

Component Symmetry and Tightness in Dimension Two

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19 October 2026

Abstract

For distinct vertices u, v of a simplicial complex X , let $c_X(u, v)$ count the components of $\text{lk}_X(v) - u$ that contain a neighbour of u in $\text{lk}_X(v)$. We prove that a finite nonempty simplicial complex of dimension at most two is tight over a field of characteristic two if and only if its one-skeleton is complete and these counts are symmetric in every induced subcomplex. The proof adapts the cycle-link argument of Bagchi, Datta and Spreer and fills induced link cycles by the mod-two fundamental chains of the resulting surfaces. Consequently, tightness over any field implies tightness over $\mathbb{F}_2$ in dimension two. A four-vertex disk shows why the neighbourliness assumption cannot be omitted from the component-symmetry question recorded in the 2015 Oberwolfach report.

1 Statement and scope

All complexes are finite and abstract. For $W \subseteq V(X)$, let $X[W]$ be the induced subcomplex, and write $X - v = X[V(X) \setminus \{v\}]$ and $L_v(X) = \text{lk}_X(v)$. A nonempty complex is *neighbourly* if every pair of distinct vertices forms an edge. This is also called 2-neighbourliness. For a field F , a complex X is F -tight if it is connected and all inclusion maps

$$H_i(X[W]; F) \longrightarrow H_i(X; F)$$

are injective, for all W and all $i \geq 0$. We use ordinary simplicial homology; the empty induced subcomplex presents no obstruction.

Following Bagchi–Datta–Spreer [2, Notation 3.1], define $c_X(u, v)$, for $u \neq v$, as the number of connected components of $L_v(X) - u$ that contain a vertex adjacent to u in $L_v(X)$. The count is zero if u is not a vertex of the link. We say that X has *hereditary component symmetry* if

$$c_{X[W]}(u, v) = c_{X[W]}(v, u) \quad (u, v \in W, u \neq v) \quad (1)$$

for every $W \subseteq V(X)$.

Theorem 1. *Let X be a finite nonempty simplicial complex with $\dim X \leq 2$, and let K be any field of characteristic two. Then X is K -tight if and only if X is neighbourly and has hereditary component symmetry.*

No manifold, purity or orientability assumption is made on X . The necessity of component symmetry for tight complexes is [2, Lemma 3.3]. The key argument below is a weakening of the hypotheses in [2, Lemma 3.4], not a new surface-link method. The additional step is to use the resulting surfaces to kill every link cycle in a vertex deletion.

Bagchi’s report [1, p. 299] asks whether hereditary component symmetry alone forces a two-dimensional complex to be $\mathbb{F}_2$ -tight. As written, that statement lacks the necessary neighbourliness hypothesis; Proposition 6 gives a counterexample. Theorem 1 establishes the formulation with that hypothesis. It also establishes the field-transfer consequence discussed immediately after the report’s conjecture.

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Corollary 2. *If a finite nonempty simplicial complex of dimension at most two is tight over some field, then it is tight over $\mathbb{F}_2$, and over every field of characteristic two.*

2 The necessary identities

We give the necessity argument to specify exactly which part of [2, Lemmas 3.2–3.3] is needed.

Lemma 3. *Over any field F , an F -tight complex is neighbourly and has hereditary component symmetry.*

Proof. Two nonadjacent vertices induce a subcomplex with $H_0 \cong F^2$, which cannot inject into the $H_0 \cong F$ of a connected complex. Hence the complex is neighbourly. Every nonempty induced subcomplex is itself tight: its homology maps injectively into the original complex, and any smaller induced inclusion factors through it.

For a tight complex Y with at least two vertices, set $g(Y, v) = \beta_0(L_v(Y); F) - 1$. Both the link and the deletion are nonempty, and the deletion is connected. Mayer–Vietoris for $Y = (Y - v) \cup (v * L_v(Y))$, using tightness, gives

$$\beta_1(Y; F) = \beta_1(Y - v; F) + g(Y, v).$$

If Y has at least three vertices, delete u and v in opposite orders and subtract. This yields

$$g(Y, v) - g(Y - u, v) = g(Y, u) - g(Y - v, u). \quad (2)$$

For any graph G and a vertex u , the component containing u splits after deletion into exactly those components that contain a former neighbour of u . If their number is c , then

$$\beta_0(G - u) - \beta_0(G) = c - 1.$$

This also covers an isolated u , where $c = 0$. Applying the identity to both links in (2) proves $c_Y(u, v) = c_Y(v, u)$. Components of a higher-dimensional link are likewise determined by its graph. For two vertices, both counts are zero. Applying this to every induced Y proves the assertion. $\square$

3 A cycle link forces a surface

Lemma 4 (Adapted from [2], Lemma 3.4). *Suppose Y is neighbourly, has dimension at most two, and has hereditary component symmetry. If $L_x(Y)$ is a simple cycle, then Y is a connected closed triangulated surface.*

Proof. Put $C = L_x(Y)$. Neighbourliness ensures $V(C) = V(Y) \setminus \{x\}$. Fix $y \in V(C)$, and let a, b be its two neighbours in C . In $L_y(Y)$, the neighbours of x are precisely a, b . Since $C - y$ is a path, symmetry gives

$$c_Y(x, y) = c_Y(y, x) = 1.$$

Thus $L_y(Y) - x$ contains an a -to- b path; choose a shortest such path P .

For any $z \notin \{x, y, a, b\}$, removing y, z from C separates a and b . Applying symmetry in $Y - z$ gives

$$c_{Y-z}(x, y) = c_{Y-z}(y, x) = 2.$$

Consequently every a -to- b path in $L_y(Y) - x$ passes through z . In particular, P contains every vertex of $L_y(Y) - x$. A shortest path is induced, so that graph is exactly P . Reattaching x , adjacent precisely to its endpoints, makes $L_y(Y)$ a cycle. If C has length three, there is no z to consider and P is simply the edge ab .

All vertex links are therefore cycles. This is the local combinatorial criterion for a closed triangulated surface, and neighbourliness gives connectedness. $\square$

Only neighbourliness and the symmetry identities for Y and its single-vertex deletions enter this proof. In [2] these facts come from tightness. Here they are the hypotheses, so invoking the lemma does not assume the conclusion of Theorem 1.

4 Filling cycles and proving sufficiency

Lemma 5. *Let X satisfy the two combinatorial hypotheses of Theorem 1, and let $v \in V(X)$. Then*

$$H_1(L_v(X); \mathbb{F}_2) \longrightarrow H_1(X - v; \mathbb{F}_2)$$

is the zero map.

Proof. The link is a graph. Its cycle space is generated by simple induced cycles: an even edge set decomposes into simple cycles, and a chord splits any non-induced simple cycle into two shorter cycles whose sum is the original cycle over $\mathbb{F}_2$.

Take such an induced cycle C , on vertex set S , and set $Y = X[S \cup \{v\}]$. Then $L_v(Y) = C$, and Lemma 4 makes Y a closed surface. Let Z be the sum of all its triangles over $\mathbb{F}_2$. Each edge occurs twice, so $\partial Z = 0$. Let D be the sum of its triangles containing v ; this is the cone chain over C , with $\partial D = C$. The chain $Z + D$ is supported entirely in $Y - v$, and

$$\partial(Z + D) = C.$$

Thus every induced link cycle bounds in $X - v$, proving the lemma. $\square$

Proof of Theorem 1 and Corollary 2. Necessity is Lemma 3. For sufficiency, first work over $\mathbb{F}_2$. Mayer–Vietoris for $X = (X - v) \cup (v * L_v(X))$ identifies the kernel of

$$H_1(X - v; \mathbb{F}_2) \longrightarrow H_1(X; \mathbb{F}_2)$$

with the image of the map in Lemma 5. The kernel is therefore zero. The degree-zero inclusion is injective whenever the deletion is nonempty, because a neighbourly complex is connected. In degree two, an included cycle cannot become a boundary, since X has no three-simplices. Higher homology vanishes. The cases of one or two vertices are immediate.

Every induced subcomplex satisfies the same combinatorial hypotheses. Given nonempty $W \subseteq V(X)$, add the vertices outside W one at a time. Each inclusion is a vertex-deletion inclusion of the kind just proved injective. Their composition is injective in every degree. This proves $\mathbb{F}_2$ -tightness. All chain and exact-sequence arguments extend by scalars to any characteristic-two field K .

Finally, tightness over an arbitrary field supplies neighbourliness and hereditary symmetry by Lemma 3. The theorem then gives Corollary 2. $\square$

5 Why the missing hypothesis matters

Proposition 6. *For $k \geq 2$, let B_k have vertices $\{a, b, z_1, \dots, z_k\}$ and maximal faces abz_i . Then B_k is connected, pure and contractible, has hereditary component symmetry, and is not tight over any field. For $k = 2$ it is a four-vertex triangulated disk.*

Proof. The k triangles meet along their common edge ab and collapse onto it. In an induced subcomplex containing a, b and exactly r of the other vertices, the counts are

$$c(a, b) = c(b, a) = r, \quad c(a, z_i) = c(z_i, a) = c(b, z_i) = c(z_i, b) = 1.$$

Distinct z_i, z_j give zero in both directions. If a or b is absent there are no triangles, and all component counts are zero. This proves hereditary symmetry. But z_1, z_2 induce two isolated points, whose H_0 cannot inject into that of the connected B_k . $\square$

The $k = 2$ example has the fewest possible vertices for a two-dimensional counterexample: on three vertices, a two-dimensional complex is the full triangle. More generally, adding an edge without adding triangles leaves every component count unchanged. It adds only isolated link vertices, none adjacent in the link to the distinguished vertex. The same holds in every induced subcomplex. Thus component symmetry alone cannot enforce neighbourliness.

The characteristic restriction in Theorem 1 is substantive. The familiar six-vertex triangulation of $\mathbb{RP}^2$ is tight over $\mathbb{F}_2$ but not over a field of odd characteristic; compare the surface criterion discussed in [2, Section 1]. Corollary 2 is a one-way implication, not field independence.

6 Reproducible finite checks

The proof above holds for every finite number of vertices and does not rely on enumeration. As supplementary checks, a C++ program enumerated every triangle subset on a fixed complete one-skeleton through six labelled vertices. A separate Python implementation used breadth-first link components and dense finite-field row reduction through five vertices. It computed images of explicit graph cycle bases in homology, independently of the C++ vertex-deletion rank formula.

Vertices	Triangle subsets	Hereditarily symmetric	Symmetric, not tight
1	1	1	0
2	1	1	0
3	2	2	0
4	16	12	0
5	1,024	299	0
6	1,048,576	37,692	0

The final column counts only hereditarily symmetric complexes. The enumeration includes complexes of dimension less than two. The Python audit also checks the explicit filling chains, the examples B_k for $2 \leq k \leq 8$, invariance under adding unfilled edges, and the coefficient-field contrast on the six-vertex surface. Normal and optimized Python modes agree. These finite tests are not a formal proof certificate.

This unrefereed note was developed with AI assistance in the mathematical derivation, literature comparison, coding and exposition. The arguments were self-audited; no independent review or absolute literature-priority certification is claimed.

References

- [1] B. Bagchi, Tight 3-manifolds, in *Geometric and Algebraic Combinatorics*, Oberwolfach Reports **12** (2015), no. 1, 297–299. Report DOI: 10.4171/OWR/2015/5.
- [2] B. Bagchi, B. Datta and J. Spreer, Tight triangulations of closed 3-manifolds, *European Journal of Combinatorics* **54** (2016), 103–120. doi:10.1016/j.ejc.2015.12.006. Preprint arXiv:1412.0412.