

On Montgomery’s Question on Frobenius–Schur Indicators and Gauge Equivalence: A Negative Answer for Group Algebras

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Abstract

In the open questions of the Oberwolfach mini-workshop on infinite dimensional Hopf algebras (Report 20/2014), Montgomery asks whether two semisimple Hopf algebras over $\mathbb{C}$ with the same set of Frobenius–Schur indicator values have gauge equivalent, that is tensor equivalent, representation categories. She notes that the question still seems to be open for group algebras, and records R. Ng’s variant that also requires equal fusion rules. Read literally as a set of numbers, the datum cannot be what is meant, and we read it as the family of indicator sequences of the simple modules. We show that the answer is negative for group algebras, also in this strongest reading and with Ng’s condition: if two groups of odd order form a Brauer pair (they are not isomorphic but have identical character tables and power maps), then there is a bijection between the simple modules of their group algebras that preserves all higher indicators and the fusion rules, the exponents are equal, and the representation categories are not tensor equivalent. The last step is the theorem of Etingof and Gelaki that groups of odd order are determined by their tensor categories of representations, which we cite. Brauer pairs of odd order are classical (Dade; Eick and Müller; Lewis; Nenciu); for a self-contained instance we give two Camina groups of order 3^6 , of a kind described by Lewis, and prove all their properties in the text. For general semisimple Hopf algebras, without Ng’s condition, an elementary pair already answers the question in the negative (the functions on the Heisenberg group of order 27 and the group algebra of $(\mathbb{Z}/3)^3$, Remark 5.1); we did not find it in print, but it is in the spirit of an observation of Mignard and Schauenburg on pointed fusion categories, and we claim nothing new about it. The main content of the note is the group-algebra case, with or without equal fusion rules. The corpus record of the question also contains Lentner’s Question 6(c) and Lu’s Question 7, which are not settled here. No printed answer to the question was found in the sources we searched; this is not a proof of novelty, and experts may regard the argument as folklore. This is an unrefereed note.

1 Introduction

1.1 The question

Let H be a semisimple Hopf algebra over $\mathbb{C}$, let $\Lambda \in H$ be its integral with $\varepsilon(\Lambda) = 1$, and for $x \in H$ let $x^{[n]} = \sum x_{(1)}x_{(2)} \cdots x_{(n)}$ be its n -th Hopf power. The n -th Frobenius–Schur indicator of an H -module V with character χ is $\nu_n(V) = \chi(\Lambda^{[n]})$; for a group algebra it is the classical higher indicator $|G|^{-1} \sum_{g \in G} \chi(g^n)$. This is the definition recalled in Montgomery’s abstract [BGLZ14, pp. 1126–1127], where a more conceptual description is attributed to Kashina, Sommerhäuser and Zhu [KSZ06]; the abstract states that the indicators are gauge invariants, that is, invariants of the tensor category of H -modules, by Mason and Ng [MN05] and Ng and Schauenburg [NS08]. (According to its abstract,

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[NS08] defines the higher indicators of semisimple quasi-Hopf algebras through the category and obtains them by applying characters to a sequence of gauge invariant central elements.)

Question 9 of the Open Questions of the report [BGLZ14, pp. 1133–1134], due to S. Montgomery, reads as follows. Let $V_1, \dots, V_r$ be the simple H -modules, let m be the exponent of H , and put

$$S = \{\nu_j(V_i) : 1 \leq i \leq r, 1 \leq j \leq m\}.$$

If two semisimple Hopf algebras H and H' have the same set S , are their module categories gauge equivalent? The source adds that $\nu_m(V_i) = \dim V_i$, so that equal sets force $H' \cong H$ as algebras; that the question is trivially true if H is the group algebra of an abelian group; that it was raised in 2006 and “still seems to be open for group algebras”; and that R. Ng suggested also requiring that the two categories have the same fusion rules. The question is record OWR-12978-004 of the UnsolvedMath collection [ula26], listed as open; the record also contains Lentner’s Question 6(c) and Lu’s Question 7 of the same report, which we do not settle (Remark 5.4).

1.2 How the question is read

The datum S . Read literally as a set of complex numbers, S cannot be what is meant. For $H = \mathbb{C}[\mathbb{Z}/2]$ ($m = 2$) the indicator sequences (ν_1, ν_2) of the two simple modules are $(1, 1)$ and $(0, 1)$; for $H' = \mathbb{C}[\mathbb{Z}/3]$ ($m = 3$) the three sequences are $(1, 1, 1)$, $(0, 0, 1)$ and $(0, 0, 1)$. Both sets are $\{0, 1\}$, but $H \cong \mathbb{C}^2$ and $H' \cong \mathbb{C}^3$ are not isomorphic as algebras. So the literal reading contradicts the proposer’s remark that equal sets force $H' \cong H$, and it would make the abelian case false rather than trivially true. We therefore read the datum as the family of indicator sequences indexed by the simple modules, up to renumbering, and we answer the question in the strongest form that covers every reading.

Definition 1.1. Let H, H' be semisimple Hopf algebras over $\mathbb{C}$. A bijection $\sigma: \text{Irr}(H) \rightarrow \text{Irr}(H')$ between the sets of isomorphism classes of simple modules *matches the indicators* if $\nu_n(\sigma(V)) = \nu_n(V)$ for all $V \in \text{Irr}(H)$ and all $n \geq 1$. It *preserves the fusion rules* if $[\sigma(V) \otimes \sigma(W) : \sigma(U)] = [V \otimes W : U]$ for all $U, V, W \in \text{Irr}(H)$, where $[X : U]$ is the multiplicity of U in X .

Question 1.2 (strong form). *Let H, H' be semisimple Hopf algebras over $\mathbb{C}$ with $\exp H = \exp H'$, and let $\sigma: \text{Irr}(H) \rightarrow \text{Irr}(H')$ be a bijection that matches the indicators and preserves the fusion rules. Are $\text{Rep}(H)$ and $\text{Rep}(H')$ equivalent as tensor categories?*

If the hypothesis of Question 1.2 holds, then H and H' have the same literal set S (the range $1 \leq j \leq m$ is the same, since the exponents agree), the same multiset of sequences $(\nu_1(V_i), \dots, \nu_m(V_i))$, and, by the fusion condition, they satisfy Ng’s extra requirement. A negative answer to Question 1.2 is therefore a negative answer to Question 9 in each of these readings at once.

Gauge equivalence. We read “gauge equivalent” as equivalent as tensor (monoidal) categories, with no braiding or symmetry; Montgomery’s abstract phrases the question as whether the indicator values determine the category “up to tensor equivalence” [BGLZ14, p. 1127]. For finite-dimensional Hopf algebras this is, by the standard reconstruction argument, the same as H' being isomorphic to a twist of H ; we use this only for group algebras, where it is [EG01, Lemma 3.1] (Section 2.3). Tensor equivalence is the weakest relation that the word can mean, so ruling it out is the strongest conclusion.

1.3 Results

A *Brauer pair* is a pair of non-isomorphic finite groups with identical character tables and identical power maps (Definition 2.2).

Theorem 1.3. *Let G_1, G_2 be finite groups of odd order that form a Brauer pair. Then $\exp \mathbb{C}[G_1] = \exp \mathbb{C}[G_2]$, and there is a bijection $\sigma: \text{Irr}(\mathbb{C}[G_1]) \rightarrow \text{Irr}(\mathbb{C}[G_2])$ that matches the indicators and preserves the fusion rules; but $\text{Rep}(G_1)$ and $\text{Rep}(G_2)$ are not equivalent as tensor categories. So $\mathbb{C}[G_1]$ and $\mathbb{C}[G_2]$ give a negative answer to Question 1.2, and hence to Question 9 of [BGLZ14] in each reading of Section 1.2, also with Ng’s condition.*

The only step that is neither proved here nor computed is the theorem of Etingof and Gelaki that groups of odd order are *categorically rigid* [EG01, Corollary 1.4]: a group whose representation category is tensor equivalent to that of a group of odd order is isomorphic to it. Brauer pairs of odd order are classical: Dade [Dad64], Eick and Müller [EM06], Lewis [Lew09], Nenciu [Nen08, Nen09] (see Remark 4.3). For a reader who wants an instance with a complete proof we give one.

Theorem 1.4. *The groups P_1, P_2 of order 3^6 defined in Section 4 are Camina groups of nilpotency class 2 and exponent 9, they form a Brauer pair, and $\exp \mathbb{C}[P_1] = \exp \mathbb{C}[P_2] = 9$. Each has 89 irreducible characters, and the indicator sequences $(\nu_1, \dots, \nu_9)$ are, for both groups,*

$$1 \times (1, 1, 1, 1, 1, 1, 1, 1, 1), \quad 80 \times (0, 0, 1, 0, 0, 1, 0, 0, 1), \quad 8 \times (0, 0, 0, 0, 0, 0, 0, 0, 9),$$

the first for the trivial character, the second for the non-trivial linear characters and the third for the characters of degree 9; ν_n depends only on $n \bmod 9$, and the literal set is $S = \{0, 1, 9\}$. The group P_1 is not isomorphic to P_2 : the elements g with $g^3 = 1$ form a subgroup of order 81 which is abelian in P_1 and not abelian in P_2 . Consequently $\mathbb{C}[P_1]$ and $\mathbb{C}[P_2]$ satisfy the hypothesis of Question 1.2 and $\text{Rep}(P_1), \text{Rep}(P_2)$ are not tensor equivalent.

All parts of Theorem 1.4 except the last sentence are proved in Section 4; the last sentence follows from Theorem 1.3. With the Brauer t -tuples of Nenciu [Nen09], Theorem 1.3 gives for every $t \geq 2$ group algebras $\mathbb{C}[G_1], \dots, \mathbb{C}[G_t]$ of odd order with pairwise matching indicators and fusion rules whose representation categories are pairwise not tensor equivalent (Remark 5.2).

For semisimple Hopf algebras in general, without Ng’s condition, the question has an elementary negative answer, $\mathbb{C}^{\text{Heis}(3)}$ against $\mathbb{C}[(\mathbb{Z}/3)^3]$ (Remark 5.1). We did not find this pair in print; it is in the spirit of an observation of Mignard and Schauenburg on pointed fusion categories [MS17, Remark 4.5], and we claim nothing new about it. The main content of this note is the case of group algebras, with or without equal fusion rules, which the source singles out as open. Credits, the sources read and the searches are given in the paragraph “Scope and priority” in Section 6.

2 Preliminaries

2.1 Group algebras

For a finite group G we write $\text{Cl}(G)$ for its set of conjugacy classes, $\text{Irr}(G)$ for its irreducible complex characters, identified with $\text{Irr}(\mathbb{C}[G])$, and, for $C \in \text{Cl}(G)$ and $n \in \mathbb{Z}$, $C^{[n]}$ for the class that contains g^n for one, hence every, $g \in C$. Following [EG99, Definition 2.1], the exponent $\exp H$ of a Hopf algebra H is the least $n \geq 1$ with $m_n \circ (\text{id} \otimes S^{-2} \otimes \dots \otimes S^{-2n+2}) \circ \Delta_n = \varepsilon \cdot 1$, where m_n and Δ_n are the iterated product and coproduct; if $S^2 = \text{id}$, as for semisimple H over $\mathbb{C}$, the condition is $x^{[n]} = \varepsilon(x)1$ for all $x \in H$ [EG99, §4]. In particular $\Lambda^{[m]} = \varepsilon(\Lambda)1 = 1$ for $m = \exp H$, which gives the remark $\nu_m(V) = \dim V$ of the source.

Lemma 2.1. *Let G be a finite group and $H = \mathbb{C}[G]$.*

(a) $\Lambda = |G|^{-1} \sum_{g \in G} g$, $\Lambda^{[n]} = |G|^{-1} \sum_{g \in G} g^n$, and for $\chi \in \text{Irr}(G)$ and $n \geq 1$

$$\nu_n(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^n) = \frac{1}{|G|} \sum_{C \in \text{Cl}(G)} |C| \chi(C^{[n]}).$$

(b) $\exp \mathbb{C}[G] = \exp G$.

(c) For $\chi, \psi, \theta \in \text{Irr}(G)$ the fusion coefficient is $[\chi\psi : \theta] = |G|^{-1} \sum_{C \in \text{Cl}(G)} |C| \chi(C) \psi(C) \overline{\theta(C)}$.

Proof. (a) The element Λ satisfies $g\Lambda = \Lambda = \Lambda g$ and $\varepsilon(\Lambda) = 1$. The Hopf power $x \mapsto x^{[n]} = m_n \Delta_n(x)$ is linear, and $g^{[n]} = g^n$ because $\Delta_n(g) = g \otimes \cdots \otimes g$. Grouping the elements by classes gives the second formula. (b) Here $S^2 = \text{id}$. By linearity, $x^{[n]} = \varepsilon(x)1$ holds for all $x \in \mathbb{C}[G]$ if and only if it holds on the basis G , that is, if and only if $g^n = 1$ for all $g \in G$. (This is [EG99, Proposition 2.2(2)].) (c) is the usual multiplicity formula $[\chi\psi : \theta] = \langle \chi\psi, \theta \rangle$. $\square$

2.2 Brauer pairs

Definition 2.2. Finite groups G_1, G_2 form a *Brauer pair* if they are not isomorphic and there are bijections $\tau: \text{Cl}(G_1) \rightarrow \text{Cl}(G_2)$ and $\sigma: \text{Irr}(G_1) \rightarrow \text{Irr}(G_2)$ such that $\sigma(\chi)(\tau(C)) = \chi(C)$ and $\tau(C^{[n]}) = \tau(C)^{[n]}$ for all $\chi \in \text{Irr}(G_1)$, $C \in \text{Cl}(G_1)$ and $n \in \mathbb{Z}$.

This is the definition stated in the zbMATH review of [EM06] (Zbl 1124.20004), with power maps for every $n \in \mathbb{Z}$; Lewis [Lew09, p. 1] says “identical character tables and identical power maps”. The case $n = 0$ gives $\tau(\{1\}) = \tau(C^{[0]}) = \tau(C)^{[0]} = \{1\}$.

2.3 Isocategorical groups

Etingof and Gelaki call finite groups G_1, G_2 *isocategorical* if $\text{Rep}(G_1)$ is equivalent to $\text{Rep}(G_2)$ as a tensor category, without regard for the symmetric structure [EG01, Definition 1.1]. By [EG01, Lemma 3.1] this holds if and only if there is a twist J for $\mathbb{C}[G_1]$ such that $\mathbb{C}[G_1]^J \cong \mathbb{C}[G_2]$ as Hopf algebras, that is, if and only if $\mathbb{C}[G_1]$ and $\mathbb{C}[G_2]$ are gauge equivalent. A group G is *categorically rigid* if every group isocategorical to G is isomorphic to G . By [EG01, Corollary 1.4], a group that is not categorically rigid has a normal abelian subgroup A of order 2^{2m} , $m \geq 1$, with a skew-symmetric G -invariant isomorphism $R: A^\vee \rightarrow A$; the last sentence of that corollary says that “all groups of orders $2k+1$, $2(2k+1)$, and all simple groups are categorically rigid”. We use the arXiv version v1 of [EG01], whose numbering is quoted here; the journal version was not compared with it. The zbMATH review of the journal version (Zbl 0988.20003) states the same structure result. We need only the following consequence.

Corollary 2.3 ([EG01, Definition 1.1 and Corollary 1.4]). *Let G_1, G_2 be finite groups, $|G_1|$ odd. If $\text{Rep}(G_1)$ and $\text{Rep}(G_2)$ are equivalent as tensor categories, then $G_1 \cong G_2$.*

Proof. The groups are isocategorical. A group of odd order has no subgroup of order 2^{2m} with $m \geq 1$, so G_1 is categorically rigid, and $G_2 \cong G_1$. $\square$

The parity is essential for this route: by [EG01, Theorem 1.2], if Y is a vector space over $\mathbb{F}_2$ of dimension at least 3 and $V = Y \oplus Y^*$ is the corresponding symplectic space, then $\text{Sp}(V) \ltimes V$ is isocategorical to a group that is not isomorphic to it; these groups have even order.

3 Proof of Theorem 1.3

Lemma 3.1. *Let (G_1, G_2) be a Brauer pair with bijections τ, σ as in Definition 2.2. Then $|G_1| = |G_2|$, $|\tau(C)| = |C|$ for every $C \in \text{Cl}(G_1)$, $\sigma(\chi)(1) = \chi(1)$, $\exp G_1 = \exp G_2$, and $\nu_n(\sigma(\chi)) = \nu_n(\chi)$ for all $\chi \in \text{Irr}(G_1)$ and $n \geq 1$.*

Proof. Since $\tau(\{1\}) = \{1\}$, we have $\sigma(\chi)(1) = \chi(1)$, and $|G_1| = \sum_{\chi} \chi(1)^2 = \sum_{\chi} \sigma(\chi)(1)^2 = |G_2|$. By column orthogonality, $|C_G(g)| = \sum_{\chi} |\chi(g)|^2$, so the centralizer orders, and with $|G|$ the class sizes $|C| = |G|/|C_G(g)|$, are read off the table: $|\tau(C)| = |C|$. By Lemma 2.1(a),

$$\begin{aligned} \nu_n(\sigma(\chi)) &= \frac{1}{|G_2|} \sum_{C \in \text{Cl}(G_1)} |\tau(C)| \sigma(\chi)(\tau(C)^{[n]}) = \frac{1}{|G_1|} \sum_{C \in \text{Cl}(G_1)} |C| \sigma(\chi)(\tau(C^{[n]})) \\ &= \frac{1}{|G_1|} \sum_{C \in \text{Cl}(G_1)} |C| \chi(C^{[n]}) = \nu_n(\chi). \end{aligned}$$

The exponent of G is the least $n \geq 1$ with $C^{[n]} = \{1\}$ for all classes C . As τ is a bijection with $\tau(\{1\}) = \{1\}$ and $\tau(C^{[n]}) = \tau(C)^{[n]}$, this condition holds in G_1 if and only if it holds in G_2 . $\square$

Lemma 3.2. *In the situation of Lemma 3.1, σ preserves the fusion rules: $[\sigma(\chi)\sigma(\psi) : \sigma(\theta)] = [\chi\psi : \theta]$ for all $\chi, \psi, \theta \in \text{Irr}(G_1)$.*

Proof. By Lemma 2.1(c), the fusion coefficients are computed from the character table and the class sizes, which τ and σ preserve by Lemma 3.1. $\square$

Proof of Theorem 1.3. By Lemma 2.1(b) and Lemma 3.1, $\exp \mathbb{C}[G_1] = \exp G_1 = \exp G_2 = \exp \mathbb{C}[G_2]$ and σ matches the indicators; by Lemma 3.2 it preserves the fusion rules. If $\text{Rep}(G_1)$ and $\text{Rep}(G_2)$ were tensor equivalent, then $G_1 \cong G_2$ by Corollary 2.3, since $|G_1|$ is odd; but the groups of a Brauer pair are not isomorphic. $\square$

Remark 3.3. (a) Only the plain tensor equivalence is excluded; no braided or symmetric structure enters. As recalled in [EG01, p. 1], $\text{Rep}(G)$ as a symmetric tensor category determines G up to isomorphism, so a symmetric version of the conclusion would simply say $G_1 \cong G_2$. (b) Brauer pairs of even order exist; by the zbMATH review of [EM06], the smallest 2-groups forming Brauer pairs have order 2^8 . For them Corollary 2.3 is not available, and we do not treat them. (c) The groups D_8 and Q_8 have the same character table and are not isocategorical [EG01, Remark 1.5], but they do not satisfy the hypothesis of Question 1.2: ν_2 of the character of degree 2 is $+1$ for D_8 and -1 for Q_8 . By Lemma 3.1 there are therefore no bijections as in Definition 2.2: no bijection of classes that preserves the table also preserves the power maps, and D_8, Q_8 are not a Brauer pair.

4 An explicit Brauer pair of order 3^6

4.1 Construction

Let $\mathbb{F}_9 = \mathbb{F}_3[i]$ with $i^2 = -1$, let $V = \mathbb{F}_9^2$ and $Z = \mathbb{F}_9$, and write $\omega = e^{2\pi\sqrt{-1}/3}$. An element $v = (v_1, v_2) \in V$ with $v_1 = a + bi$, $v_2 = c + di$ is identified with $(a, b, c, d) \in \mathbb{F}_3^4$; let $e_1, \dots, e_4$ be the standard basis of $\mathbb{F}_3^4$, and let $\hat{v}_1, \dots, \hat{v}_4 \in \{0, 1, 2\}$ be the integers representing a, b, c, d . For $s, t \in \{0, 1, 2\}$ put $\text{carry}(s, t) = 1$ if $s + t \geq 3$ and $\text{carry}(s, t) = 0$ otherwise. For an $\mathbb{F}_3$ -linear map $\varphi: V \rightarrow Z$ let

$$c_{\varphi}(v, w) = v_1 w_2 + \sum_{k=1}^4 \text{carry}(\hat{v}_k, \hat{w}_k) \varphi(e_k) \in Z \quad (v_1 w_2 \text{ the product in } \mathbb{F}_9),$$

and let $P(\varphi)$ be the set $V \times Z$ with the product $(v, z)(w, z') = (v + w, z + z' + c_{\varphi}(v, w))$. We take

$$\varphi_1(v_1, v_2) = v_2, \quad \varphi_2(a + bi, c + di) = b + di, \quad P_1 = P(\varphi_1), \quad P_2 = P(\varphi_2).$$

Both maps are onto ($\varphi_1(e_3) = \varphi_2(e_2) = 1$, $\varphi_1(e_4) = \varphi_2(e_4) = i$). Let $\pi: P(\varphi) \rightarrow V$ be the first projection, $Z_0 = \{0\} \times Z$, $K_v = \pi^{-1}(v)$ for $v \in V$, and $\beta(v, w) = v_1w_2 - v_2w_1 \in \mathbb{F}_9$. For $\xi \in V^* = \text{Hom}_{\mathbb{F}_3}(V, \mathbb{F}_3)$ and $\lambda \in Z^* = \text{Hom}_{\mathbb{F}_3}(Z, \mathbb{F}_3)$ define functions on $P(\varphi)$ by

$$\psi_\xi(v, z) = \omega^{\xi(v)}, \quad \chi_\lambda(0, z) = 9\omega^{\lambda(z)}, \quad \chi_\lambda(v, z) = 0 \quad (v \neq 0).$$

Proposition 4.1. *Let $\varphi: V \rightarrow Z$ be $\mathbb{F}_3$ -linear and onto, and $P = P(\varphi)$.*

- (a) *P is a group of order 729 with identity $(0, 0)$; Z_0 is central and π is a homomorphism with kernel Z_0 .*
- (b) *$[g, h] = (0, \beta(\pi g, \pi h))$ for all $g, h \in P$. Hence $Z(P) = P' = Z_0$, P has nilpotency class 2, and the conjugacy classes are the 9 sets $\{(0, z)\}$ and the 80 cosets K_v , $v \neq 0$; in particular P is a Camina group: the class of every $g \notin P'$ is gP' .*
- (c) *$g^3 = (0, \varphi(\pi g))$ for all g . The map $g \mapsto g^3$ is a homomorphism onto Z_0 with kernel $\Omega(P) = \{g : g^3 = 1\} = \pi^{-1}(\ker \varphi)$ of order 81; P has exponent 9, with 80 elements of order 3 and 648 of order 9.*
- (d) *$\text{Irr}(P)$ consists of the 81 linear characters ψ_ξ , $\xi \in V^*$, and the 8 characters χ_λ , $\lambda \in Z^* \setminus \{0\}$, of degree 9. The products are $\psi_\xi \psi_\eta = \psi_{\xi+\eta}$, $\psi_\xi \chi_\lambda = \chi_\lambda$, $\chi_\lambda \chi_\mu = 9\chi_{\lambda+\mu}$ if $\lambda + \mu \neq 0$, and $\chi_\lambda \chi_{-\lambda} = \sum_{\xi \in V^*} \psi_\xi$.*
- (e) *For $n \in \mathbb{Z}$: $\{(0, z)\}^{[n]} = \{(0, nz)\}$; if $3 \nmid n$ and $v \neq 0$, then $K_v^{[n]} = K_{nv}$; if $n = 3k$, then $K_v^{[n]} = \{(0, k\varphi(v))\}$.*
- (f) *$\nu_n(\psi_0) = 1$ for all n ; for $\xi \neq 0$, $\nu_n(\psi_\xi) = 1$ if $3 \mid n$ and 0 otherwise; $\nu_n(\chi_\lambda) = 9$ if $9 \mid n$ and 0 otherwise.*

Proof. (a) For a map $c: V \times V \rightarrow Z$ consider the identity

$$c(x, y) + c(x + y, u) = c(y, u) + c(x, y + u) \quad (x, y, u \in V). \quad (1)$$

It holds for the bilinear map $(v, w) \mapsto v_1w_2$. For $s \in \mathbb{F}_3$ let $\hat{s} \in \{0, 1, 2\}$ represent it; then $\hat{s} + \hat{t} = \widehat{s + t + 3 \text{ carry}(\hat{s}, \hat{t})}$ in $\mathbb{Z}$, and computing $\hat{s} + \hat{t} + \hat{u}$ in two ways gives the analogue of (1) for carry on $\mathbb{F}_3$, as an identity of integers. Composing with the k -th coordinate $V \rightarrow \mathbb{F}_3$ and with $t \mapsto t\varphi(e_k)$ preserves (1), so c_φ satisfies (1), and $c_\varphi(0, w) = c_\varphi(v, 0) = 0$. Identity (1) is associativity of the product, $(0, 0)$ is the identity, and (v, z) has the inverse $(-v, -z - c_\varphi(v, -v))$ on both sides, since (1) for $(v, -v, v)$ gives $c_\varphi(v, -v) = c_\varphi(-v, v)$. As c_φ vanishes when an argument is 0, the elements $(0, t)$ are central; π is clearly a homomorphism.

(b) Let $g = (v, z)$ and $h = (w, z')$. The products gh and hg have the same first coordinate and second coordinates differing by $c_\varphi(v, w) - c_\varphi(w, v)$; as Z_0 is central, $[g, h] = gh(hg)^{-1} = (0, c_\varphi(v, w) - c_\varphi(w, v))$. The carry terms are symmetric in v, w , so $[g, h] = (0, v_1w_2 - w_1v_2) = (0, \beta(v, w))$. If $v \neq 0$, then $w \mapsto \beta(v, w)$ is $\mathbb{F}_9$ -linear and non-zero ($\beta(v, (0, 1)) = v_1$, $\beta(v, (1, 0)) = -v_2$), hence onto $\mathbb{F}_9$. So no element outside Z_0 is central, every element of Z_0 is a commutator, and $P/Z_0 \cong V$ is abelian: $Z(P) = P' = Z_0$. For $v \neq 0$ the conjugates $xgx^{-1} = [x, g]g = (0, -\beta(v, \pi x))g$ run through $Z_0g = K_v$.

(c) We have $(v, z)^3 = (3v, 3z + c_\varphi(v, v) + \widehat{c_\varphi(2v, v)}) = (0, c_\varphi(v, v) + c_\varphi(2v, v))$. The bilinear part contributes $3v_1v_2 = 0$. The k -th digit of $2v$ is $2\hat{v}_k$, and $\text{carry}(t, t) + \text{carry}(\widehat{2t}, t) = t$ for $t = 0, 1, 2$ (the values are $0 + 0, 0 + 1, 1 + 1$); so the carry part is $\sum_k \hat{v}_k \varphi(e_k) = \varphi(v)$. Thus the cube map is $\varphi \circ \pi$ followed by $z \mapsto (0, z)$, a homomorphism onto Z_0 whose kernel $\pi^{-1}(\ker \varphi)$ has order $9 \cdot 9 = 81$. Since $g^9 = (0, 3\varphi(\pi g)) = 1$, the elements of $\Omega(P) \setminus \{1\}$ have order 3 and the other 648 have order 9.

(d) The ψ_ξ are the 81 distinct linear characters of $P/P' \cong V$. Let χ be a non-linear irreducible character. By Schur's lemma there is $\lambda \in Z^*$ with $\chi(zg) = \omega^{\lambda(z)}\chi(g)$ for all $(0, z)$ and g (we write zg for $(0, z)g$), and $\lambda \neq 0$, since otherwise χ factors through the abelian group P/Z_0 . For $g \notin Z_0$ every zg is conjugate to g by (b), so $\chi(g) = \omega^{\lambda(z)}\chi(g)$ for all z , and $\chi(g) = 0$. Then $1 = \langle \chi, \chi \rangle = 729^{-1} \cdot 9 \chi(1)^2$, so $\chi(1) = 9$ and $\chi = \chi_\lambda$. Conversely, for $\lambda \neq 0$ an irreducible constituent of $\text{Ind}_{Z_0}^P \omega^\lambda$ contains ω^λ in its restriction to Z_0 (Frobenius reciprocity), so it is not linear, and it is χ_λ by the above. This gives $81 + 8 = 89$ irreducible characters, the number of classes. The products are checked pointwise, using $\sum_\xi \psi_\xi(v, z) = 81$ for $v = 0$ and 0 for $v \neq 0$.

(e) $(0, z)^n = (0, nz)$; if $3 \nmid n$ then $\pi(g^n) = n\pi(g) \neq 0$ for $g \in K_v$, $v \neq 0$; if $n = 3k$ then $g^n = (g^3)^k = (0, k\varphi(v))$ by (c).

(f) By Lemma 2.1(a), $\nu_n(\psi) = \langle \psi^n, 1 \rangle$ for a linear character ψ , which is 1 if $\psi^n = 1$ and 0 otherwise; the ψ_ξ with $\xi \neq 0$ have order 3. If $3 \nmid n$, then n is invertible modulo $9 = \exp P$, so $g \mapsto g^n$ is a bijection of P and $\nu_n(\chi_\lambda) = \langle \chi_\lambda, 1 \rangle = 0$. If $n = 3k$ with $3 \nmid k$, then by (c)

$$\nu_n(\chi_\lambda) = \frac{1}{729} \sum_{v \in V} \sum_{z \in Z} 9 \omega^{k\lambda(\varphi(v))} = \frac{1}{9} \sum_{v \in V} \omega^{k\lambda(\varphi(v))} = 0,$$

because $v \mapsto k\lambda(\varphi(v))$ is a non-zero linear form on V (φ is onto and $\lambda \neq 0$), which takes each value 27 times. If $9 \mid n$, then $g^n = 1$ and $\nu_n(\chi_\lambda) = \chi_\lambda(1) = 9$. $\square$

Proposition 4.2. *Let $\alpha: V \rightarrow V$, $\alpha(a, b, c, d) = (a, c, b, d)$. Then α is an $\mathbb{F}_3$ -linear involution with $\varphi_2 \circ \alpha = \varphi_1$. Define $\tau: \text{Cl}(P_1) \rightarrow \text{Cl}(P_2)$ by $\tau(\{(0, z)\}) = \{(0, z)\}$ and $\tau(K_v) = K_{\alpha(v)}$ ($v \neq 0$), and $\sigma: \text{Irr}(P_1) \rightarrow \text{Irr}(P_2)$ by $\sigma(\psi_\xi) = \psi_{\xi \circ \alpha}$ and $\sigma(\chi_\lambda) = \chi_\lambda$. These are bijections, $\sigma(\chi)(\tau(C)) = \chi(C)$ and $\tau(C^{[n]}) = \tau(C)^{[n]}$ for all χ , C and $n \in \mathbb{Z}$, and $P_1 \not\cong P_2$.*

Proof. $\varphi_2(\alpha(a, b, c, d)) = \varphi_2(a + ci, b + di) = c + di = \varphi_1(a + bi, c + di)$. The maps τ, σ are bijections because α permutes $V \setminus \{0\}$ and $\xi \mapsto \xi \circ \alpha$ permutes V^* (Proposition 4.1(b),(d)). Character values: $\psi_{\xi \circ \alpha}(K_{\alpha(v)}) = \omega^{\xi(\alpha(\alpha(v)))} = \omega^{\xi(v)}$; on the central classes the ψ 's are 1 and χ_λ takes the value $9\omega^{\lambda(z)}$ in both groups; off the centre $\chi_\lambda = 0$ in both. Power maps, by Proposition 4.1(e): both sides are $\{(0, nz)\}$ for central classes; for $3 \nmid n$, $\tau(K_v^{[n]}) = \tau(K_{nv}) = K_{\alpha(nv)} = K_{n\alpha(v)} = K_{\alpha(v)}^{[n]}$; for $n = 3k$, $\tau(K_v^{[n]}) = \{(0, k\varphi_1(v))\}$ and $K_{\alpha(v)}^{[n]} = \{(0, k\varphi_2(\alpha v))\} = \{(0, k\varphi_1(v))\}$.

An isomorphism $P_1 \rightarrow P_2$ would map $\Omega(P_1)$ onto $\Omega(P_2)$. In P_1 , $\pi(\Omega(P_1)) = \ker \varphi_1 = \mathbb{F}_9 \times \{0\}$ and $\beta((x, 0), (y, 0)) = 0$, so $\Omega(P_1)$ is abelian by Proposition 4.1(b). In P_2 , $\ker \varphi_2 = \{(a, c) : a, c \in \mathbb{F}_3\}$ contains $(1, 0)$ and $(0, 1)$, and $\beta((1, 0), (0, 1)) = 1 \neq 0$, so $\Omega(P_2)$ is not abelian. $\square$

Proof of Theorem 1.4. By Proposition 4.1(b),(c), P_1 and P_2 are Camina groups of class 2 and exponent 9, so $\exp \mathbb{C}[P_i] = 9$ by Lemma 2.1(b). By Proposition 4.2 they form a Brauer pair, and $\Omega(P_1), \Omega(P_2)$ are as stated. The irreducible characters and their indicator sequences are given by Proposition 4.1(d),(f); the sequences have period dividing 9 and the values for $1 \leq j \leq 9$ form the set $\{0, 1, 9\}$. The last sentence follows from Theorem 1.3. (By Proposition 4.1(d) the fusion rules are also visibly the same, σ being additive on V^* and the identity on the χ_λ .) $\square$

Remark 4.3 (Relation to printed Brauer pairs). Lewis's Main Theorem [Lew09]: for an odd prime p , two non-isomorphic Camina p -groups P, Q of nilpotency class 2 form a Brauer pair if and only if $|P : P'| = |Q : Q'|$, $|P'| = |Q'|$ and $|\mathcal{U}_1(P)| = |\mathcal{U}_1(Q)|$. By Proposition 4.1(b),(c), $|P_i : P'_i| = 3^4$, $|P'_i| = 3^2$ and $|\mathcal{U}_1(P_i)| = 3^2$, so this theorem gives a second proof that P_1, P_2 form a Brauer pair. Lewis reports a MAGMA search (primes up to 31): for $|P : P'| = p^4$, $|P'| = p^2$ there are $p + 3$ Camina groups, $p + 1$ of them with $|\mathcal{U}_1(P)| = p^2$, exactly one of which has $\Omega_1(P)$ non-abelian; he also says that two examples of Nenciu [Nen08] meet his hypotheses, one with Ω_1 abelian and one with Ω_1 non-abelian. The pair P_1, P_2 is of this kind; we have not compared it with the groups of [Nen08],

and we do not claim that it is new as a Brauer pair. Further odd-order Brauer pairs, as stated in the zbMATH review of [EM06]: Dade’s pairs of order p^7 and exponent p for all primes $p \geq 7$ [Dad64]; the three series of order p^5 and exponent p^2 , $p \geq 5$, of [EM06]; and the smallest 3-groups forming Brauer pairs have order 3^6 . Any of these, put into Theorem 1.3, gives the conclusion of Theorem 1.4; so Theorem 1.3 does not depend on this section or on any computation.

5 Remarks

Remark 5.1 (General semisimple Hopf algebras). Let G be a finite group and $H = \mathbb{C}^G$ the algebra of functions on G , with $\Delta(\delta_x) = \sum_{yz=x} \delta_y \otimes \delta_z$. Its simple modules are the one-dimensional modules S_g ($g \in G$), on which f acts by $f(g)$, and $S_g \otimes S_h \cong S_{gh}$; so the Grothendieck ring of $\text{Rep}(\mathbb{C}^G)$, with the simple modules as basis, is $\mathbb{Z}[G]$. The integral is $\Lambda = \delta_1$, and $\delta_x^{[n]} = \sum_{y_1 \cdots y_n = x} \delta_{y_1} \cdots \delta_{y_n} = \sum_{y^n = x} \delta_y$. Hence $\nu_n(S_g)$ is 1 if $g^n = 1$ and 0 otherwise, which is also the case $\omega = 1$ of [MS17, Corollary 3.3]; and $f^{[n]}(y) = f(y^n)$, so $\exp \mathbb{C}^G = \exp G$ (also [EG99, Proposition 2.2(2),(4)]). Take for G the Heisenberg group $\text{Heis}(3)$ of upper unitriangular 3×3 matrices over $\mathbb{F}_3$ (order 27, exponent 3) and $H' = \mathbb{C}[A]$, $A = (\mathbb{Z}/3)^3$, whose simple modules are the 27 characters θ of A , with $\nu_n(\theta) = 1$ if $\theta^n = 1$ and 0 otherwise (Lemma 2.1(a)). Both Hopf algebras have exponent 3 and the indexed indicator data $1 \times (1, 1, 1)$ and $26 \times (0, 0, 1)$. A tensor equivalence would induce an isomorphism of Grothendieck rings mapping simple objects to simple objects; but $\mathbb{Z}[\text{Heis}(3)]$ is not commutative, and the Grothendieck ring of $\text{Rep}(\mathbb{C}[A])$ is.

So for semisimple Hopf algebras in general, without Ng’s condition, Question 9 has an elementary negative answer in every reading of Section 1.2; Ng’s condition excludes this pair. The pair also shows that the source’s remark that the question is trivially true when H is the group algebra of an abelian group needs H' to be one as well (take $H = \mathbb{C}[A]$, $H' = \mathbb{C}^{\text{Heis}(3)}$). When both are, the indexed data give, for each n , the number of characters θ with $\theta^n = 1$, and two finite abelian groups with the same number of elements of each order are isomorphic. The closest printed observation that we found is [MS17, Remark 4.5]: Frobenius–Schur indicators do not classify pointed fusion categories $\mathcal{C}(G, \omega)$ in general; the smallest example is in rank 8, where there are 47 non-equivalent categories but only 34 different sets of indicators. For a non-trivial 3-cocycle ω these categories are representation categories of quasi-Hopf algebras, not of Hopf algebras (a fibre functor on $\mathcal{C}(G, \omega)$ would trivialize ω). No Hopf-algebra pair occurs among these coincidences: the five categories with trivial cocycle in rank 8, the $\text{Rep}(\mathbb{C}^G)$ for the groups G of order 8, have pairwise different indicator data counted with multiplicity, since these data count the elements of each order, and the five groups have different such counts. The paper [MS17], read in full, contains no pair of non-equivalent pointed categories with trivial cocycle and equal indicators (the coincidences it names explicitly are between two cocycles on the same group), and in particular not the pair $\mathbb{C}^{\text{Heis}(3)}, \mathbb{C}[(\mathbb{Z}/3)^3]$. We give this pair as an elementary example in the same spirit and claim nothing new about it.

Remark 5.2 (Many algebras with the same data). According to the zbMATH review of [Nen09] (Zbl 1181.20007), Nenciu proves that for every $t \geq 2$ and every prime $p \geq 3$ there are Brauer t -tuples of p -groups, that is, groups $G_1, \dots, G_t$ any two of which form a Brauer pair; by the same review, Eick and Müller had found Brauer 4-tuples of order p^5 for $p \geq 5$. With Theorem 1.3 this gives, for every $t \geq 2$, group algebras $\mathbb{C}[G_1], \dots, \mathbb{C}[G_t]$ of odd order, any two of which have equal exponents and a bijection of simple modules matching the indicators and the fusion rules, while no two of $\text{Rep}(G_1), \dots, \text{Rep}(G_t)$ are tensor equivalent. We know [Nen08, Nen09] only through their zbMATH reviews.

Remark 5.3 (Further invariants). (a) By the zbMATH review (Zbl 1469.18021) of [MS21a], Meir and Szymik prove that a monoidal equivalence $\text{Rep}(G) \rightarrow \text{Rep}(G')$ induces an isomorphism of representation rings that preserves the Adams operations Ψ^p for all odd p . For a Brauer pair, $(\Psi^n \chi)(g) = \chi(g^n)$

is preserved by σ and τ , so this necessary condition holds for P_1, P_2 , consistently with Theorem 1.4; note that $\nu_n(\chi) = \langle \Psi^n \chi, 1 \rangle$.

(b) *A computation, not used in any proof.* A tensor equivalence $\text{Rep}(P_1) \simeq \text{Rep}(P_2)$ would induce a braided equivalence of the centres $\text{Rep } D(P_1) \simeq \text{Rep } D(P_2)$, and hence equal indicators and twists of the simple modules of the Drinfeld doubles. These do not separate the pair either. In $D(G)$, with basis $\delta_x g$, product $(\delta_x g)(\delta_y h) = \delta_x \delta_{gy} g^{-1} gh$, coproduct $\Delta(\delta_x g) = \sum_{x_1 x_2 = x} \delta_{x_1} g \otimes \delta_{x_2} g$ and integral $\Lambda = |G|^{-1} \sum_g \delta_1 g$, the product $(\delta_{x_1} g) \cdots (\delta_{x_n} g)$ vanishes unless $x_k = g^{-(k-1)} x_1 g^{k-1}$ for all k , and then $x_1 \cdots x_n = (x_1 g^{-1})^n g^n$; hence $\Lambda^{[n]} = |G|^{-1} \sum_g \sum_{x: (xg^{-1})^n = g^{-n}} \delta_x g^n$. Evaluating the character of the simple module $V(a, \eta)$ induced from $\eta \in \text{Irr}(C_G(a))$ gives

$$\nu_n(V(a, \eta)) = \frac{1}{|C_G(a)|} \sum_{g \in G: (ag^{-1})^n = g^{-n}} \eta(g^n).$$

This formula is derived here and was not compared with the printed formulas [Kei12, IMM14, Sch16]. One verification run validated it against a brute-force evaluation of the Hopf power of the integral on explicitly constructed induced modules for S_3, Q_8 and D_8 ($n \leq 4$, floating point, deviation below 10^{-14}). The program `double_indicators.py` (exact arithmetic in $\mathbb{Q}(\zeta_9)$) finds 7281 simple modules for each of $D(P_1), D(P_2)$, and equal multisets of $(\dim, \text{twist}, \nu_1, \dots, \nu_9)$, with 14 distinct types; the second verification run obtained the same with its own program (floating point, rounded to integers). So the step through [EG01] cannot be replaced by these invariants of the centre.

(c) Stronger invariants that the source does not mention, such as generalized indicators $\nu_{n,k}$, the S -matrix of $D(P_i)$, or the Kuperberg-type invariants of Chang, Ng and Wang [CNW25], were not examined for this pair.

Remark 5.4 (The other questions of the record). Lentner’s Question 6(c) asks for an interpretation of the diagram folding construction under the correspondence between small quantum groups and the representation theory of the groups $G(\mathfrak{g})$ over $\mathbb{F}_q$, and Lu’s Question 7(b) asks which algebraic and homological relations hold between H and H' ; both are open-ended and not treated here. Lu’s Question 7(a) asks, for the maximal pointed Hopf subalgebra H' of a Hopf algebra H , whether there is a Hopf ideal I with $H = H' \oplus I$. The source introduces H' in Lu’s talk, joint work with Z.-P. Fan and X.-L. Yu [BGLZ14, p. 1125], and describes it only as based on a generalized direct sum decomposition of coalgebras and the filtration of Bakalov, D’Andrea and Kac; it cites no paper for it. The definition is presumably in the paper [FLY16] of these authors, which we could not read. What follows is conditional and is not part of our results. Suppose that for cosemisimple H the subalgebra H' is the largest pointed Hopf subalgebra, and let $H = \mathbb{C}^G$. A pointed Hopf subalgebra of the cosemisimple coalgebra $\mathbb{C}^G$ is cosemisimple and pointed, hence spanned by grouplike elements of $\mathbb{C}^G$, which are the linear characters $G \rightarrow \mathbb{C}^\times$; so H' is their span, the algebra of functions constant on the cosets of G' . Hopf ideals I of $\mathbb{C}^G$ correspond to Hopf quotients, whose duals are Hopf subalgebras of $\mathbb{C}[G]$, that is, the $\mathbb{C}[K]$ for subgroups K ; so the Hopf ideals are the $I_K = \{f : f|_K = 0\}$. For $G = Q_8$ we have $G' = \{\pm 1\}$ and $\dim H' = 4$; $H = H' \oplus I_K$ needs $|K| = 4$ and the restriction $H' \rightarrow \mathbb{C}^K$ to be injective, but each of the three subgroups of order 4 contains -1 , so the restriction has rank 2. Under this reading, $\mathbb{C}^{Q_8}$ has no such Hopf ideal. The authors may define H' differently, for instance through the coradical filtration when H is not cosemisimple, and then the example need not apply.

6 Computations and reproducibility

No proof in this note depends on a computation; Theorem 1.4 is a finite statement, and the programs confirm it. They are in the folder `reproducibility/` of the source archive, with their recorded outputs. *The exact program of this note, check_note.py* (Python 3, standard library only, exact

arithmetic in $\mathbb{Z}$ and $\mathbb{Z}[\omega]$, about 10 seconds), checks for P_1 and P_2 the identity (1) on all 81^3 triples, the identity and inverses, Proposition 4.1(b) and (c) (commutators, centre, the 89 classes, cubes, element orders, $|\bar{U}_1| = 9$), the induced characters $\text{Ind}_{Z_0}^P \omega^\lambda = 9\chi_\lambda$ and the exact orthonormality of the 89 characters, the products of Proposition 4.1(d), the bijections of Proposition 4.2 with the power maps for $n = 0, \dots, 18$, the indicators $\nu_1, \dots, \nu_{18}$ of all irreducible characters by summation over the 729 elements and by the class formula of Lemma 2.1(a), and the subgroups $\Omega(P_i)$; and the finite statements of Remark 3.3(c), Remark 5.1 and Remark 5.4. *The program with which the result was first obtained*, `brauer_pair_729.py` (with `numpy`, exact integer and $\mathbb{Z}[\omega]$ arithmetic, about 25 seconds), checks in addition associativity on all 729^3 triples, and its output ends with “ALL CHECKS PASSED”; `double_indicators.py` is the computation of Remark 5.3(b). The programs of the three verification runs described below are in `verification_run_A/`, `verification_run_B/` and `independent_run_2/`.

Verification. The result was first obtained, with proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the first written version of the result. The first run read the question and Montgomery’s abstract in the report, re-derived Lemmas 2.1, 3.1 and 3.2 and all parts of Propositions 4.1 and 4.2 line by line, and checked [EG01, Definition 1.1, Lemma 3.1, Corollary 1.4] and the statements of [Lew09] in their arXiv texts. Its program, in exact arithmetic in $\mathbb{Z}[\zeta_9]$, certified the eight characters of degree 9 as characters induced from an abelian normal subgroup of order 81, and checked the character tables, the class sizes, the power maps for $n = 0, \dots, 18$, all fusion coefficients and the indicators. The second run computed the character tables from scratch by Burnside’s algorithm (rounded into $\mathbb{Z}[\omega]$, then checked exactly), all 89^3 class structure constants, the power maps and the indicators, with five negative controls (a broken cocycle, D_8 against Q_8 , a map φ of rank one, a wrong class bijection, a perturbed table) that its tests detect; it derived and validated the formula of Remark 5.3(b), recomputed the Drinfeld doubles, checked the bibliography with Crossref and zbMATH, and found the example of Remark 5.1 and the related remark of [MS17]. Both runs re-ran the original programs and obtained identical outputs. Neither found a mathematical error; their corrections, all of presentation, were made (among them the proof that $\exp \mathbb{C}[G] = \exp G$, the exact wording of [EG01, Corollary 1.4], Remark 5.1 with the scope it implies, and the justifications in Remark 5.4). When this note was written, `check_note.py` was added, and all programs were re-run from a copy of the archive; every recorded output was reproduced.

The third run examined the text of this note line by line. It read the sources again (the report, pp. 1125–1127 and 1131–1134; [EG01, Definition 1.1, Theorems 1.2 and 1.3, Corollary 1.4, Remark 1.5, Lemmas 3.1 and 3.6]; [EG99]; [Lew09]; the zbMATH reviews used here; [MS17] in full), re-derived all lemmas, propositions and remarks, and wrote two programs of its own (`independent_run_2/`). In the first, each group P_i is defined by the presentation read off from Proposition 4.1(b),(c) and enumerated by Todd–Coxeter coset enumeration, which returns the order 729; a bijective homomorphism onto $P(\varphi_i)$ is checked on all pairs of elements; the characters of degree 9 are traces of explicit monomial representations induced from linear characters of the abelian subgroup $\pi^{-1}(\mathbb{F}_9 \cdot (1, 1))$, checked to be representations on all pairs; then, in exact arithmetic in $\mathbb{Z}[\zeta_9]$, it checks the orthonormality of the 89 characters, the power maps for $n = 0, \dots, 18$, $\nu_1, \dots, \nu_{18}$, all 89^3 fusion coefficients, the bijections of Proposition 4.2 and the subgroups $\Omega(P_i)$, with two negative controls. The second program checks the finite statements of Remark 3.3(c), Remark 5.1 (including the rank-8 statement) and Remark 5.4. The run re-ran all programs from an extracted copy of the source archive; every recorded output was reproduced, that of `q7_q8_check.py` up to the order of three lines, which depends on the hash seed of Python (see the README). It found no mathematical error; its corrections were made (among them the attribution of H' in Remark 5.4, the addition on [MS17] in Remark 5.1, the description of

[EG01, Theorem 1.2], and the record of the searches).

Scope and priority. The note answers Question 9 of [BGLZ14] negatively for group algebras (Theorems 1.3 and 1.4), in every reading of Section 1.2 and also with Ng’s condition. For general semisimple Hopf algebras without Ng’s condition the elementary pair of Remark 5.1 also gives a negative answer; we did not find it in print, it is in the spirit of [MS17, Remark 4.5], and we claim nothing new about it. The main content of the note is the group-algebra case, with or without equal fusion rules. Questions 6(c) and 7 of the same corpus record are not settled (Remark 5.4), so the record as a whole remains partly open. The question is Montgomery’s, raised in 2006 [BGLZ14, p. 1134], and the fusion-rule variant is R. Ng’s. The higher indicators of semisimple Hopf algebras are those of Kashina, Sommerhäuser and Zhu [KSZ06], and their gauge invariance is due to Mason and Ng [MN05] and Ng and Schauenburg [NS08]; the exponent of a Hopf algebra is taken from [EG99]. The non-equivalence rests entirely on Etingof and Gelaki [EG01]. Brauer pairs of odd order are due to Dade [Dad64], Eick and Müller [EM06], Lewis [Lew09] and Nenciu [Nen08, Nen09]; the pair of Section 4 is not claimed to be new as a Brauer pair. Related printed statements are [MS17, Remark 4.5] (indicators do not classify pointed fusion categories; quasi-Hopf case), the theorem of Mignard and Schauenburg [MS21b] that modular data do not determine modular categories (the braided analogue), and [MS21a] (Adams operations).

What was read: [BGLZ14], pp. 1125–1127 and 1131–1134, in the publisher’s PDF; [EG01] in the arXiv version v1 and the zbMATH review of the journal version; [EG99] in the arXiv version v2 (Definition 2.1, Proposition 2.2, Section 4); [Lew09] in the arXiv version v1; [MS17] in the arXiv version v1, in full; [MS21b] in the arXiv version. Of [NS08] and [KMN12] the abstracts were read; of [Kei17, IMM14, Sch16] the abstracts and text searches, by the second verification run; of [CNW25] the abstract. [Dad64, EM06, Nen08, Nen09, MS21a] are known to us only through their zbMATH reviews, and only what the reviews state is used. We did not read [KSZ06], [MN05] (cited, as in the source, for the definition and the gauge invariance), [Kei12, Kei14], or [FLY16], so the definition of H' in Question 7 is unknown to us.

The searches were made on 20 October 2026: queries to the arXiv interface (Frobenius–Schur indicators together with Brauer pairs, gauge equivalence or gauge invariants, character tables, tensor equivalence, complete invariants or pointed fusion categories; isocategorical groups; Brauer pairs), some of which were refused with HTTP status 429 and replaced by single abstract pages; queries to zbMATH Open (isocategorical groups, Brauer pairs, indicators as gauge invariants or complete invariants, the later work of the proposer, and the authors named above); Crossref for the bibliography; nine web searches; and the manuscripts of a public release of AI-generated mathematical results (github.com/openai/math, October 2026). OpenAlex returned the records of the report and of [EG01] but refused the lists of citing works (HTTP status 429). Semantic Scholar listed five works citing the report (2015–2020, on Nichols algebras and Gelfand–Kirillov dimension) and 53 works citing [EG01], whose titles were screened; none of them concerns Question 9. Works citing Montgomery’s 2006 formulation could not be traced (the source gives no reference for it), and the journal version of [EG01] was not compared. No printed answer to the question was found in these sources, and none of them links Brauer pairs or isocategorical groups to it. A search that finds nothing is not a proof of novelty; both ingredients are in print and the combination is short, so experts may regard it as folklore, and we claim only that we found no place where it is written down.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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