

Border Observations and the Number of Rank-Two Matrix Completions

Alper Ferudun*

24 October 2026

Abstract

Let H be a connected bipartite graph specifying interior observations of a generic rank-two matrix. Suppose every entry in a border row and a border column is also observed, except their intersection. We determine the number of completions from the cycle structure of H . A tree gives a one-dimensional generic fiber; a unicyclic graph whose cycle has length $2k$ gives exactly $k - 1$ complex completions; and a graph with at least two independent cycles gives a unique completion. For a cycle with the border intersection restored, we give an irreducible circuit polynomial of degree $2k - 1$, of degree $k - 1$ in that intersection and degree one in each other variable. The proof uses a Schur complement and a greatest common divisor calculation for two cycle polynomials. The rank-one ingredients and small circuit examples are known and credited. This restricted theorem does not classify all rank-two circuits or settle random-regular matrix completion.

1 The observation pattern and its completion count

Throughout, $\mathbb{K}$ is an algebraically closed field of characteristic zero. A generic assertion means an assertion on a nonempty Zariski open subset of the rank-at-most-two determinantal variety. Finite counts refer to distinct complex completions when $\mathbb{K} = \mathbb{C}$, not to all real data.

Let $H = ([m], [n], E)$ be a connected simple bipartite graph, where $m, n \geq 1$. Index the matrix rows by $0, 1, \dots, m$ and its columns by $0, 1, \dots, n$. We observe the entries in

$$\Omega_H = \{(i, 0) : 1 \leq i \leq m\} \cup \{(0, j) : 1 \leq j \leq n\} \cup E.$$

In particular, the anchor x_{00} is missing. Write

$$\beta(H) = |E| - m - n + 1$$

for the cycle rank. A completion always has rank at most two and must agree with all the observations.

Theorem 1. *For generic observations from a rank-two matrix, the completion fiber of Ω_H has the following description.*

1. If $\beta(H) = 0$, its dimension is one.
2. If $\beta(H) = 1$ and the unique cycle has length $2k$, it consists of exactly $k - 1$ distinct matrices.

*Mercury Software GmbH. Email: alper@mercurycodelab.com.

3. If $\beta(H) \geq 2$, it consists of exactly one matrix.

Thus generic completion is finite precisely when H contains a cycle. It is unique precisely when H has at least two independent cycles or is unicyclic with a four-cycle.

This problem is a restricted instance of the circuit and completion questions in Király's Oberwolfach contribution [1]. The general algebraic-matroid framework and rank-one cycle equations are developed in [2]. Bernstein characterizes rank-two independence in [4]; the later problem list [5] distinguishes this from circuit-polynomial and completion-degree questions. The theorem here does not resolve their unrestricted versions.

2 An explicit circuit family

Take $m = n = k$, and let the interior graph be the cycle with edges (i, i) and $(i, i + 1)$, using cyclic indices in $[k]$. Restore the anchor to the observed set. Put

$$a = x_{00}, \quad u_i = x_{i0}, \quad v_i = x_{0i}, \quad p_i = x_{ii}, \quad q_i = x_{i,i+1}.$$

Theorem 2. *For every $k \geq 2$, these $4k + 1$ observed positions form a circuit of the rank-two determinantal matroid. Its circuit polynomial, up to a nonzero scalar, is*

$$F_k = \frac{1}{a} \left[\prod_{i=1}^k (ap_i - u_i v_i) - \prod_{i=1}^k (aq_i - u_i v_{i+1}) \right]. \quad (1)$$

The polynomial is irreducible, has total degree $2k - 1$, has degree $k - 1$ in a , and has degree one in every other variable. It has exactly $2^{k+1} - 2$ monomials, each with coefficient 1 or -1 . Deleting the anchor gives $k - 1$ generic full completions; deleting any other circuit entry gives a unique generic full completion.

Proof. On the open set $a \neq 0$, elementary row operations give

$$\text{rank} \begin{pmatrix} a & v^\top \\ u & X \end{pmatrix} = 1 + \text{rank}(X - uv^\top/a). \quad (2)$$

Consequently, writing $z_{ij} = ax_{ij} - u_i v_j$, rank at most two means rank at most one for $Z = (z_{ij})$. Over the ring containing a, a^{-1}, u, v , this is an invertible coordinate change.

For a rank-one matrix, the entries of a cycle satisfy the difference of the two alternating matching products. There is no other independent relation: the factors on a spanning path can be recovered successively from nonzero edge values, up to a reciprocal rescaling of the row and column factors. The last edge imposes the matching-product equation. The binomial is irreducible, since its monomials are coprime and their exponent-difference vector is primitive. This is the classical rank-one cycle relation; see also [2, Theorem 34].

The numerator in (1) has zero constant coefficient in a . Precisely its two terms containing no p_i or q_i cancel. Every remaining term has a distinct support among the interior variables, so no other cancellation occurs. Some surviving terms have a -exponent one. Thus F_k is a polynomial and a does not divide it.

In the localization at a , the coordinate change identifies F_k with a unit times the irreducible rank-one cycle binomial. Any factorization before localization would therefore have a factor that becomes a unit upon inverting a , hence a scalar times a power of a . The preceding paragraph excludes that factor. Thus F_k is irreducible.

The rank-two observed image has localized ideal (F_k) . The subset of the determinantal variety with $a \neq 0$ is dense. Moreover, (F_k) is saturated with respect to a , since it is prime and does not contain a . Hence the full elimination ideal of the observed set is (F_k) .

The expansion gives its total degree, anchor degree, and monomial count. The leading matching terms contain every p_i, q_i . Terms containing $k-1$ interior variables show that every u_i, v_i also occurs. Their degrees are all one. A nonzero multiple of F_k cannot omit any of its variables: degree in one variable is additive under multiplication in an integral domain. Thus no proper subset of the observed positions has a nonzero polynomial relation. This proves the circuit assertion.

When $a \neq 0$ and the observed residuals are nonzero, the cycle determines the complete rank-one residual matrix uniquely. The projection from the rank-two variety to the circuit hypersurface is therefore birational. Forgetting one coordinate of that hypersurface has generic degree equal to its degree in the irreducible polynomial F_k , by Gauss's lemma. These degrees are $k-1$ for a and one otherwise. Characteristic zero ensures separability. Restricting to a suitable nonempty open subset excludes exceptional residuals and points missing from the projection image, so these degrees count actual distinct full completions. $\square$

For $k=2$, F_k is the ordinary 3-by-3 determinant. The $k=3$ and $k=4$ patterns, with degrees five and seven, occur in the circuit tables of Király–Rosen–Theran [3, Example 3.3 and Table 2]. These small examples are not new. The formula above treats every k and will be used for the unicyclic case of Theorem 1.

3 Two cycle equations have only the true common root

Let s_i and t_j be algebraically independent row and column parameters. For a simple bipartite cycle C , denote its two alternating matchings by $M_0(C)$ and $M_1(C)$, and define

$$P_C(z) = \prod_{ij \in M_0(C)} (z + s_i t_j) - \prod_{ij \in M_1(C)} (z + s_i t_j). \quad (3)$$

All polynomial greatest common divisors in this section are over the rational function field in the vertex parameters.

Lemma 3. *If C has length $2k$, then P_C has degree $k-1$, and zero is a simple root. For any row p and column q , including vertices outside C , the polynomial $P_C(-s_p t_q)$ is not identically zero.*

Proof. Both matching products have the same constant term and the same leading term z^k . The coefficient of z^{k-1} is

$$\sum_{ij \in M_0(C)} s_i t_j - \sum_{ij \in M_1(C)} s_i t_j,$$

which is nonzero. The coefficient of z is the product of all row and column parameters on C , multiplied by

$$\sum_{ij \in M_0(C)} (s_i t_j)^{-1} - \sum_{ij \in M_1(C)} (s_i t_j)^{-1}.$$

The distinct Laurent monomials show that this too is nonzero.

If p or q lies outside C , the product $s_p t_q$ contains a transcendental variable absent from P_C , so substitution cannot make this nonzero polynomial vanish identically. If pq is an edge of C , exactly one matching product vanishes at $-s_p t_q$, and the other does not.

Suppose instead that p, q lie on C but pq is not an edge. Let $pj \in M_0(C)$ and $p\ell \in M_1(C)$. The vertices j, ℓ, q are distinct. After the substitution $z = -s_p t_q$, the first matching product has a

factor $t_j - t_q$. The second does not. Indeed, on setting $t_j = t_q$, its factor at row p is $s_p(t_\ell - t_q) \neq 0$; all other factors involve a row parameter different from s_p , so none vanishes identically. Hence the two evaluated products are different. $\square$

Lemma 4. *If cycles C, D have a vertex in D that is not in C , then $\gcd(P_C, P_D) = z$, up to multiplication by a nonzero scalar.*

Proof. Transpose if necessary so that an outside vertex is a row r , with neighbors j, ℓ in D . Let K be the rational function field in all vertex parameters except s_r . Then $P_C \in K[z]$, and

$$P_D(z) = (z + s_r t_j)A(z) - (z + s_r t_\ell)B(z),$$

where $A, B \in K[z]$ are products of the other matching factors. Irreducibility over K is preserved under the purely transcendental extension $K(s_r)$. A common irreducible factor $f \neq z$ of P_C and P_D must therefore divide both coefficients of P_D , regarded as a polynomial in s_r :

$$z(A - B), \quad t_j A - t_\ell B.$$

As $f \neq z$ and $t_j - t_\ell \neq 0$ is a unit in K , this implies that f divides both A and B . In particular, f is one of the linear factors $z + s_p t_q$ of A , contradicting Lemma 3. Both polynomials have a simple root at zero, which proves the result. $\square$

Lemma 5. *Every simple graph with at least two distinct cycles contains cycles C, D such that D has a vertex outside C .*

Proof. Choose a shortest cycle C . It has no chord, so the induced subgraph on its vertices is exactly the cycle. Any distinct cycle D must therefore use a vertex outside C . $\square$

The conclusion does not require disjoint cycles. They may share an edge, a path, or several vertices.

4 Proof of the completion classification

Proof of Theorem 1. First consider $\beta(H) \geq 2$. Invertible row and column scalings normalize the generic underlying matrix so that its true anchor and all its border entries are one. The interior then has the form

$$x_{ij} = 1 + s_i t_j.$$

The independent vertex-parameter model dominates this normalized rank-two variety; its reciprocal rescaling redundancy causes no difficulty for the polynomial arguments.

For a candidate completion with anchor $a \neq 0$, the observed Schur residuals are

$$a(1 + s_i t_j) - 1 = a(z + s_i t_j), \quad z = 1 - \frac{1}{a}.$$

Every cycle of H imposes $P_C(z) = 0$. Lemmas 5 and 4 force $z = 0$, hence $a = 1$. Since H is connected and the true residuals $s_i t_j$ are nonzero, propagation along a spanning tree determines the entire rank-one residual matrix. The completed matrix is unique.

It remains to exclude an anchor-zero completion. Such a matrix has column zero nonzero, top-left entry zero, and all other entries in its border row equal to one. The difference of two interior columns has top coordinate zero. In a column space of dimension at most two, that difference is a scalar multiple of column zero. The interior must therefore have the additive form $\alpha_i + \gamma_j$.

Its alternating sum around every cycle vanishes. For the actual observed interior $1 + s_{it_j}$, that alternating sum is a nonzero polynomial in independent parameters. Thus anchor zero is excluded for generic observations.

If H is a tree, choose arbitrary generic border and tree-edge observations. For every a outside finitely many values, the residuals on the tree are nonzero. They specify a unique full rank-one residual matrix by propagation. The anchor is consequently a free parameter. There are $2(m+n)-1$ observations, and their projection is dominant by this construction. The rank-at-most-two variety has dimension $2(m+n)$. The generic fiber has dimension one.

Finally, suppose H is unicyclic with a cycle of length $2k$. Theorem 2 supplies exactly $k-1$ generic anchor values on that cycle and exactly one completion of its interior per anchor. All those anchors are nonzero and their cycle residuals are nonzero on a suitable open set. Each attached tree edge gives an additional independent observation and hence has nonzero residual at every one of these finitely many anchors generically. Tree propagation extends each cycle completion uniquely, without further constraints on the anchor. The number of full completions is therefore $k-1$. $\square$

Corollary 6. *For every positive integer d , a border-observation pattern for a generic rank-two matrix has exactly d complex completions.*

Proof. Take the interior graph to be the cycle of length $2(d+1)$. $\square$

For a concrete six-cycle, consider the observations

$$\begin{pmatrix} ? & 1 & 1 & 1 \\ 1 & 3 & 6 & ? \\ 1 & ? & 11 & 23 \\ 1 & 7 & ? & 34 \end{pmatrix}.$$

The anchor equation is $12(a-1)(13a-1) = 0$. Its two rank-two completions are

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 12 \\ 1 & 5 & 11 & 23 \\ 1 & 7 & 16 & 34 \end{pmatrix}, \quad \begin{pmatrix} 1/13 & 1 & 1 & 1 \\ 1 & 3 & 6 & 48 \\ 1 & 71/7 & 11 & 23 \\ 1 & 7 & 44/5 & 34 \end{pmatrix}.$$

Both anchors have nonzero residuals on the cycle, so propagation proves that these are all the completions. This illustrates the known four-by-four circuit case of the general formula.

5 Scope, attribution, and verification

The Schur complement, rank-one cycle equations, and propagation of rank-one factors are standard. Wang [6] also uses Schur complements to study special staircase observation patterns. The bipartite cone lemma is known; see [7, Corollary 3.12]. Neither operation is presented here as a new method. Our explicit missing-anchor classification concerns a different specified family.

The original Oberwolfach questions ask for general rank-two circuit descriptions and random $2r$ -regular completability. The complete restricted theorem above leaves both broader questions open. In particular, this note must not be counted as a complete solution of the combined frozen record OWR-13346-003.

A reproducibility script checks the circuit expansion for $2 \leq k \leq 7$, including its support, degrees and monomial count. It also checks 114 single-edge deletion ranks over the prime field with $p = 1\,000\,000\,007$, and cycle-polynomial behavior for all 512 3-by-3 masks and 1,200 additional

deterministic random masks. These finite checks supplement, rather than prove, the general results. The analytical proofs above establish the universal statements.

The note was prepared with AI assistance for derivation, literature search, exposition and computation. It is an unrefereed preprint; the originating research process has checked the arguments, but no independent review or formal proof verification is claimed. The targeted literature comparison does not establish absolute novelty or priority.

References

- [1] F. J. Király, Algebraic Combinatorial Methods in Statistics and Machine Learning, in *Mini-Workshop: Asymptotic Statistics on Stratified Spaces*, Oberwolfach Reports 44/2014, 2499–2501. [doi:10.4171/owr/2014/44](https://doi.org/10.4171/owr/2014/44).
- [2] F. J. Király, L. Theran and R. Tomioka, The Algebraic Combinatorial Approach for Low-Rank Matrix Completion, *Journal of Machine Learning Research* 16 (2015), 1391–1436. <https://jmlr.org/papers/v16/kiraly15a.html>.
- [3] F. J. Király, Z. Rosen and L. Theran, Algebraic Matroids with Graph Symmetry, 2013. [arXiv:1312.3777](https://arxiv.org/abs/1312.3777).
- [4] D. I. Bernstein, Completion of Tree Metrics and Rank 2 Matrices, *Linear Algebra and its Applications* 533 (2017), 1–13. [doi:10.1016/j.laa.2017.07.016](https://doi.org/10.1016/j.laa.2017.07.016).
- [5] D. I. Bernstein, G. Blekherman and R. Sinn, Algebraic Matrix Completion Problems, 25 October 2018. <https://dibernstein.github.io/pdfs/matrixCompletionProblems.pdf>.
- [6] F. Wang, Uniqueness of Low Rank Matrix Completion and Schur Complement, 2022. [arXiv:2206.15018](https://arxiv.org/abs/2206.15018).
- [7] J. Brakensiek, M. Dhar, J. Gao, S. Gopi and M. Larson, Rigidity Matroids and Linear Algebraic Matroids with Applications to Matrix Completion and Tensor Codes, *Combinatorica* 46 (2026), article 15. [doi:10.1007/s00493-026-00208-z](https://doi.org/10.1007/s00493-026-00208-z).