

Projective Units and Witt Vanishing for Stable Module Categories

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19 October 2026

Abstract

Let G be a finite group and let k be any field of odd positive characteristic. We prove that the symmetric and skew-symmetric Cartan homomorphisms from the Witt groups of projective kG -modules to those of all finite-dimensional kG -modules are isomorphisms. The proof constructs a projective symmetric form whose module Witt class is the trivial line. An explicit middle-layer calculation supplies this form for groups with normal cyclic Sylow subgroups; Dress's classical 2-local induction theorem then gives it for arbitrary G . Tensoring by a projective lift of the unit inverts both Cartan maps. Balmer's kernel-cokernel formulas consequently imply $W^i(kG\text{-stab}) = 0$ for every integer i . No splitting-field hypothesis is required.

1 Statement and conventions

Throughout, k has odd positive characteristic p , and all modules are finite-dimensional. Write $\mathcal{M}_G = kG\text{-mod}$ and $\mathcal{P}_G = kG\text{-proj}$. Both categories carry the contragredient duality

$$M^\vee = \operatorname{Hom}_k(M, k), \quad (g\varphi)(m) = \varphi(g^{-1}m).$$

A nonsingular ε -form, for $\varepsilon \in \{+1, -1\}$, is a G -invariant k -bilinear form satisfying $b(x, y) = \varepsilon b(y, x)$. In $W^\varepsilon(\mathcal{M}_G)$ we quotient by forms admitting a module Lagrangian. In $W^\varepsilon(\mathcal{P}_G)$ the Lagrangian must be projective and split. The latter is the Witt group of the split exact category of projectives. Inclusion induces

$$c_G^\varepsilon : W^\varepsilon(\mathcal{P}_G) \longrightarrow W^\varepsilon(\mathcal{M}_G).$$

Theorem 1. *For every finite group G and every field k of odd positive characteristic, c_G^+ and c_G^- are isomorphisms.*

Balmer's report [1, pp. 1756–1757] identifies the ordinary triangular Witt groups of the stable module category as

$$\begin{aligned} W^{-1}(kG\text{-stab}) &= \ker c_G^+, & W^0(kG\text{-stab}) &= \operatorname{coker} c_G^+, \\ W^1(kG\text{-stab}) &= \ker c_G^-, & W^2(kG\text{-stab}) &= \operatorname{coker} c_G^-. \end{aligned}$$

The report states vanishing for p -nilpotent groups and asks whether it holds for arbitrary finite groups. We use its ordinary duality and the characteristic-not-two framework of triangular Witt theory [2].

Corollary 2. *Under the hypotheses of Theorem 1, $W^i(kG\text{-stab}) = 0$ for every $i \in \mathbb{Z}$.*

Indeed, the four displayed groups vanish, and the theory is four-periodic. This statement does not concern Grothendieck–Witt groups, characteristic-two variants, support-restricted categories, or dualities twisted by endotrivial modules.

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2 A projective lift of the unit

Put $R_G = W^+(\mathcal{M}_G)$. Tensor product with diagonal G -action makes R_G a commutative ring with unit the trivial line $\langle 1 \rangle$.

Lemma 3. *If P is projective and $M \in \mathcal{M}_G$, then $P \otimes_k M$ is projective. Tensor product induces pairings*

$$W^+(\mathcal{P}_G) \times W^\varepsilon(\mathcal{M}_G) \longrightarrow W^\varepsilon(\mathcal{P}_G)$$

and an R_G -module structure on $W^\varepsilon(\mathcal{P}_G)$.

Proof. For a free projective, the map

$$kG \otimes_k M \longrightarrow kG \otimes_k M, \quad g \otimes m \longmapsto g \otimes g^{-1}m$$

intertwines the diagonal action on the source and the action on the first factor alone on the target. The assertion for a projective summand follows.

Let (P, b) be a projective form, and let $L \subseteq M$ be a Lagrangian for a module form. Then $P \otimes_k L$ is a Lagrangian for the tensor form. Both this submodule and the quotient $P \otimes_k L^\vee$ are projective, so it is a split projective Lagrangian. Thus tensoring by P respects arbitrary module metabolic relations. Conversely, tensoring a split projective Lagrangian by an arbitrary module preserves projectivity and the splitting. This proves the required well-definedness. The usual associativity and symmetry isometries give the module structure. $\square$

Lemma 4. *If $q \in W^+(\mathcal{P}_G)$ satisfies $c_G^+(q) = 1$, then both Cartan homomorphisms are isomorphisms.*

Proof. Tensoring by q defines $T_q^\varepsilon : W^\varepsilon(\mathcal{M}_G) \rightarrow W^\varepsilon(\mathcal{P}_G)$. We have $c_G^\varepsilon T_q^\varepsilon = \text{id}$. For a projective ε -form x , the pairing of Lemma 3, with x fixed, also gives

$$T_q^\varepsilon(c_G^\varepsilon(x)) = T_x(c_G^+(q)) = T_x(1) = x.$$

This is the other inverse identity. In particular, injectivity does not have to be deduced from a separate dimension or parity argument. $\square$

3 The normal cyclic Sylow calculation

Proposition 5. *Suppose that H has a normal cyclic Sylow p -subgroup. Then $1 \in \text{im } c_H^+$.*

Proof. Let $N = \langle n \rangle$ be this subgroup, of order $h = p^a = 2m + 1$. If $h = 1$, the trivial line is projective by Maschke's theorem. Otherwise, Schur–Zassenhaus gives $H = N \rtimes T$, where $p \nmid |T|$. For $q \in T$, write

$$qnq^{-1} = n^{a(q)}, \quad \chi(q) = a(q) \bmod p \in \mathbb{F}_p^\times.$$

The character $\theta = \chi^m$ has square one, since $p - 1$ divides $2m = p^a - 1$. Take the projective module

$$P = \text{Ind}_T^H k_\theta.$$

It is projective because kT is semisimple and induction preserves projectives. With basis $e_i = n^i \otimes 1$, $0 \leq i < h$, its action is

$$ne_i = e_{i+1}, \quad qe_i = \theta(q)e_{a(q)i},$$

where indices are taken modulo h . Therefore

$$b(e_i, e_j) = (-1)^m \delta_{ij}$$

is a nonsingular invariant symmetric form.

Put $t = n - 1$. As an N -module, P is the regular module $k[t]/(t^h)$, and normality of N makes every $t^r P$ H -stable. The adjoint of t for b is $n^{-1} - 1 = -n^{-1}t$. Hence

$$(t^r P)^\perp = \ker(t^r : P \rightarrow P) = t^{h-r} P. \quad (1)$$

Set $U = t^{m+1} P$. It is isotropic, with $U^\perp = t^m P$, and $U^\perp/U$ is one-dimensional. It is represented by $v = t^m e_0$. The element n acts trivially on this quotient, and for $q \in T$,

$$qv = \theta(q)(n^{a(q)} - 1)^m e_0 \equiv \theta(q)\chi(q)^m v = v \pmod{t^{m+1} P}.$$

Thus the quotient is the trivial representation.

Vandermonde's identity gives

$$b(v, v) = (-1)^m \sum_{j=0}^m \binom{m}{j}^2 = (-1)^m \binom{h-1}{m} = 1 \quad \text{in } k. \quad (2)$$

For the last equality, compare coefficients of x^m in

$$(1+x)^{h-1} = \frac{1+x^h}{1+x} = \sum_{j=0}^{h-1} (-x)^j \quad \text{in } \mathbb{F}_p[x].$$

For clarity, the sublagrangian reduction here takes place in $\mathcal{M}_H$, not in $\mathcal{P}_H$. The graph

$$\{(x, \bar{x}) : x \in U^\perp\} \subseteq (P, b) \perp -(U^\perp/U, \bar{b})$$

is invariant, isotropic, and of half the ambient dimension. It is a module Lagrangian, so $[P, b] = [U^\perp/U, \bar{b}] = 1$ in R_H . No projectivity of U has been assumed. $\square$

4 Induction to arbitrary finite groups

We use a classical theorem, rather than establish a new induction principle. Dress's equivariant Witt ring $GW(G, \mathbb{Z})$ consists of unimodular symmetric forms on $\mathbb{Z}$ -free G -lattices modulo metabolic forms [3, p. 294]. In particular, his notation must not be read as the modern Grothendieck–Witt group before quotienting by metabolic forms. A group is 2-hyperelementary if it has a cyclic normal subgroup with 2-group quotient. The cyclic subgroup can be taken to have odd order, by replacing it with its characteristic odd-order part.

Dress's 2-local induction theorem [3, Theorem 2(c)] states that induction from 2-hyperelementary subgroups generates $GW(G, \mathbb{Z}) \otimes \mathbb{Z}_{(2)}$. The equivalent formulation in terms of the Dress ring is also recorded in [4, Proposition 2.14 and Example 7.5]. In particular, for some odd integer d , there are classes $\beta_H \in GW(H, \mathbb{Z})$ such that

$$d \cdot 1 = \sum_H \text{Ind}_H^G \beta_H \quad \text{in } GW(G, \mathbb{Z}), \quad (3)$$

where the sum is over 2-hyperelementary subgroups. To obtain this integral identity, first clear the odd denominators in a localized unit expression, then annihilate its possible localization-kernel error by another odd integer. Injectivity of localization is not required.

Base change sends (3) to an identity in R_G . Indeed, for an integral Lagrangian $L = L^\perp \subseteq M$, the quotient M/L is torsion-free, since L is the kernel of $M \rightarrow \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. Thus L is a $\mathbb{Z}$ -summand, and unimodularity identifies M/L with L^* . The defining exact sequence stays exact after tensoring with k . This does not assume that $\mathbb{Z} \rightarrow k$ is flat. Induced forms are orthogonal sums over cosets, so induction commutes with base change.

Proof of Theorem 1. Let $I_G = \text{im } c_G^+ \subseteq R_G$. By Lemma 3, this is an ideal. Every 2-hyperelementary subgroup H has a normal cyclic Sylow p -subgroup: it is the unique p -primary subgroup of the normal odd-order cyclic subgroup, since p is odd. Proposition 5 gives $1 \in I_H$, so $I_H = R_H$. Induction preserves projective modules and split projective Lagrangians. Therefore $\text{Ind}_H^G(I_H) \subseteq I_G$, and the base change of (3) yields $d \cdot 1 \in I_G$.

We recall why $4\langle 1 \rangle = 0$ in $W(k)$. In $\mathbb{F}_p$, the sets of squares and of -1 minus squares both have $(p+1)/2$ elements. Their intersection gives $u^2 + v^2 = -1$. Consequently $\langle 1, 1, 1, 1 \rangle$ is isotropic. Removing a hyperbolic plane leaves a nonsingular binary form of determinant -1 modulo squares, hence another hyperbolic plane. Extension to k proves the identity. Inflating these forms to trivial G -representations gives $4 \cdot 1 = 0$ in R_G . For odd d , $d^2 \equiv 1 \pmod{4}$; hence

$$1 = d(d \cdot 1) \in I_G.$$

A preimage of 1 is represented by an actual projective symmetric form: negative classes are represented by negative forms and sums by orthogonal sums. Lemma 4 now proves both Cartan isomorphisms. $\square$

Only the scalar odd integer d has been inverted in this proof. No assertion that every odd-dimensional equivariant Witt class is invertible is needed.

5 An example and exact controls

For $H = S_3 = C_3 \rtimes C_2$ in characteristic three, one has $m = 1$ and $\theta = \text{sgn}|_{C_2}$. The construction is the three-dimensional module $\text{Ind}_{C_2}^{S_3} \text{sgn}$ with the negative dot product. Here U is spanned by $e_0 + e_1 + e_2$, while the quotient $U^\perp/U$ is represented by $e_1 - e_0$. Its norm is $-2 = 1$, and its action is trivial. This illustrates the reason for both the character twist and the scalar sign in Proposition 5.

The accompanying exact-arithmetic script checks the filtration, perpendiculars, middle norm, and automorphism action for $h = 3, 9, 27, 5, 25, 7, 49, 11$ in their respective characteristics. It includes 108 automorphisms, 3,276 action checks and 25,896 perpendicular pairings. A two-dimensional nonsplit C_4 -module over $\mathbb{F}_3$ tests the tensor reduction for both signs. These are finite regression controls, not the proof of the general theorem or of Dress's induction theorem.

The argument combines the explicit projective-unit construction with classical induction and Balmer's stable-category formulas. A bounded literature search did not locate the same general conclusion; no exhaustive novelty or absolute-priority claim is made. This work was developed with AI assistance and checked by the originating research process. It is an unrefereed preprint; no independent expert review or proof-assistant certification is claimed.

References

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