \subseteq I^2$ directly, by showing that $(I^2)_t$ is the space of all forms F of degree t with $\text{div}(\pi^*F) \geq 2Z$, and both detect the known failure of containment for the ideal of the twelve vertices [DSTG13]. Five programs compute $\mathbb{P}^1[6]$ or $\mathbb{P}^1[9]$ (11 and 26 parameters; exactly, with division polynomials, in the first two verification runs and in the run on the final text) and the minimal polynomial of τ , and `b5_symbolic.py` and `r2_symbolic.py` check the identities of Sections 4 and 5 symbolically in m and t . Three exploratory programs of the second verification run (for instance on the readings of Remark 1.2(c), (d) for $m = 2$) play no role in this note.

Command lines. The recorded outputs of `original/` were produced, in `original/scripts`, by `python3 -I param_check.py`, `python3 -I m1_exact.py`, `python3 -I m1_alpha3_modp.py` (default prime 1000039), `python3 -I m2_modp.py 1000039 46`, `python3 -I m2_modp.py 1000303 49 52` and `python3 -I m3_modp.py` (which searches upwards from 2000000 and uses 2000521). The output for $p = 1000039$ was produced with the first degree 46 given on the command line; by default the program starts at $t = 44$. The command lines of all other recorded outputs are listed in `reproducibility/README.md`; the program of this note is run as `python3 check_note.py`.

Validity of the modular computations. The matrices have entries in a ring of algebraic integers with finitely many primes inverted, and reduction modulo p is a ring homomorphism; so full rank, or

a zero kernel, modulo p is a certificate in characteristic 0. For $m = 1$ this applies directly. For $m = 2$ and $m = 3$ it also needs that the points computed over $\mathbb{F}_p$ are reductions of the points in characteristic 0, which was not proved; these results are corroboration only. We record as an observation that at each of the primes 1000039, 1000303, 1000171 and 1000291 all intersection points of pairs of the 44 cubics, among them the 297 quadruple points, were rational over $\mathbb{F}_p$, and that at 1002073 and 1002091, where only the pairs from two of the pencils were intersected, the 297 points were rational as well; we do not claim that the quadruple points are defined over $\mathbb{Q}(\varepsilon, 2^{1/3})$.

Re-runs. On 14 October 2026 all programs with a recorded output were run again with these command lines, once from a copy of the package when this note was written and once from the archive by the run on the final text; both times the outputs were identical to the recorded ones up to fields that record running times. Each program takes at most 20 seconds, except five programs of run B (up to 3 minutes), `b3_numeric.py` 3 (15 to 22 minutes), and three programs of the run on the final text: `r2_m1_exact.py` (about 1 minute), `r2_m2_modp.py` (about 30 seconds) and `r2_numeric.py` (2.5, 4 and 30 minutes for the arguments 2, 2 wrong and 3).

Verification. The results were first obtained, with proofs, together with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs, and neither saw the other's report. Both compared the statement with the source and found the reading of Definition 1.1 to be the natural extension of the source's description of C_3 ; both re-derived the proofs of Sections 3 to 6 line by line, with every intersection number; compared the cited statements of [BH10] and [BHRS17] with these papers; and recomputed the certificate for $m = 1$ exactly, each with its own construction of the configuration. The first run corroborated Theorem 2.2 modulo primes for $m = 2$ and $m = 3$. The second used other methods (complex numerics for $m = 2, 3$, a symbolic check of the identities in m and t , exact intersection multiplicities for $m = 1$), ran the original programs again, searched the literature, and found the misprinted class of Remark 7.4(a) and the direct argument of Proposition 4.1(b). Neither run found a mathematical error. All corrections they required were made: the naming of the ideal and of our extension, the status of the two clauses, the range of n with references, the misprinted class, the command lines of the recorded outputs, the bibliography, and some wording. The monotonicity argument in Lemma 5.1, Proposition 7.2 and `check_note.py` were added when this note was written.

A further independent verification run, also AI-assisted, then examined the final text of this note, with programs of its own (`independent_run_2/`). It read the source, [BHRS17], [BH10], [DSTG13], [Rou17] and [RU15] again, and the passages of [Har18] on containment and on the configurations of cubics; re-derived the proofs of Sections 2 to 7 line by line, including the monotonicity argument in Lemma 5.1 and Proposition 7.2; and checked all identities symbolically in m and t . Its programs build the configuration for $m = 1$ from scratch and give the ranks of Proposition 5.4 exactly over $\mathbb{Q}(\varepsilon)$; for $m = 1$ modulo a prime they find no nonzero form of degree at most 23 with $\text{div}(\pi^*F) \geq 3Z$ and check $\Phi \in I^2$ directly in degree 24; they compute $\mathbb{P}^1[3]$, $\mathbb{P}^1[6]$ and $\mathbb{P}^1[9]$ with division polynomials and confirm both misprints of Remark 7.4; they corroborate Theorem 2.2 for $m = 2$ and $m = 3$ by complex numerics with 80 digits; and for $m = 2$ they find the values of $h^1(\mathcal{I}_Z(t))$ stated above at two further primes. The run also ran all other programs of the package again from the archive, with the recorded outputs reproduced, and repeated the literature search. It found no mathematical error. The corrections it required were made: the ground field in Theorem 6.1 is algebraically closed, as in [BH10]; in the definition of $\mathbb{P}^1[3m]$ the origin of $\mathcal{E}$ is a ramification point, and the uniqueness of f is stated as in [BHRS17, Lemma 1.1 and p. 6]; Remark 1.2(b) is stated in the local form that Lemma 3.1(c) proves; and this account of the verification and of the package was brought up to date. The run also executed `check_note.py`, whose results agree with those of its own programs, but did not review that program line by line.

Scope and priority. The note answers the first clause of record OWR-13494-009 under the reading of Definition 1.1: $I^{(3)} \subseteq I^2$ for every $n = 3m$. The second clause, Szemberg’s question on configurations of conics, is not addressed. The question is Harbourne’s [Har15], with the conics clause attributed there to Szemberg; the case C_3 and its numbers $\alpha(I^{(3)}) = 24$ and $\text{reg}(I) = 12$ are Harbourne’s [Har15, p. 512]; the geometry of C_n is due to Bauer, Harbourne, Ro   and Szemberg [BHRS17, Theorems 1, 1.5 and 2], cited here and corroborated by computation only for $m = 2$ and $m = 3$; the criterion is that of Bocci and Harbourne [BH10]. New, to our knowledge and relative to the literature we found, is the containment for every $m \geq 2$ under the reading of Definition 1.1. No answer for $m \geq 2$ was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and no priority is claimed.

What was read: [Har15], pp. 510–513, in the publisher’s open file; [BHRS17] in arXiv version 1, pp. 1–10, to which our numbering refers (the journal version was not read); [BH10] in arXiv version 2, Sections 1.1, 1.2 and 2.3; [DSTG13], [Rou17] and [RU15] in their arXiv versions, the sections cited; the passages of [Har18] and [SS24] on configurations of cubics and conics and on containment; the abstract of [PTG16]. Not read: Harbourne’s chapter [Har21], which is closed access, cites [BHRS17], and may restate the question or report on it, so a reader with access should check it; [GGP95], cited after [BH10]; and [Laz04], not consulted for this note, whose statements used here are standard and are cited under their usual theorem numbers. The searches (14 October 2026: when the results were first obtained, in the two earlier verification runs, in part again when this note was written, and in the run on the final text) used the arXiv API (Halphen cubics, Roulleau, elliptic curve configurations, infinitely near points and the Hesse configuration, each with symbolic powers or containment; cubics or conics with containment), Crossref, zbMATH, Semantic Scholar, OpenAlex (the works citing [BHRS17], [Rou17] or the source, and the works citing [BH10] found with the words “cubics” or “Roulleau”), and six general web searches. OpenAlex lists six works citing [BHRS17] or the report; none of their titles, and none of the abstracts available there, announces an answer, and neither does any title among the other works found.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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