

A Sharp Linear Computability Gap for Direct Sums of Binary Cubics

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Abstract

Linear computability detects Waring rank by an apolar colon quotient associated with an ideal generated by linear forms and a general linear element of that ideal. We show that it is not preserved by sums in disjoint variables. Over an algebraically closed field of characteristic zero, let $F_m = \sum_{i=1}^m x_i y_i^2$. Its Waring rank is the known value $3m$, whereas the maximum of all linear-computability bounds is exactly $2m + 1$. We calculate the quotient length for every linear ideal and identify the unique maximizing subspace of linear generators. In particular, $xy^2 + zw^2$ has rank six but every such bound is at most five, although each summand is linearly computable. This answers the closure question in Catalisano's 2015 Oberwolfach contribution. The argument uses elementary inverse systems and includes a self-contained proof of the rank value.

1 Linear computability and the main result

Throughout, $\mathbb{k}$ is algebraically closed of characteristic zero. Put

$$S = \mathbb{k}[x_1, y_1, \dots, x_m, y_m], \quad T = \mathbb{k}[X_1, Y_1, \dots, X_m, Y_m],$$

where capital letters act on S by differentiation. For $F \in S$, its apolar ideal is $F^\perp = \{D \in T : DF = 0\}$. The Waring rank of a cubic is the least number of cubes of linear forms whose sum is the cubic; scalar coefficients can be absorbed into the linear forms over $\mathbb{k}$.

For a nonzero subspace $K \subseteq T_1$, let $I = (K)$ be its generated ideal and define

$$B_F(K) = \dim_{\mathbb{k}} T / ((F^\perp : I) + (q)), \quad q \text{ general in } K. \quad (1)$$

The value is constant on a nonempty Zariski-open subset of K . Set $B_F(0) = 0$. A form is *linearly computable* if $B_F(K) = \text{rk}(F)$ for some K . This is the definition in [1, p. 503], and the case $e = 1$ of [2, Definition 3.5]. In particular, I must be generated in degree one. The associated lower bound is $B_F(K) \leq \text{rk}(F)$.

Catalisano asks whether the sum of two linearly computable forms remains linearly computable [1, p. 503]. The surrounding discussion concerns disjoint variable sets. We answer negatively even in that setting.

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Theorem 1.1. *For every $m \geq 1$, let*

$$F_m = \sum_{i=1}^m x_i y_i^2, \quad E = \langle X_1, \dots, X_m \rangle \subset T_1.$$

Then

$$\text{rk}(F_m) = 3m, \quad \max_{K \subseteq T_1} B_{F_m}(K) = 2m + 1. \quad (2)$$

Moreover, E is the unique maximizing subspace. Consequently F_1 is linearly computable and F_m is not linearly computable for $m \geq 2$.

The rank equality in (2) is already a special case of the rank formula for sums of coprime monomials [3]. It is not a new rank-additivity assertion. The point here is the exact maximum over *all* linear ideals; evaluating the ideal (E) alone would not prove noncomputability.

2 An elementary rank calculation

Proposition 2.1. *The Waring rank of F_m is $3m$.*

Proof. The identity

$$xy^2 = \frac{(x+y)^3 + (x-y)^3 - 2x^3}{6}$$

gives the upper bound $3m$.

For the lower bound, write $F_m = \sum_{j=1}^r L_j^3$, and let A be the $2m$ -by- r coefficient matrix of the linear forms L_j . The $2m$ first derivatives of F_m are

$$y_1^2, 2x_1 y_1, \dots, y_m^2, 2x_m y_m,$$

which are linearly independent. Thus F_m is concise: no nonzero operator in T_1 annihilates it. If A failed to have row rank $2m$, a nonzero linear differential operator would annihilate every L_j , and hence F_m . Therefore $\text{rank}(A) = 2m$.

Let V be the m -dimensional vector subspace defined by $y_1 = \dots = y_m = 0$. All first derivatives of F_m vanish on V . Differentiating the decomposition gives

$$A v(p) = 0 \quad (p \in V), \quad v(p) = (L_1(p)^2, \dots, L_r(p)^2)^t.$$

The restrictions $L_j|_V$ span V^* : the x -rows of the full-row-rank matrix A are independent. Select m independent restrictions. Their squares are linearly independent quadratic polynomials, as is seen by using the selected restrictions as coordinates on V .

For polynomial functions $g_1, \dots, g_r$ on a vector space over an infinite field, the dimension of the span of their evaluation vectors equals the dimension of their linear span as polynomial functions. Indeed, the linear relations among the coordinate functions are exactly the linear functionals vanishing on every evaluation vector. It follows that $\dim \langle v(p) : p \in V \rangle \geq m$. This space lies in $\ker A$, so $r - 2m \geq m$, proving $r \geq 3m$. $\square$

3 The length for every linear ideal

We first recall the elementary inverse-system translation of (1).

Lemma 3.1. *Let $K \subseteq T_1$ and let $M = T(KF_m)$ be the vector space generated by all derivatives of the quadrics uF_m , $u \in K$. For every $q \in K$,*

$$\dim_{\mathbb{k}} T/((F_m^\perp : (K)) + (q)) = \dim_{\mathbb{k}} M - \dim_{\mathbb{k}}(qM). \quad (3)$$

Proof. An operator D annihilates each uF_m exactly when $Du \in F_m^\perp$ for every $u \in K$. Differential operators commute, so the same condition annihilates all further derivatives. Hence $\text{Ann}_T(M) = F_m^\perp : (K) = J$. The perfect apolar pairing between homogeneous operators and forms identifies M , degree by degree, with the dual of the finite-dimensional algebra T/J . Multiplication by q on T/J is dual to differentiation by q on M . Taking the dimension of the cokernel of multiplication proves (3). $\square$

Write a linear operator as $u = \sum_i (a_i(u)X_i + b_i(u)Y_i)$. For a fixed nonzero K , define

$$\mathcal{S} = \{i : b_i|_K \neq 0\}, \quad \mathcal{T} = \{i : b_i|_K = 0, a_i|_K \neq 0\}.$$

Set

$$s = |\mathcal{S}|, \quad t = |\mathcal{T}|, \quad k = \dim K, \quad v = \dim(K \cap \langle X_i : i \in \mathcal{T} \rangle).$$

All coordinates of K on the remaining blocks vanish.

Proposition 3.2. *With this notation, the exact general-element length is*

$$B_{F_m}(K) = \begin{cases} 2s + t + v, & s > 0, \\ 1 + t + k, & s = 0. \end{cases} \quad (4)$$

Proof. The degree-two part $M_2 = KF_m$ has dimension k , by concision. For $u \in K$,

$$uF_m = \sum_i (a_i(u)y_i^2 + 2b_i(u)x_iy_i).$$

If $i \in \mathcal{S}$, differentiating a suitable such quadric by X_i and Y_i yields both y_i and x_i . If $i \in \mathcal{T}$, the corresponding derivatives yield only y_i . Blockwise differentiation isolates these coordinates even when K couples different blocks. Hence $\dim M_1 = 2s + t$. Also $\dim M_0 = 1$, because a nonzero quadratic has a nonzero second derivative in characteristic zero. Thus

$$\dim M = 1 + 2s + t + k. \quad (5)$$

Choose $q \in K$ with $b_i(q) \neq 0$ for $i \in \mathcal{S}$ and $a_i(q) \neq 0$ for $i \in \mathcal{T}$. These conditions define a nonempty open subset of K . Apart from a common factor 2, the coefficients of quF_m on the block (x_i, y_i) are

$$\begin{pmatrix} 0 & b_i(q) \\ b_i(q) & a_i(q) \end{pmatrix} \begin{pmatrix} a_i(u) \\ b_i(u) \end{pmatrix}. \quad (6)$$

The block matrix is invertible on $\mathcal{S}$. On $\mathcal{T}$ its kernel is the X_i -direction. Inactive blocks contribute no coordinates from K . Consequently the kernel of $q : M_2 \rightarrow M_1$ has dimension v , and

$$\dim(qM)_1 = k - v. \quad (7)$$

If $s > 0$, then $q^2F_m \neq 0$: its x_i coefficient is $2b_i(q)^2$ for $i \in \mathcal{S}$. Thus qM contains a nonzero linear form. It is derivative-closed, since differentiation commutes with q , and therefore $\dim(qM)_0 = 1$. Equations (3)–(7) give $B_{F_m}(K) = 2s + t + v$.

If $s = 0$, then $K \subseteq E$, while M contains only y -variables and constants. Thus $qM = 0$, and (5) gives the second formula. $\square$

Proof of Theorem 1.1. The rank equality is Proposition 2.1. If $s > 0$, then $v \leq t$ and $s + t \leq m$, whence

$$B_{F_m}(K) = 2s + t + v \leq 2(s + t) \leq 2m.$$

If $s = 0$, then $k \leq t \leq m$, so

$$B_{F_m}(K) = 1 + t + k \leq 2m + 1.$$

Equality forces $k = t = m$, hence $K = E$. Conversely, for $K = E$, the module M has basis

$$1, y_1, \dots, y_m, y_1^2, \dots, y_m^2,$$

and every $q \in E$ annihilates it. Thus $B_{F_m}(E) = 2m + 1$. The zero subspace has value zero and cannot maximize. Finally, $2m + 1 = 3m$ precisely when $m = 1$. $\square$

4 The four-variable counterexample and scope

Corollary 4.1. *The forms xy^2 and zw^2 are linearly computable in disjoint variable sets, but their sum is not linearly computable.*

Proof. Each summand has rank three and a linear-computability bound of three. Their sum is F_2 , for which Theorem 1.1 gives rank six and maximum bound five. $\square$

For the maximizing ideal $I = (X_1, X_2)$ in this example, the colon is explicitly

$$F_2^\perp : I = (X_1, X_2, Y_1 Y_2, Y_1^3, Y_2^3).$$

Its quotient has basis $1, Y_1, Y_2, Y_1^2, Y_2^2$, and adding any $q \in I_1$ changes nothing. Proposition 3.2, rather than this single calculation, rules out every competing linear ideal.

Two conditions in the definition are essential. First, when $K = T_1$ for $m = 2$, the special element $q = X_1$ gives length seven, whereas a general q gives length four. For the special choice, M has graded dimensions $(1, 4, 4)$ and qM has dimension two. The length seven is not a valid general-element rank bound.

Second, if higher-degree generators are permitted, take

$$I = (X_1, X_2, Y_1^2, Y_2^2).$$

The inverse module $T(IF_2)$ has basis $1, x_1, y_1, x_2, y_2, y_1^2, y_2^2$. A general $q \in I_1 = \langle X_1, X_2 \rangle$ maps this module onto the constants, giving length six. This ideal is not generated in degree one and does not meet the definition of linear computability.

The failure established here concerns computability, not symmetric rank additivity. Indeed $\text{rk}(F_2) = \text{rk}(xy^2) + \text{rk}(zw^2)$, in agreement with the known results [3, 2].

5 Reproducibility and attribution

The accompanying standard-library Python checker differentiates sparse polynomials and performs rational row reduction to compute the inverse module and the image of q independently of (4). For $m = 1, 2, 3, 4$, it checks respectively 9, 114, 150, 356 distinct subspaces, with two admissible choices of q for each. Including the sharpness and scope controls gives 1264 exact quotient-length calculations. Normal and optimized Python executions agree. These finite checks are diagnostics; Proposition 3.2 proves the assertion for every subspace over $\mathbb{k}$.

The definition and apolar rank-bound framework are from [1, 2], and the rank value is from the established coprime-monomial theory [3]. A targeted literature search did not identify this exact maximal-length formula, but no absolute priority claim is made. OpenAI Codex assisted with derivation, source review, exact checks and manuscript preparation. The result has undergone detailed self-audit; it is an unrefereed preprint, not an independently reviewed or proof-assistant-verified theorem.

References

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