

Computing the Odd-Ear Prefix Parameter Is NP-Complete

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23 October 2026

Abstract

For a 2-vertex-connected graph G , let $\beta(G)$ be the maximum number of initial odd ears in an open ear decomposition, counting its initial cycle. We prove that deciding $\beta(G) \geq k$, with k part of the input, is NP-complete. This gives a complexity classification for a question of Benchetrit and Sebő. Hardness holds even for simple, 2-connected, 2-degenerate graphs whose deletion of one vertex is bipartite, and whose girth is at least any prescribed fixed bound. The proof identifies β on a cone over a bipartite multigraph with the largest induced edge count on a connected vertex set admitting a perfect matching. Two polynomially bounded edge multiplicities adapt a known connected-matching satisfiability gadget to this different objective. Odd subdivision then removes parallel edges without changing β .

1 The question and the result

All graphs are finite and undirected. Intermediate graphs may have parallel edges but no loops; edges are given explicitly. Connectivity means vertex connectivity. A 2-connected graph has at least three vertices and remains connected after the deletion of any one vertex.

An open ear decomposition consists of an initial cycle followed by paths whose distinct endpoints lie in the preceding graph and whose internal vertices are new. The ears partition the edge set; a single unused edge is an allowed ear. An ear is odd when its number of edges is odd. We count the initial cycle as one ear. For a 2-connected graph G , the parameter $\beta(G)$ is the largest number of consecutive odd ears at the beginning of such a decomposition.

Sebő's report [1, p. 2914, Question 4] asks for the complexity of computing this parameter. Benchetrit and Sebő [2, Question 3.3] ask whether it can be computed in polynomial time. Their work gives polynomial-time recognition of $\beta \leq 1$ and fixed-parameter tractability in β . These results do not settle the case of an input-dependent threshold.

Theorem 1. *The following decision problem is NP-complete: given a simple 2-connected graph G and an integer k , decide whether $\beta(G) \geq k$. For every fixed integer $g \geq 3$, it remains NP-complete when G is required to be 2-degenerate, to have girth at least g , and to have a vertex a such that $G - a$ is bipartite.*

Here 2-degenerate means that every nonempty subgraph has a vertex of degree at most two. The graph $G - a$ need not be connected. In particular, computing β exactly is NP-hard. A polynomial-time algorithm would imply $P = NP$; we do not assert an unconditional separation of these classes. The result concerns the *initial* odd ears, not the maximum total number of odd ears in a decomposition.

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2 Factor-critical subgraphs and cones

A graph H is factor-critical if $H - v$ has a perfect matching for every vertex v . Write

$$\rho(H) = |E(H)| - |V(H)| + 1$$

for the cycle rank of a connected graph, counting edge multiplicities. We use Lovász's characterization that a 2-connected graph is factor-critical if and only if it has an odd open ear decomposition; see [2, Theorem 3.2].

Lemma 2. *For a 2-connected graph G ,*

$$\beta(G) = \max(\{0\} \cup \{\rho(G[U]) : G[U] \text{ is 2-connected and factor-critical}\}).$$

Proof. Every ear increases the cycle rank by one. The graph formed by an odd initial sequence is 2-connected and factor-critical, so its number of ears is its cycle rank. Conversely, an odd ear decomposition of a 2-connected factor-critical subgraph extends to an ear decomposition of G . To see the extension, a component outside the current vertex set has at least two distinct neighbors in that set, since G has no cut vertex. A path through the component between two such attachment vertices supplies a new ear. After all vertices have been included, add missing edges as single-edge ears. This proves the equivalent subgraph formulation in [2, Section 3.1].

Adding edges on an unchanged vertex set preserves 2-connectivity and factor-criticality and does not decrease cycle rank. Thus a maximizing subgraph can be replaced by the induced graph on its vertices. $\square$

For a loopless graph B , let $C(B)$ be its cone: add a new vertex a and exactly one edge from a to each vertex of B . If B is connected and has at least two vertices, then $C(B)$ is 2-connected.

Lemma 3. *Let B be a connected bipartite multigraph with at least two vertices. Then*

$$\beta(C(B)) = \max\{|E(B[S])| : |S| \geq 2, B[S] \text{ is connected and has a perfect matching}\}. \quad (1)$$

Proof. A bipartite graph of order at least three cannot be factor-critical: deleting a vertex from each of its two nonempty parts would require two incompatible equalities between the part sizes. Consequently every induced 2-connected factor-critical subgraph of $C(B)$ contains a .

For $S \subseteq V(B)$, the cone on $B[S]$ is 2-connected if and only if $B[S]$ is connected and $|S| \geq 2$. One direction follows by deleting a ; the other follows because a joins all remaining vertices after deletion of a vertex of S .

The cone on $B[S]$ is factor-critical if and only if $B[S]$ has a perfect matching M . Necessity follows by deleting a . For sufficiency, deletion of a leaves M ; after deleting $x \in S$, replace the edge of M incident with x by the edge joining a to its other endpoint. Finally,

$$\rho(C(B)[S \cup \{a\}]) = (|E(B[S])| + |S|) - (|S| + 1) + 1 = |E(B[S])|.$$

Apply Lemma 2. $\square$

The objective in (1) counts *all induced edges*, not merely the edges of a chosen perfect matching. This distinction is essential in the reduction below.

3 A satisfiability reduction with two multiplicities

We reduce from 3-SAT. A formula has variables $x_1, \dots, x_n$, with $n \geq 1$, and nonempty clauses $D_1, \dots, D_m$, each containing at most three literals. Repeated literals within a clause are removed. This version of 3-SAT is NP-complete; instances with an empty clause can be handled separately. Let L be the total number of literal occurrences after removal of repetitions.

The underlying graph is the bipartite variable/clause construction used by Gomes, Masquieu, Pinto, dos Santos and Szwarcfiter for weighted connected matchings [3, Section 3.1]. We use the same support but a different objective and two levels of multiplicities. Define

$$Z = n + L, \quad A = Z + 1, \quad B = (n + 1)A, \quad T = (m + 1)B + nA. \quad (2)$$

Construct a multigraph F as follows.

- Introduce h^+, h^- and B parallel edges between them.
- For each i , introduce x_i, x_i^+, x_i^- . Put A parallel edges between x_i and each of x_i^+, x_i^- , and one edge $h^+ x_i$.
- For each j , introduce c_j^+, c_j^- and B parallel edges between them. Join c_j^+ by one edge to x_i^+ when $x_i \in D_j$, and to x_i^- when $\neg x_i \in D_j$.

The single edges in the second and third items are called connectors; there are exactly Z of them. A bipartition is

$$\{h^+\} \cup \{x_i^+, x_i^- : 1 \leq i \leq n\} \cup \{c_j^- : 1 \leq j \leq m\}$$

and its complement. The graph is connected: the variable gadgets attach to h^+ , and each clause has at least one literal.

Lemma 4. *The formula is satisfiable if and only if there is $S \subseteq V(F)$ such that $F[S]$ is connected, has a perfect matching, and satisfies $|E(F[S])| \geq T$.*

Proof. For a satisfying assignment, select h^+, h^- , all c_j^+, c_j^- , and, for each variable, x_i together with the literal vertex representing its truth value. The hub pair, all clause pairs and the selected variable pairs form a perfect matching. The hub and variable gadgets induce a connected graph, and each clause pair attaches to it through a true literal. The selected bundles already contribute $(m + 1)B + nA = T$ edges.

Conversely, suppose $F[S]$ has the stated properties. There are $m + 1$ bundles of multiplicity B . If either endpoint of any such bundle is absent, then even including every variable bundle and every connector gives at most

$$\begin{aligned} |E(F[S])| &\leq mB + 2nA + Z \\ &= T - B + nA + Z = T - 1, \end{aligned}$$

by (2). Therefore S contains h^+, h^- and every c_j^+, c_j^- .

In every perfect matching of $F[S]$, the vertex h^- must match h^+ , and c_j^- must match c_j^+ : these vertices have only one distinct neighbor. No connector incident with h^+ or a c_j^+ can therefore belong to that matching. Hence each variable gadget contributes either no selected vertices, or exactly x_i and one of x_i^+, x_i^- . Indeed, a selected literal vertex must match x_i , and x_i cannot match two literal vertices.

If any variable gadget is absent, the induced edge count is at most

$$(m+1)B + (n-1)A + Z = T - 1.$$

Thus each variable selects exactly one literal vertex. Use these choices as a truth assignment. The hub and chosen variable vertices form a connected core. If a clause has no selected literal neighbor, its two vertices form a separate component of $F[S]$. Connectivity excludes this, so every clause is satisfied. $\square$

Combining Lemmas 3 and 4 gives

$$\text{formula satisfiable} \iff \beta(C(F)) \geq T. \quad (3)$$

This is an ordinary explicit-graph reduction, not a succinct encoding of large weights. The graph F has $2 + 3n + 2m$ vertices and

$$(m+1)B + 2nA + Z$$

edges, with multiplicities counted. Since $L \leq 3m$, this size and the threshold are polynomial in $n + m$. The cone adds only one vertex and $2 + 3n + 2m$ edges.

4 Odd subdivisions and sparse simple graphs

Lemma 5. *If a loopless multigraph H' is obtained from a loopless multigraph H by replacing its edges with internally disjoint paths of positive odd lengths, then $\beta(H') = \beta(H)$, where for non-2-connected graphs β is defined by the maximum cycle rank of a 2-connected factor-critical subgraph, or zero if none exists.*

Proof. Inserting two vertices into an edge of a factor-critical graph preserves factor-criticality. To verify this, replace uv by $upqv$. After deleting an old vertex, a near-perfect matching extends by pq if it did not use uv , and by up, qv if it did. After deleting p , take a perfect matching of $H - v$ and add qv ; after deleting q , take one of $H - u$ and add up . Repetition treats every positive odd length. Subdivision also preserves 2-connectivity and cycle rank. This proves $\beta(H') \geq \beta(H)$.

For the reverse inequality, let Q' be a 2-connected factor-critical subgraph of H' . Every vertex of Q' has degree at least two. An internal vertex of a replacement path has degree two in the ambient graph, so including it forces both incident edges. Propagation along the path forces the entire path. Thus Q' is a subdivision of a subgraph Q of H .

There are at least three old vertices in Q . With no old vertex, the subgraph would lie inside a path. One old vertex cannot support a cycle because H has no loops. With exactly two old vertices, Q' would be a union of odd paths between them; its order would be even, contradicting factor-criticality.

Connectivity descends under suppression. If Q had a cut vertex, deleting it would also disconnect Q' , so Q is 2-connected. Fix an old vertex v . A perfect matching of $Q' - v$ projects to a perfect matching of $Q - v$. Indeed, on a replacement path not incident with v , its even number of internal vertices forces either both old endpoints or neither to be matched into the path. In the first case project to the corresponding edge of Q ; in the second case project to no edge. On a path incident with v , the remaining internal vertices pair consecutively, so its other old endpoint is matched elsewhere. Each old vertex other than v is consequently covered once. Thus Q is factor-critical. Finally, $\rho(Q') = \rho(Q)$, giving the reverse inequality. $\square$

Proof of Theorem 1. For NP membership, thresholds $k \leq 0$ are immediate yes-instances. For $k > 0$, a certificate consists of a vertex set U , a perfect matching of $G[U] - v$ for each $v \in U$, and the assertion $\rho(G[U]) \geq k$. Matching validity, cycle rank and 2-connectivity can all be checked in polynomial time. Lemma 2 proves the certificate criterion.

For hardness, form the cone $C(F)$ in (3). Choose a fixed odd integer $\ell \geq 3$ with $2\ell \geq g$, and replace *every copy of every edge* by a distinct path of length ℓ . The resulting graph G is simple and 2-connected. Its size is polynomial because ℓ is fixed. Lemma 5 preserves the threshold equivalence.

Every cycle of G traverses at least two entire replacement paths, so its length is at least $2\ell \geq g$. Every subgraph containing a new internal vertex has a vertex of degree at most two; a subgraph containing only old vertices has no edges. Hence G is 2-degenerate. Delete the old cone vertex a . What remains is an odd subdivision of the bipartite graph F , with pendant paths attached to its vertices from the old cone edges. This graph is bipartite. All asserted restrictions hold simultaneously. $\square$

5 Diagnostics and scope

The reduction is a written proof for arbitrary input formulas. Exact finite computations provide additional checks of its ingredients. A dependency-free Python checker accompanies this preprint. It checks the cone identity for all 512 labelled bipartite supports on a fixed $3 + 3$ partition with nonuniform multiplicities, 97 small CNF instances by exhaustive vertex-set optimization, and 154 formula instances by a generic exact subset branch-and-bound procedure. The latter include 25 unsatisfiable instances and the formula containing all eight three-literal clauses on three variables. It also checks 200 odd-subdivision identities, including multigraphs.

As a negative control, the unweighted support for $(x) \wedge (\neg x)$ has cone parameter 7, although the corresponding matching-weight threshold is only 4. Thus substituting induced edge count into the existing weighted-connected-matching reduction without the multiplicity hierarchy gives an invalid proof. The hierarchy in (2) removes precisely this problem. The checker gives identical results with and without Python optimization; it is not a formal verification of the theorem.

The graph support is credited to [3]; the cone identity, hierarchical induced-edge forcing and subdivision transfer supply the present argument. The complexity classification is for the exact input-dependent threshold question. It does not contradict fixed-parameter tractability, and it makes no claim about optimal approximation ratios or bounded maximum degree.

The author used OpenAI's Codex for literature searches, proof development, computational diagnostics and manuscript preparation. This is a self-audited, unrefereed preprint; no independent or formal verification, or absolute priority claim, is made.

References

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