

A Rank Threshold for Exact Sums of Squares in Virtually Abelian Group Algebras

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Abstract

Let G have a finite-index subgroup isomorphic to $\mathbb{Z}^d$. We give an explicit finite-index compression formula and use it to show that positivity in all finite-dimensional unitary representations is equivalent to a sum of Hermitian squares at every matrix level exactly when $d \leq 2$. In the positive cases, the representations induced by characters of the abelian subgroup suffice. The essential input is the corrected bivariate matrix factorization theorem of Dritschel, Klep, McCullough and Volčič. For $d \geq 3$, a trigonometric version of the classical Motzkin form gives an explicit obstruction, and coefficient projection transfers it to every group containing $\mathbb{Z}^3$. We spell out the resulting two-dimensional representation test and factor-transfer formula for the Klein bottle group. The result is a deduction from credited factorization and positivity methods; it does not settle the exact-squares conjecture for general closed surface groups. No factor-degree bound or absolute priority claim is made.

1 Statement and prior input

Write $\mathbb{C}[G]$ for the complex group algebra of a discrete group G , with the involution conjugating coefficients and sending g to g^{-1} . For $m \geq 1$, let

$$\mathcal{S}_m(G) = \left\{ \sum_{\nu=1}^N B_\nu^* B_\nu : N < \infty, B_\nu \in M_m(\mathbb{C}[G]) \right\}.$$

A rectangular factor $B \in M_{k,m}(\mathbb{C}[G])$ also gives an element of this cone: split B into rows and pad each row by zero rows. Let $\mathcal{P}_m(G)$ be the Hermitian matrices A for which $\pi^{(m)}(A) \succeq 0$ in every unitary representation π of G on a complex Hilbert space. Let $\mathcal{P}_m^{\text{fd}}(G)$ use just the finite-dimensional representations. Evidently,

$$\mathcal{S}_m(G) \subseteq \mathcal{P}_m(G) \subseteq \mathcal{P}_m^{\text{fd}}(G). \tag{1}$$

All positivity in this note is positive semidefiniteness.

Theorem 1. *Suppose $H \leq G$ has finite index r and $H \cong \mathbb{Z}^d$. The following are equivalent:*

1. $d \leq 2$;
2. $\mathcal{S}_m(G) = \mathcal{P}_m(G) = \mathcal{P}_m^{\text{fd}}(G)$ for every $m \geq 1$;
3. $\mathcal{S}_1(G) = \mathcal{P}_1(G)$.

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For $d \leq 2$, testing the r -dimensional unitary representations induced by characters of H already suffices at every matrix level. For $d \geq 3$, the failure in (3) is witnessed by the explicit element in (7), embedded in a subgroup $\mathbb{Z}^3 \leq H$.

Every finitely generated virtually abelian group has a finite-index free abelian subgroup. Indeed, a finite-index abelian subgroup is finitely generated, and taking a suitable integer multiple kills its finite torsion subgroup while preserving finite index. Thus Theorem 1 gives a rank threshold for that class.

The nontrivial factorization input is Dritschel–Klep–McCullough–Volčič [1, Theorem 1.1]. It states that a finite matrix Laurent polynomial which is positive semidefinite on $\mathbb{T}^2$ has a finite rectangular analytic polynomial factor. The 2026 addendum corrects a difficulty and a degree bound in Dritschel’s earlier proof. We use only the corrected existence result.

Conditional expectations for transferring sums of squares are developed by Savchuk–Schmüdgen [3, Section 5]. The addendum [1, Corollary 4.2] already applies that method to $(\mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)$. The scalar $\mathbb{Z}^2$ theorem and the failure for groups containing $\mathbb{Z}^3$ are recorded by Netzer–Thom [2, Section 6], with credit to Scheiderer and Rudin. The purpose here is to give a direct formula valid for arbitrary finite-index subgroups, without a normality or splitting assumption, and to organize these inputs into Theorem 1. The factorization theorem and the rank-three obstruction principle are not new claims of this note.

2 An explicit finite-index compression

Let $H \leq G$ have finite index r , and choose representatives $t_1 = e, t_2, \dots, t_r$ for the left cosets G/H . For $g \in G$, define $\Phi(g) \in M_r(\mathbb{C}[H])$ by

$$\Phi(g)_{ij} = \begin{cases} t_i^{-1} g t_j, & g t_j H = t_i H, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Every row and column has exactly one nonzero entry. Uniqueness of coset decomposition gives $\Phi(gh) = \Phi(g)\Phi(h)$, and the inverse monomial matrix gives $\Phi(g^{-1}) = \Phi(g)^*$. Hence Φ extends to a unital $*$ -homomorphism. It is injective: its first column recovers the coefficients in the unique expression $\sum_i t_i a_i$ with $a_i \in \mathbb{C}[H]$.

For any unitary representation σ of H , applying σ entrywise in (2) gives a unitary representation $\text{Ind}_H^G \sigma$ of G . This observation also applies when the Hilbert space of σ is infinite dimensional.

Lemma 2 (Compression identity). *Set $v = (t_1^{-1}, \dots, t_r^{-1})^T$, a column over $\mathbb{C}[G]$. Then*

$$\Phi(a)v = va, \quad v^*v = r, \quad v^*\Phi(a)v = ra \quad (a \in \mathbb{C}[G]). \quad (3)$$

For $A = (A_{pq}) \in M_m(\mathbb{C}[G])$, define

$$\Phi_m(A)_{(i,p),(j,q)} = \Phi(A_{pq})_{ij}, \quad W_{(i,p),q} = \delta_{pq} t_i^{-1}.$$

Thus $\Phi_m(A)$ is rm by rm and W is rm by m . Then

$$\Phi_m(A)W = WA, \quad W^*W = rI_m, \quad W^*\Phi_m(A)W = rA. \quad (4)$$

Proof. For fixed i and g , the unique nonzero term in the i -th entry of $\Phi(g)v$ is

$$(t_i^{-1} g t_j) t_j^{-1} = t_i^{-1} g.$$

Linearity and $\sum_i t_i t_i^{-1} = r$ prove (3). The $(i,p), q$ entry of $\Phi_m(A)W$ is

$$\sum_j \Phi(A_{pq})_{ij} t_j^{-1} = t_i^{-1} A_{pq},$$

which is the same entry of WA . The remaining matrix identities follow from $W^*W = rI_m$. All products have the displayed order; commutativity of $\mathbb{C}[G]$ is not assumed. $\square$

Proposition 3 (Finite-index ascent). *If $\mathcal{P}_n(H) = \mathcal{S}_n(H)$ for every $n \geq 1$, then $\mathcal{P}_m(G) = \mathcal{S}_m(G)$ for every $m \geq 1$. More precisely, every factorization $\Phi_m(A) = Q^*Q$ over $\mathbb{C}[H]$ gives*

$$A = (r^{-1/2}QW)^*(r^{-1/2}QW). \quad (5)$$

Proof. If $A \in \mathcal{P}_m(G)$, then applying any unitary representation σ of H to $\Phi_m(A)$ evaluates A in the amplification of $\text{Ind}_H^G \sigma$, up to a fixed ordering of the indices. It is therefore positive semidefinite. Thus $\Phi_m(A) \in \mathcal{P}_{rm}(H) = \mathcal{S}_{rm}(H)$, so it has a rectangular factor Q over $\mathbb{C}[H]$. Equation (4) gives (5). Splitting the rows of its right-hand factor puts A in $\mathcal{S}_m(G)$. The reverse inclusion always holds. $\square$

3 The positive half of the rank threshold

Identify $\mathbb{C}[\mathbb{Z}^d]$ with the Laurent polynomial algebra in d commuting unitary variables. For $d = 2$, the corrected factorization theorem [1, Theorem 1.1] gives

$$P(\xi) \succeq 0 \text{ for every } \xi \in \mathbb{T}^2 \implies P = Q^*Q \quad (6)$$

for a finite rectangular analytic matrix polynomial Q . Equality on the torus is equality in the Laurent polynomial ring, by uniqueness of finite Fourier expansions. The same assertion for $d = 1$ follows either from matrix Fejér–Riesz factorization or by introducing a dummy variable in (6) and specializing it to one in Q . For $d = 0$ it is the ordinary positive semidefinite matrix square root.

Now let $d \leq 2$ and $A \in \mathcal{P}_m^{\text{fd}}(G)$. For every character $\chi : H \rightarrow \mathbb{T}$, entrywise evaluation of $\Phi(g)$ gives a unitary representation π_χ on $\mathbb{C}^r$. Consequently $\Phi_m(A)(\chi) \succeq 0$. The appropriate factorization above yields $\Phi_m(A) = Q^*Q$, and (5) puts A in $\mathcal{S}_m(G)$. Together with (1), this proves (1) $\Rightarrow$ (2) in Theorem 1. It also proves the assertion about testing just the induced-character family. The implication (2) $\Rightarrow$ (3) is immediate.

Remark 4. The existence of Q is sufficient here. We obtain no uniform bound on its degree or number of rows. In particular, the previous bounds from the corrected theorem are not transferred to group algebra certificates. Formula (5) is explicit once a factorization is available; it is not a bounded-degree algorithm for finding one.

4 A concrete rank-three obstruction

Let u_1, u_2, u_3 be the standard generators of $\mathbb{Z}^3$, and put $h_j = 2 - u_j - u_j^{-1}$. Consider

$$P = h_1^2 h_2 + h_1 h_2^2 + h_3^3 - 3h_1 h_2 h_3 \in \mathbb{C}[\mathbb{Z}^3]. \quad (7)$$

Lemma 5. *The element P is positive in every unitary representation of $\mathbb{Z}^3$, but $P \notin \mathcal{S}_1(\mathbb{Z}^3)$.*

Proof. Each h_j is nonnegative on $\mathbb{T}^3$. The product of the three nonnegative terms $h_1^2 h_2, h_1 h_2^2, h_3^3$ is $(h_1 h_2 h_3)^3$. AM–GM therefore proves $P \geq 0$ on the torus. The joint functional calculus for three commuting unitaries proves positivity in every unitary representation.

To disprove a sum-of-squares representation, put $u_j = e^{i\theta_j}$. Since $h_j = \theta_j^2 + O(\theta_j^4)$, the leading homogeneous Taylor term of P is the Motzkin form

$$M(X, Y, Z) = X^4 Y^2 + X^2 Y^4 + Z^6 - 3X^2 Y^2 Z^2. \quad (8)$$

Suppose $P = \sum_{\nu} f_{\nu}^* f_{\nu}$ for Laurent polynomials f_{ν} . Locally this is a sum of squares of the real and imaginary parts of finitely many real analytic functions. Their homogeneous terms of degree below three all vanish: otherwise choose their least nonzero degree $\ell < 3$. The degree- 2ℓ term of the sum would be a sum of squares of nonzero real homogeneous forms, and so could not vanish identically. But P has no term below degree six. It follows that M would be a sum of squares of real homogeneous cubics.

For completeness, this is impossible by coefficient comparison. If $M = \sum_{\nu} q_{\nu}^2$, the zero coefficients of X^6 and Y^6 force every X^3 and Y^3 coefficient in the cubics to be zero. The zero coefficients of X^4Z^2 and Y^4Z^2 next eliminate X^2Z and Y^2Z : their only other possible cross terms involve the already absent pure cubes. The zero coefficients of X^2Z^4 and Y^2Z^4 then eliminate XZ^2 and YZ^2 . The remaining monomials are

$$X^2Y, \quad XY^2, \quad Z^3, \quad XYZ.$$

The coefficient of $X^2Y^2Z^2$ in the sum of their squares is therefore the sum of the squares of their XYZ coefficients. It is nonnegative, contradicting its value -3 in (8). $\square$

Lemma 6 (Subgroup projection). *For any subgroup $L \leq G$, the coefficient projection $E_L : \mathbb{C}[G] \rightarrow \mathbb{C}[L]$ preserves sums of Hermitian squares.*

Proof. Write $q = \sum_t tq_t$ over distinct left L -cosets, with $q_t \in \mathbb{C}[L]$ and only finitely many nonzero terms. If the cosets tL, sL differ, $t^{-1}s \notin L$, and multiplication on the left and right by elements of L cannot put it in L . Consequently

$$E_L(q^*q) = \sum_t q_t^* q_t. \quad (9)$$

Apply this identity to each square. $\square$

If G contains $L \cong \mathbb{Z}^3$, regard P as an element of $\mathbb{C}[G]$. It is positive in every representation of G , by restriction to L . Were it a sum of squares in $\mathbb{C}[G]$, applying E_L would contradict Lemma 5. For $d \geq 3$, the subgroup H contains such an L . Thus (3) fails, proving (3) $\Rightarrow$ (1) and completing the theorem. The general obstruction for groups containing $\mathbb{Z}^3$ is classical; the calculation supplies an explicit representative and proof.

5 The Klein bottle group and the surface question

Let

$$K = \langle x, y \mid yxy^{-1} = x^{-1} \rangle.$$

Its elements have unique normal form $x^a y^b$ with $a, b \in \mathbb{Z}$, and the corresponding multiplication is

$$(a, b)(c, d) = (a + (-1)^b c, b + d).$$

Thus $H = \langle x, z = y^2 \rangle \cong \mathbb{Z}^2$ has index two. For representatives $(1, y)$, the construction above reads

$$\Phi(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad \Phi(y) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ y^{-1} \end{pmatrix}. \quad (10)$$

After replacing (x, z) by $(\xi, \zeta) \in \mathbb{T}^2$, both matrices are unitary and obey the defining relation.

Corollary 7. *A Hermitian $a \in \mathbb{C}[K]$ is a sum of Hermitian squares if and only if $\Phi(a)(\xi, \zeta) \succeq 0$ for every $(\xi, \zeta) \in \mathbb{T}^2$. If a factorization $\Phi(a) = Q^*Q$ has rows (p_j, q_j) over $\mathbb{C}[x^{\pm 1}, z^{\pm 1}]$, then*

$$a = \frac{1}{2} \sum_j (p_j + q_j y^{-1})^* (p_j + q_j y^{-1}). \quad (11)$$

The analogous equivalence holds at every matrix level.

Proof. Apply Theorem 1 and (5) with $r = 2$ and the column in (10). The converse follows by evaluating squares in these unitary representations. $\square$

Thom’s Conjecture 0.4 in the Oberwolfach report [4, p. 1023] asks for exact sums of Hermitian squares for fundamental groups of closed surfaces; compare [2, Conjecture 6.3]. The equivalence of all-unitary and finite-dimensional positivity was already credited there to Lubotzky–Shalom. The theorem above covers the Klein bottle, the torus, and the finite fundamental groups of the sphere and the real projective plane. It does not cover closed orientable surfaces of genus at least two or closed nonorientable surfaces of genus at least three, whose groups are not virtually abelian. In particular, the virtual abelian rank threshold is not a criterion in terms of cohomological dimension. The general surface-group conjecture remains unproved here.

The accompanying standard-library Python script checks 905 exact finite identities and coefficient comparisons, including complex integer coefficients, the nonsplit Klein bottle extension, a nonnormal subgroup of S_3 , matrix compression, subgroup projection, and the local Motzkin obstruction. Ordinary and optimized Python runs agree. These are algebraic regressions, not a formal proof or a verification of the cited factorization theorem. AI tools assisted the derivation, source search, verification code and exposition. This preprint is unrefereed; no independent review or absolute novelty certification is claimed.

References

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