

On the Albert–Bousquet–Mélou Conjecture for Corners in Quadrant Loops: The Case of Nonnegative Corner Weight

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Abstract

For a real number a let $Q_n(a)$ be the sum of $a^{c(w)}$ over the walks w of length $2n$ with steps N, E, S, W that start and end at the origin and stay in the quadrant $\{x \geq 0, y \geq 0\}$, where $c(w)$ is the number of factors NW and ES of w , and let $L_n(a)$ be the same sum without the quadrant condition. Albert and Bousquet–Mélou conjectured, and Bousquet–Mélou asked in the problem session of Oberwolfach Report 12/2014, that for every $a \geq -1$ the series $\sum_n Q_n(a)u^n$ has the same radius of convergence as $\sum_n L_n(a)u^n$, whose radius they determined. They proved this for $a = 1$ and $a = -1$, and for $a \geq 0$ they proved that the radius for quadrant loops is at least $(2 + \sqrt{2 + 2a})^{-2}$. We prove the conjecture for every $a \geq 0$: the limits of $Q_n(a)^{1/n}$ and of $L_n(a)^{1/n}$ exist and both are equal to $(2 + \sqrt{2 + 2a})^2$, and $Q_n(a) \leq (2 + \sqrt{2 + 2a})^{2n}$ for all n . The proof is elementary and self-contained. It combines a formula for corner-weighted walks under an exponential tilt, a large-deviation bound for the loops that go far to the left of or below the origin, a padding without corners that moves the other loops into the quadrant, and Fekete’s lemma; all these tools are standard. The question remains open for $-1 < a < 0$, and this range contains the values in $[-1/3, 0)$ on which the conjectural growth rate of permutations sortable by two stacks in parallel depends. We show where the argument needs $a \geq 0$ and report numerical evidence, not a proof, for the open range. No proof of the case $a \geq 0$ was found in the sources we searched; for $a \geq 1$ the result seems to follow, after a short argument, from known theorems on weighted walks in cones. This is an unrefereed note.

1 Introduction

1.1 The question

We read words over the alphabet $\{N, E, S, W\}$ as walks on $\mathbb{Z}^2$ that start at the origin, with the steps $N = (0, 1)$, $E = (1, 0)$, $S = (0, -1)$ and $W = (-1, 0)$. A *loop* is a word whose walk ends at the origin; its length is even, say $2n$, and n is its *half-length*. A *quadrant loop* is a loop whose walk stays in the closed quadrant $\{(i, j) : i \geq 0, j \geq 0\}$. For a word $w = w_1 \cdots w_m$ let

$$c(w) = \#\{t : (w_t, w_{t+1}) \in \{(N, W), (E, S)\}\}$$

be the number of its factors NW and ES, which we call *corners*. For instance, the quadrant loop NEENSESWWW has one corner [BM14]. For a real number a put $\text{wt}(w) = a^{c(w)}$ (with $0^0 = 1$) and

$$Q_n(a) = \sum_{\substack{w \text{ quadrant loop} \\ |w|=2n}} \text{wt}(w), \quad L_n(a) = \sum_{\substack{w \text{ loop} \\ |w|=2n}} \text{wt}(w), \quad Q(a, u) = \sum_{n \geq 0} Q_n(a)u^n.$$

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Thus $Q_1 = 2$, $Q_2 = 8 + 2a$, $Q_3 = 44 + 24a + 2a^2$, $Q_4 = 288 + 246a + 52a^2 + 2a^3$ and $Q_5 = 2100 + 2464a + 882a^2 + 96a^3 + 2a^4$. By an involution due to Bernardi and Courtiel [ABM15, Proposition 14], which acts inside each shuffle class (the set of words with given projections on $\{N, S\}$ and on $\{E, W\}$), replacing the pair (NW, ES) by (WN, ES) , (WN, SE) or (NW, SE) changes neither $Q_n(a)$ nor $L_n(a)$, so everything below holds for these pairs as well. We denote by $\rho_Q(a)$ and $\rho_L(a)$ the radii of convergence of $Q(a, \cdot)$ and of $\sum_n L_n(a)u^n$.

These series were introduced by Albert and Bousquet-Mélou [ABM15] in their solution of the enumeration of permutations sortable by two stacks in parallel. Throughout, the numbers of results of [ABM15] are those of its arXiv version v3, in which all statements share one counter. The series $\mathcal{Q}(a, u; x, y)$ of quadrant walks counted by length (u), corners (a) and endpoint (x, y) is characterised by the functional equation [ABM15, Theorem 7, Eq. (1)]

$$(1 - u(x + \bar{x} + y + \bar{y}) - u^2(a-1)(x\bar{y} + y\bar{x})) \mathcal{Q}(x, y) = 1 - u\bar{y}(1 + ux(a-1)) \mathcal{Q}(x, 0) - u\bar{x}(1 + uy(a-1)) \mathcal{Q}(0, y),$$

with $\bar{x} = 1/x$, $\bar{y} = 1/y$, and $Q(a, u) = \mathcal{Q}(a, \sqrt{u}; 0, 0)$.

Conjecture A (Albert and Bousquet-Mélou [ABM15, Conjecture 11]). *For $a \geq -1$ the radius of convergence of $Q(a, \cdot)$ is*

$$\rho_Q(a) = \frac{1}{(2 + \sqrt{2 + 2a})^2} \quad \text{if } a \geq -\frac{1}{2}, \quad \rho_Q(a) = -\frac{a}{2(a-1)^2} \quad \text{if } -1 \leq a \leq -\frac{1}{2}.$$

Albert and Bousquet-Mélou proved that the right-hand side is the radius $\rho_L(a)$ of the series of unconfined loops for every $a \geq -1$ [ABM15, Proposition 21], and that the same holds for loops confined to the upper half-plane [ABM15, Proposition 22]. So the conjecture says that the confinement to the quadrant does not change the radius. This is the second question of Bousquet-Mélou's problem-session contribution [BM14]: is the exponential growth rate of the numbers $Q_n(a)$, $a \geq -1$, the same as for unconfined loops? For $a < 0$ the numbers $Q_n(a)$ are signed sums, and the growth rate is $\limsup_n |Q_n(a)|^{1/n} = 1/\rho_Q(a)$; that $Q_n(a) > 0$ for $a \geq -1$ is believed [BM14] and would follow from the $(a+1)$ -positivity of $Q(a, u)$ [ABM15, Conjecture 10], the first question of [BM14].

What was known. At $a = 1$ and $a = -1$ there are closed forms, $Q_n(1) = C_n C_{n+1}$ and $Q_n(-1) = \sum_i \binom{n}{i} C_i C_{n-i}$ with the Catalan numbers C_i [ABM15, Proposition 15], and they give $\rho_Q(1) = \frac{1}{16}$ and $\rho_Q(-1) = \frac{1}{8}$ [ABM15, Proposition 17]: the conjecture holds at $a = \pm 1$. For $a \geq 0$, Proposition 17 of [ABM15] also proves the inequality $\rho_Q(a) \geq (2 + \sqrt{2 + 2a})^{-2}$, because the quadrant loops are dominated by all walks, whose series is $1/(1 - 4u - 2u^2(a-1))$; the conjecture says that this bound is tight. The authors report that Raschel predicted the values of $\rho_Q(a)$ from an application, not yet rigorous, of analytic methods for walks in the quarter plane, and that the conjecture was checked numerically [ABM15, Section 4]; Elvey Price and Guttmann gave further numerical evidence [EG17, Section 5.3]. Elvey Price proved, without the conjectures, that permutations sortable by deque and by two stacks in parallel have the same growth rate, and he writes that he could not prove the three conjectures of [ABM15] [Elv19].

Why the conjecture matters. Assuming [ABM15, Conjecture 10] and the convergence of $\partial_u Q(a, \rho_Q(a))$ for $-1/3 \leq a \leq 0$ [ABM15, Conjecture 12], the radius t_c of the generating function $S(t)$ of permutations sortable by two stacks in parallel is reached when the point $(-S^\bullet(t), t(1 + S^\bullet(t))^{-2})$ meets the curve $\{(a, \rho_Q(a))\}$, where $S = 1/(1 - S^\bullet)$, and $S^\bullet(t_c) \leq 1/3$ [ABM15, Theorem 23]. With the first branch of Conjecture A this gives $t_c = (1 - \sqrt{2 + 2a}/2)^2$, where $a = -S^\bullet(t_c) \in [-1/3, 0)$ is determined by an equation in $Q(a, \cdot)$ [ABM15, Corollary 24]; numerically $-0.15 \leq a \leq -0.148$ and $8.25 \leq 1/t_c \leq 8.29$ [ABM15, Section 5.1]. So the application needs Conjecture A for negative a .

1.2 Result

Write $\lambda(a) = 2 + \sqrt{2 + 2a}$ and $G(a) = \lambda(a)^2$; thus $1/G(a)$ is the conjectured radius for $a \geq -\frac{1}{2}$.

Theorem 1.1 (the case $a \geq 0$). *Let $a \geq 0$. Then the limits $\lim_n Q_n(a)^{1/n}$ and $\lim_n L_n(a)^{1/n}$ exist and*

$$\lim_{n \rightarrow \infty} Q_n(a)^{1/n} = \lim_{n \rightarrow \infty} L_n(a)^{1/n} = G(a) = (2 + \sqrt{2 + 2a})^2.$$

Hence $\rho_Q(a) = \rho_L(a) = (2 + \sqrt{2 + 2a})^{-2}$: Conjecture A holds for every $a \geq 0$, and the second question of [BM14] has a positive answer for every $a \geq 0$. Moreover $Q_n(a) \leq G(a)^n$ for every $n \geq 0$.

Together with the known case $a = -1$ [ABM15, Propositions 15 and 17], the question is now settled for $a \in \{-1\} \cup [0, \infty)$. It remains **open** for $-1 < a < 0$, and in particular for the values $a \in [-1/3, 0)$ that the application to two stacks in parallel needs; so Theorem 1.1 does not make the growth rate of these permutations unconditional. At $a = 1$ the theorem recovers $\rho_Q(1) = \frac{1}{16}$. At $a = 0$ it says that the number of quadrant loops of length $2n$ without any factor NW or ES is $(6 + 4\sqrt{2})^{n+o(n)}$; this is the growth constant $(2 + \sqrt{2})^2 \approx 11.66$ that [ABM15, Section 5.1] derive from Conjecture A for the operation sequences that output eagerly, and it agrees with the numerical value $u_c = (6 + 4\sqrt{2})^{-1}$ of [EG17, Section 5.3]. As a by-product the proof re-proves the formula for $\rho_L(a)$ of [ABM15, Proposition 21], but only for $a \geq 0$. Proposition 3.1 adds a lower bound with the error factor $e^{-O(\sqrt{n \log n})}$.

1.3 Method, credit and scope

The proof (Section 2) is elementary. The generating function of all corner-weighted walks under an exponential tilt (x, y) satisfies a two-term recurrence whose characteristic polynomial is the kernel of the functional equation above (Lemma 2.1). A Cauchy–Schwarz argument gives a lower bound for unconfined loops (Lemma 2.3); an exponential tilt bounds the weight of the loops that reach the line $x = -k - 1$ or $y = -k - 1$ (Lemma 2.4); a padding without corners moves the other loops into the quadrant at the cost of $4k + 4$ letters (Lemma 2.5); and supermultiplicativity with Fekete’s lemma [Fek23] gives the existence of the limit (Lemma 2.6). These tools (Fekete’s lemma, exponential tilting or change of measure, Cauchy–Schwarz, concatenation and padding of walks) are standard, and we claim no new method. The argument uses $a \geq 0$ in three essential ways (Remark 3.2), and for $-1 \leq a < -\frac{1}{2}$ no argument of this kind can work (Section 5).

We found no proof of the case $a \geq 0$, other than $a = 1$, in the sources we searched (the paragraph “Scope and priority” in Section 6); [ABM15, EG17, Elv19] treat it as open. For $a \geq 1$ the result seems to follow after a short argument from known results on weighted walks in cones, and a general theorem of Ozawa on Markov-modulated walks nearly covers $a \geq 0$ (Remark 4.1). A search that finds nothing is not a proof of novelty.

2 Proof of Theorem 1.1

Notation. For a word w let $|w|$ be its length and $(X(w), Y(w))$ the endpoint of its walk. Put $b = a - 1$ and, for real $x, y > 0$,

$$S = x + \frac{1}{x} + y + \frac{1}{y}, \quad T = \frac{x}{y} + \frac{y}{x}, \quad W_m(x, y) = \sum_{|w|=m} \text{wt}(w) x^{X(w)} y^{Y(w)}.$$

We use one observation throughout:

$$\text{the first letter of a corner is N or E, and its second letter is W or S.} \tag{1}$$

So two corners never overlap, a letter N or E is never the second letter of a corner, and a letter S or W is never the first letter of a corner. The identities in the lemmas below hold for every real a ; the inequalities need $a \geq 0$, and where they do we say so.

Lemma 2.1 (all walks). *For every real a and $x, y > 0$ we have $W_0 = 1$, $W_1 = S$ and*

$$W_{m+1} = S W_m + b T W_{m-1} \quad (m \geq 1).$$

Let $a \geq 0$. Then the roots $\lambda_+ \geq \lambda_-$ of $z^2 - Sz - bT$ are real, $\lambda_+ > |\lambda_-|$, and with $\Lambda(x, y) := \lambda_+ = \frac{1}{2}(S + \sqrt{S^2 + 4bT})$

$$0 \leq W_m(x, y) \leq (m+1) \Lambda(x, y)^m \quad (m \geq 0). \quad (2)$$

At $x = y = 1$ we have $S = 4$, $T = 2$, $\Lambda(1, 1) = \lambda(a)$, and $W_m(1, 1) \geq c_a \lambda(a)^m$ for all $m \geq 0$, where $c_a = 1$ if $0 \leq a \leq 1$ and $c_a = 2/\sqrt{2+2a}$ if $a > 1$.

Proof. For $m \geq 1$ and a letter ℓ let $A_m(\ell)$ be the part of W_m that comes from the words ending with ℓ , and put $z_N = y$, $z_S = 1/y$, $z_E = x$, $z_W = 1/x$. Appending a letter ℓ' to a word ending with ℓ multiplies its term by $z_{\ell'}$, and also by $a = 1 + b$ when (ℓ, ℓ') is (N, W) or (E, S). Hence

$$A_{m+1}(\ell') = z_{\ell'}(W_m + b[\ell' = W] A_m(N) + b[\ell' = S] A_m(E)).$$

By (1), appending N or E never creates a corner, so $A_m(N) = y W_{m-1}$ and $A_m(E) = x W_{m-1}$ for $m \geq 1$. Summing over ℓ' gives $W_{m+1} = S W_m + b(y/x + x/y) W_{m-1}$, the recurrence.

Let $a \geq 0$, so $b \geq -1$. With $p = x + 1/x$ and $q = y + 1/y$ we have $pq = xy + 1/(xy) + T \geq T + 2$, hence $S^2 = (p+q)^2 \geq 4pq \geq 4T+8$, and the discriminant satisfies $S^2 + 4bT \geq S^2 - 4T \geq 8 > 0$. So the roots are real and distinct, and $\lambda_+ + \lambda_- = S > 0$ gives $\lambda_+ > |\lambda_-|$. The sequence $h_m = \sum_{k=0}^m \lambda_+^k \lambda_-^{m-k}$ satisfies $h_0 = 1$, $h_1 = S$ and $h_{m+1} = S h_m - \lambda_+ \lambda_- h_{m-1} = S h_m + b T h_{m-1}$, so $W_m = h_m$, and each of its $m+1$ terms has modulus at most λ_+^m . Finally $W_m \geq 0$ because all weights are nonnegative for $a \geq 0$.

At $(1, 1)$ the roots of $z^2 - 4z - 2b$ are $2 \pm s$ with $s = \sqrt{2+2a}$, so $\Lambda(1, 1) = \lambda(a)$. If $a \leq 1$, then $\lambda_- = 2 - s \geq 0$ and $W_m \geq \lambda_+^m$. If $a > 1$, then $\lambda_- = -r\lambda$ with $r = (s-2)/(s+2) \in (0, 1)$, and $W_m = \lambda^m \sum_{j=0}^m (-r)^j = \lambda^m (1 - (-r)^{m+1})/(1+r) \geq \lambda^m (1-r)/(1+r) = \lambda^m 2/s$. $\square$

At $x = y = 1$ the series $\sum_m W_m(1, 1) u^m$ is $1/(1 - 4u - 2u^2(a-1))$, the series of all walks used in [ABM15, Proposition 17], and the characteristic polynomial $z^2 - Sz - bT$ corresponds to the kernel of the functional equation of [ABM15, Theorem 7].

Lemma 2.2 (reversal). *For $w = w_1 \cdots w_m$ let $R(w) = (-w_m) \cdots (-w_1)$, where $-N = S$, $-S = N$, $-E = W$, $-W = E$. Then R is an involution, $c(R(w)) = c(w)$, and $(X, Y)(R(w)) = -(X, Y)(w)$.*

Proof. The factor (p, q) of w in positions $t, t+1$ becomes the factor $(-q, -p)$ of $R(w)$ in positions $m-t, m-t+1$; and $(-q, -p) = (N, W)$ if and only if $(p, q) = (E, S)$, while $(-q, -p) = (E, S)$ if and only if $(p, q) = (N, W)$. The rest is clear. $\square$

Lemma 2.3 (unconfined loops). *Let $a \geq 0$. For $m \geq 0$,*

$$L_{m+1}(a) \geq \frac{W_m(1, 1)^2}{(m+1)^2} \geq \frac{c_a^2 \lambda(a)^{2m}}{(m+1)^2}, \quad \text{hence} \quad L_n(a) \geq \frac{c_a^2 \lambda(a)^{2n-2}}{n^2} \quad (n \geq 1).$$

Proof. Let $N_m(P)$ be the total weight of the words of length m with endpoint P . If w_1, w_2 have length m and endpoints P and $-P$, then $w_1 N S w_2$ is a loop of length $2m+2$ with $c(w_1 N S w_2) = c(w_1) + c(w_2)$: by (1) the factor (last letter of w_1 , N) is not a corner, NS is not a corner, and (S, first letter of w_2) is not a corner. The map $(w_1, w_2) \mapsto w_1 N S w_2$ is injective. Since all weights are

nonnegative, $L_{m+1} \geq \sum_P N_m(P) N_m(-P) = \sum_P N_m(P)^2$, by Lemma 2.2. The endpoints of words of length m are the points (i, j) with $|i| + |j| \leq m$ and $i + j \equiv m \pmod{2}$; there are $(m+1)^2$ of them (the map $(i, j) \mapsto (\frac{m+i+j}{2}, \frac{m+i-j}{2})$ is a bijection onto $\{0, \dots, m\}^2$). By Cauchy–Schwarz, $\sum_P N_m(P)^2 \geq (\sum_P N_m(P))^2 / (m+1)^2 = W_m(1, 1)^2 / (m+1)^2$, and Lemma 2.1 gives the second inequality. $\square$

Lemma 2.4 (far excursions are rare). *Let $a \geq 0$, $k \geq 0$ and $n \geq 1$. Let $B_n^x(k)$ (resp. $B_n^y(k)$) be the total weight of the loops of length $2n$ whose walk visits the line $\{x = -k - 1\}$ (resp. $\{y = -k - 1\}$). For every $\delta > 0$, with $\Lambda_\delta := \Lambda(e^\delta, 1)$,*

$$B_n^x(k), B_n^y(k) \leq \max(1, a) (2n - 1) (n + 1)^2 e^{-2\delta(k+1)} \Lambda_\delta^{2n}.$$

Proof. Let w be a loop of length $2n$ that visits $\{x = -k - 1\}$, and let τ be the first time at which its walk has abscissa $-k - 1$; since the walk starts and ends at abscissa 0, $1 \leq \tau \leq 2n - 1$. Cut $w = w_1 w_2$ with $|w_1| = \tau$. Then $c(w) = c(w_1) + c(w_2) + e$ with $e \in \{0, 1\}$ (the junction factor), so $\text{wt}(w) \leq \max(1, a) \text{wt}(w_1) \text{wt}(w_2)$, and $X(w_1) = -(k+1)$, $X(w_2) = k+1$. The map $w \mapsto (\tau, w_1, w_2)$ is injective. We bound the total weight of these triples by the sum, over all τ and over *all* pairs of words (v_1, v_2) with $|v_1| = \tau$, $|v_2| = 2n - \tau$, $X(v_1) = -(k+1)$ and $X(v_2) = k+1$, of $\max(1, a) \text{wt}(v_1) \text{wt}(v_2)$; the choice of the first passage serves only to make the map injective, and the bound would hold for any cut point. For $X(v) = -(k+1)$ and $x = e^{-\delta}$ we have $\text{wt}(v) = e^{-\delta(k+1)} \text{wt}(v) x^{X(v)}$, so, all terms of W_τ being nonnegative, the words v_1 weigh at most $e^{-\delta(k+1)} W_\tau(e^{-\delta}, 1) \leq e^{-\delta(k+1)} (\tau + 1) \Lambda(e^{-\delta}, 1)^\tau$ by (2). In the same way, with $x = e^\delta$, the words v_2 weigh at most $e^{-\delta(k+1)} (2n - \tau + 1) \Lambda(e^\delta, 1)^{2n-\tau}$. At $(e^{\pm\delta}, 1)$ and at $(1, e^{\pm\delta})$ we have $S = 2 \cosh \delta + 2$ and $T = 2 \cosh \delta$, so the four values of Λ are all equal to Λ_δ . Multiplying, using $(\tau + 1)(2n - \tau + 1) \leq (n + 1)^2$ and summing over the $2n - 1$ values of τ gives the bound for $B_n^x(k)$.

The bound for $B_n^y(k)$ is proved in the same way, with the line $\{y = -k - 1\}$, the ordinate and the tilts $(1, e^{\pm\delta})$. This uses only (2) and the fact that S and T are symmetric under $x \leftrightarrow y$, so that $\Lambda(1, e^{\pm\delta}) = \Lambda_\delta$. (In fact $B_n^y(k) = B_n^x(k)$: the reflection in the diagonal, which exchanges $\mathbf{N}$ with $\mathbf{E}$ and $\mathbf{S}$ with $\mathbf{W}$, maps the factors $\mathbf{NW}$ and $\mathbf{ES}$ to each other; but we do not need this.) $\square$

Lemma 2.5 (padding without corners). *Let $k \geq 0$, $P = \mathbf{N}^k \mathbf{E}^{k+1} \mathbf{W}$ and $F = \mathbf{E} \mathbf{W}^{k+1} \mathbf{S}^k$. If w is a loop of length $2n$ whose walk stays in $[-k, \infty)^2$, then PwF is a quadrant loop of length $2n + 4k + 4$ with $c(PwF) = c(w)$, and $w \mapsto PwF$ is injective.*

Proof. The word P goes $(0, 0) \rightarrow (0, k) \rightarrow (k+1, k) \rightarrow (k, k)$ inside the quadrant; then the translate of the walk of w by (k, k) stays in $(k, k) + [-k, \infty)^2 = [0, \infty)^2$ and returns to (k, k) ; and F goes $(k, k) \rightarrow (k+1, k) \rightarrow (0, k) \rightarrow (0, 0)$. The factors inside P and F are among $\mathbf{NN}$, $\mathbf{NE}$, $\mathbf{EE}$, $\mathbf{EW}$, $\mathbf{WW}$, $\mathbf{WS}$, $\mathbf{SS}$, none of which is a corner. By (1) the junction $(\mathbf{W}$, first letter of $w)$ is not a corner, and neither is the junction (last letter of w , $\mathbf{E}$). Injectivity is clear. $\square$

Lemma 2.6 (supermultiplicativity). *Let $a \geq 0$. Then $Q_{m+n}(a) \geq Q_m(a) Q_n(a)$ for $m, n \geq 0$, and $Q_n(a) \geq 1$ for $n \geq 0$.*

Proof. A quadrant loop of positive length starts with $\mathbf{N}$ or $\mathbf{E}$ and ends with $\mathbf{S}$ or $\mathbf{W}$. So the concatenation uv of two quadrant loops is a quadrant loop with $c(uv) = c(u) + c(v)$, by (1), and uv determines (u, v) once $|u|$ is known. The weights are nonnegative, which gives the first claim. The quadrant loop $(\mathbf{NS})^n$ has no corner. $\square$

2.1 Proof of Theorem 1.1

Let $a \geq 0$ and write $\lambda = \lambda(a)$.

Upper bound. All weights are nonnegative, so by Lemma 2.1

$$Q_n(a) \leq L_n(a) \leq W_{2n}(1, 1) \leq (2n+1)\lambda^{2n}. \quad (3)$$

Existence of the limit. By Lemma 2.6 the sequence $\log Q_n$ is superadditive and finite, so by Fekete's lemma [Fek23] $\gamma := \lim_n \frac{1}{n} \log Q_n$ exists and equals $\sup_{n \geq 1} \frac{1}{n} \log Q_n$. By (3), $\gamma \leq 2 \log \lambda$, and therefore $Q_n \leq e^{n\gamma} \leq \lambda^{2n} = G(a)^n$ for every $n \geq 1$.

Lower bound. Fix $\varepsilon \in (0, 1)$, let $n \geq 1$ and put $k = \lceil \varepsilon n \rceil$. By Lemma 2.1, $\Lambda_\delta = f(2 \cosh \delta)$ with

$$f(p) = \frac{1}{2} \left(p + 2 + \sqrt{(p+2)^2 + 4bp} \right),$$

because $S = p+2$ and $T = p$ at $(e^\delta, 1)$ with $p = 2 \cosh \delta$. The radicand is at least $(p+2)^2 - 4p = p^2 + 4 > 0$, so f is smooth on $[2, \infty)$, $f(2) = \lambda$, and $f'(p) = \frac{1}{2} + \frac{p+2+2b}{2\sqrt{(p+2)^2 + 4bp}} > \frac{1}{2}$ there (as $p+2+2b \geq p > 0$).

Let $K = 2 \max_{[2, 2 \cosh 1]} |f'|$; then $K > 1$. Since $2 \cosh \delta - 2 = 2 \sum_{j \geq 1} \delta^{2j} / (2j)! \leq 2\delta^2$ for $0 < \delta \leq 1$, the mean value theorem gives

$$\Lambda_\delta \leq \lambda + K\delta^2 \quad (0 < \delta \leq 1). \quad (4)$$

Choose $\delta = \min(1, \varepsilon \lambda / (2K))$. Using $1 + s \leq e^s$, $k+1 \geq \varepsilon n$ and $2nK\delta^2 / \lambda \leq n\delta\varepsilon$,

$$e^{-2\delta(k+1)} \Lambda_\delta^{2n} \leq \lambda^{2n} \exp\left(\frac{2nK\delta^2}{\lambda} - 2\delta\varepsilon n\right) \leq \lambda^{2n} e^{-\delta\varepsilon n}. \quad (5)$$

A loop whose walk leaves $[-k, \infty)^2$ visits the line $\{x = -k-1\}$ or the line $\{y = -k-1\}$, since its steps are unit steps and it starts at the origin. As all weights are nonnegative, the loops of length $2n$ that stay in $[-k, \infty)^2$ weigh at least $L_n - B_n^x(k) - B_n^y(k)$, and by Lemma 2.5 they are mapped injectively, with the same weight, to quadrant loops of half-length $n+2k+2$. With Lemma 2.3, Lemma 2.4 and (5),

$$Q_{n+2k+2} \geq L_n - B_n^x(k) - B_n^y(k) \geq \lambda^{2n} \left(\frac{c_a^2}{\lambda^2 n^2} - 2 \max(1, a) (2n-1)(n+1)^2 e^{-\delta\varepsilon n} \right) \geq \frac{c_a^2}{2\lambda^2 n^2} \lambda^{2n}$$

for all $n \geq n_0(a, \varepsilon)$. Put $M = n+2k+2 \leq (1+2\varepsilon)n+4$. Taking logarithms, dividing by M and letting $n \rightarrow \infty$ (the limit γ exists, so a subsequence suffices) gives $\gamma \geq 2 \log \lambda / (1+2\varepsilon)$. Letting $\varepsilon \rightarrow 0$ gives $\gamma = 2 \log \lambda$, that is, $\lim_n Q_n(a)^{1/n} = \lambda^2 = G(a)$.

Unconfined loops. By (3) and Lemma 2.3, $c_a^2 \lambda^{2n-2} n^{-2} \leq L_n(a) \leq (2n+1)\lambda^{2n}$, so $\lim_n L_n(a)^{1/n} = G(a)$ as well. The statement about the radii follows from the Cauchy–Hadamard formula. $\square$

3 A quantitative lower bound, and where $a \geq 0$ is used

The same argument with k of order $\sqrt{n \log n}$ gives a lower bound with an explicit error term.

Proposition 3.1. *Let $a \geq 0$. There is a constant C_a such that*

$$G(a)^n e^{-C_a \sqrt{n \log n}} \leq Q_n(a) \leq G(a)^n \quad (n \geq 2).$$

Proof. The upper bound is part of Theorem 1.1. Let λ , K and (4) be as in the proof of Theorem 1.1, and put $C = \sqrt{12K/\lambda}$, so that $C^2 \lambda / (2K) = 6$. For $n \geq 2$ let $k = k(n) = \lceil C \sqrt{n \log n} \rceil$ and $\delta = k\lambda / (2nK)$; then $\delta \leq 1$ for $n \geq n_1$. By (4) and $1 + s \leq e^s$,

$$e^{-2\delta(k+1)} \Lambda_\delta^{2n} \leq \lambda^{2n} \exp\left(\frac{2nK\delta^2}{\lambda} - 2\delta k\right) = \lambda^{2n} \exp\left(\frac{k^2 \lambda}{2nK} - \frac{k^2 \lambda}{nK}\right) = \lambda^{2n} e^{-k^2 \lambda / (2nK)} \leq \lambda^{2n} n^{-6},$$

since $k^2 \geq C^2 n \log n$. As in the proof of Theorem 1.1, with Lemmas 2.3 to 2.5,

$$Q_{n+2k+2} \geq \lambda^{2n} \left(\frac{c_a^2}{\lambda^2 n^2} - 2 \max(1, a)(2n-1)(n+1)^2 n^{-6} \right) \geq \frac{c_a^2}{2\lambda^2 n^2} \lambda^{2n} \quad (n \geq n_2),$$

because $(2n-1)(n+1)^2 n^{-6} = O(n^{-3})$. Put $N(n) = n + 2k(n) + 2$. The function $C\sqrt{t \log t}$ has derivative tending to 0, so $k(n+1) - k(n) \leq 1$ and $1 \leq N(n+1) - N(n) \leq 3$ for $n \geq n_3$. Let $n_4 = \max(n_1, n_2, n_3)$ and $M \geq N(n_4)$, and let $n \geq n_4$ be the largest integer with $N(n) \leq M$. Then $M - N(n) \leq 2$, and by Lemma 2.6 $Q_M \geq Q_{N(n)} Q_{M-N(n)} \geq Q_{N(n)}$. Since $n \leq M$, $k(n) \leq k(M)$ and $n \geq M - 2k(n) - 4$, we get

$$Q_M \geq \frac{c_a^2}{2\lambda^2 M^2} \lambda^{2M-4k(M)-8} = G(a)^M \exp(-O(\sqrt{M \log M})).$$

For $2 \leq M < N(n_4)$ we use $Q_M \geq 1$ and enlarge the constant. $\square$

This does not give the polynomial correction. Elvey Price and Guttmann conjecture, from their numerical analysis, that $Q_n(0)$ behaves like a constant times $G(0)^n n^{-1-\theta}$ with an explicit θ , and they report a conjectured formula of Raschel for the exponent for all $a \geq 0$, which agrees with their numerical estimates [EG17, Section 5.3]; at $a = 1$ the closed form gives $Q_n(1) \sim \frac{4}{\pi} 16^n n^{-3}$ [ABM15, Proposition 17].

Remark 3.2 (where $a \geq 0$ is used). The identities of Section 2 (the recurrence of Lemma 2.1, Lemma 2.2, and the corner counts in Lemmas 2.3 to 2.6) hold for every real a . The hypothesis $a \geq 0$ is used in three essential ways.

1. The terms of $W_m(x, y)$ are nonnegative: this gives $W_m \geq 0$ and (2), and it allows the sum over a set of words with prescribed abscissa to be bounded by the tilted sum $W_\tau(e^{\pm\delta}, 1)$ in Lemma 2.4.
2. A sub-family weighs at most the whole family: in Lemma 2.3; in Lemma 2.4, where the first-passage condition is dropped; in Lemma 2.6; in (3); and in the step “the loops that stay in $[-k, \infty)^2$ weigh at least $L_n - B_n^x(k) - B_n^y(k)$ ”, together with the bound of Q_{n+2k+2} by the weight of the padded loops.
3. $a^e \leq \max(1, a)$ for $e \in \{0, 1\}$, in Lemma 2.4.

The inequality $b = a - 1 \geq -1$ is also used, for convenience only, in the elementary estimates of the discriminant in Lemma 2.1 and of f and f' in the proof of Theorem 1.1. For $a < 0$ the weights $a^{c(w)}$ have both signs and all three fail; see Section 5.

4 Relation to known results

Theorem 1.1 is an elementary statement about a walk with a one-step memory, and two known routes come close to it. We describe them as they were found during the verification of this note; neither is in print in a form that applies directly, and neither was checked in detail.

Remark 4.1 (close routes). (a) *The case $a \geq 1$ through a six-step model.* Since two corners never overlap, for every real a

$$a^{c(w)} = \sum_{M \subseteq \{\text{corners of } w\}} (a-1)^{|M|}.$$

Replace each marked corner NW by a diagonal step $(-1, 1)$ and each marked corner ES by a diagonal step $(1, -1)$, each counted with length 2. The intermediate point of a marked corner lies in the

quadrant when its two ends do, so this is a bijection between pairs (w, M) , with w a quadrant loop, and quadrant excursions of the six-step model with steps N, E, S, W and $(-1, 1), (1, -1)$; and $Q_n(a)$ is the sum of $(a - 1)^{\#\text{diagonal steps}}$ over the excursions of total length $2n$ (the exact program of this note checks this identity, Section 6). With weight t for each unit step and $(a - 1)t^2$ for each diagonal step, $t^2 = u$, the kernel of this model is the kernel of the functional equation of [ABM15, Theorem 7], although [ABM15] do not read it in this way. For $a \geq 1$ all weights are nonnegative, and the results of Denisov and Wachtel on zero-drift walks in cones [DW15], after the usual exponential change of measure and a short argument that sums over the number of steps at fixed t , should give $\rho_Q(a) = 1/G(a)$. We did not find this written anywhere, and the argument is only a sketch. So the case $a \geq 1$ looks like a short corollary of known results. For $0 \leq a < 1$ the diagonal weight $a - 1$ is negative, and the results on walks in cones with independent increments [DW15, GR16, Dur14] do not apply directly.

(b) *A Markov-additive view of $a \geq 0$.* For $a \geq 0$, the walk with the last letter as a phase is a skip-free Markov-modulated walk on $\mathbb{Z}^2$ with nonnegative weights. Ozawa [Oza21, Section 2] proves, for irreducible nonnegative block tridiagonal matrices Q with countable blocks (his conditions (a5) and (a6)), that the convergence parameter satisfies $\text{cp}(Q) = \sup_{\theta} \text{cp}(A_*(\theta))$. Applied twice, it would give Theorem 1.1. But the irreducibility condition (a5) fails for the natural encoding of quadrant walks: the states $(0, j, \text{E})$ and $(i, 0, \text{N})$ cannot be reached, and removing them breaks the homogeneous block form at level 0. So this general theory nearly covers $a \geq 0$ but does not apply directly.

(c) *Markov-modulated walks in cones.* Denisov and Zhang [DZ25] treat Markov chains on $\mathbb{R}^d$ in the domain of attraction of Brownian motion in cones; the position process of the corner-weighted walk is not a Markov chain, since the weight of a step depends on the previous step, so their results do not apply directly. Padhi [Pad26] (unrefereed) proves an excursion local limit theorem for run-modulated walks with the three fixed steps $(1, 0), (-1, 1), (0, -1)$; it does not apply to the present model either.

[ABM15, Section 4] note that the equality of the radii for unconfined and half-plane loops is reminiscent of the result of Denisov and Wachtel that the growth constant of unweighted loops in a wedge does not depend on the wedge (they cite [DW15, Section 1.5]). Theorem 1.1 is a weighted analogue for the quadrant, for nonnegative corner weight.

5 The open range $-1 < a < 0$

Nothing in this section is a result about the open range: we explain why the proof stops at $a = 0$, and we report numerical evidence and an obstruction to one approach.

Why the proof stops at 0. For $a < 0$ the weights $a^{c(w)}$ have both signs, and each comparison between a sub-family and the whole family in Section 2 fails (Remark 3.2). For $-1 \leq a < -\frac{1}{2}$ this cannot be repaired by better estimates of the same kind. Write $s = \sqrt{2 + 2a} \in [0, 1]$, so $a = s^2/2 - 1$. The recurrence of Lemma 2.1 holds for every real a , and at $(1, 1)$ its roots are $2 \pm s$, so $\lim_n W_{2n}(1, 1)^{1/n} = (2 + s)^2 = \lambda(a)^2$. The conjectured growth rate on this range is

$$\frac{2(1 - a)^2}{-a} = \frac{(2 + s)^2(2 - s)^2}{2 - s^2} = \lambda(a)^2 \left(1 + \frac{2(1 - s)^2}{2 - s^2}\right) > \lambda(a)^2,$$

and it is the growth rate of the unconfined loops [ABM15, Proposition 21]. So the signed sum over the loops, a sub-family of the words of length $2n$, grows faster than the signed sum over all these words. With weights of one sign this would be impossible; so no comparison with sums over all words, such as the bounds of Section 2, can give the conjectured growth on this range, and cancellation between words is essential. For unconfined and half-plane loops, [ABM15] handle all $a \geq -1$ through the sum over the loops with a fixed vertical projection, which is $(a + 1)$ -positive and depends only on the length

of the projection [ABM15, Propositions 19, 20 and 22]. The quadrant analogue is the open property (P_2) of [ABM15, Section 6.3], and the $(a+1)$ -positivity of $Q(a, u)$ itself is [ABM15, Conjecture 10].

Numerical evidence (not a proof). The exact values $Q_n(a)$ for $n \leq 200$ and the seven values $a = -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, 1, 2$ were computed by an integer transfer matrix (Section 6); all of them are positive. With $r_n = Q_n/Q_{n-1}$, Table 1 compares the second-order extrapolant

$$\frac{1}{2}(n^2 r_n - 2(n-1)^2 r_{n-1} + (n-2)^2 r_{n-2})$$

at $n = 200$ with the conjectured growth rate. For $-1 < a < 0$ the agreement is evidence only. The slower convergence at $a = -\frac{1}{2}$ is consistent with the change of the critical exponent there found numerically in [EG17, Section 5.3].

a	conjectured growth rate	extrapolant at $n = 200$	relative gap
-1	8	7.999985	$-1.9 \cdot 10^{-6}$
$-3/4$	$49/6 \approx 8.166667$	8.166589	$-9.5 \cdot 10^{-6}$
$-1/2$	9	9.000336	$3.7 \cdot 10^{-5}$
$-1/4$	10.398979	10.398842	$-1.3 \cdot 10^{-5}$
0	$6 + 4\sqrt{2} \approx 11.656854$	11.656832	$-1.9 \cdot 10^{-6}$
1	16	15.999972	$-1.8 \cdot 10^{-6}$
2	$10 + 4\sqrt{6} \approx 19.797959$	19.797844	$-5.8 \cdot 10^{-6}$

Table 1: Ratio estimates of the growth rate of $Q_n(a)$ from the exact values for $n \leq 200$. The rows $a = -1$ and $a \geq 0$ are covered by [ABM15, Proposition 17] and by Theorem 1.1; the rows $-1 < a < 0$ are numerical evidence only.

Shuffle classes (an obstruction to one approach). For a word V over $\{N, S\}$ and a word H over $\{E, W\}$ let $S_a(V, H)$ be the sum of $a^{c(w)}$ over the shuffles w of V and H . A quadrant loop is a shuffle of two Dyck paths, one on $\{N, S\}$ and one on $\{E, W\}$ [ABM15, proof of Proposition 15], so $Q_n(a)$ is the sum of $S_a(V, H)$ over the pairs of Dyck paths with $|V| + |H| = 2n$. At $a = -1$, property (P_1) of [ABM15, Section 6.3], observed by Courtiel and listed there among the properties that may hold, says that $S_{-1}(V, H) = \binom{i+j}{i}$ for all bilateral Dyck paths (one-dimensional walks from 0 to 0) H and V of half-lengths i and j . According to its abstract, Kuzmaul's paper [Kus17] resolves this conjecture of [ABM15] on quarter-plane walks with fixed horizontal and vertical projections counted by their upper-right corners modulo 2 (the corners NW and ES are upper-right turns); we did not read its full text, and its exact scope, Dyck or bilateral Dyck paths, is not visible from the abstract. One might hope that $S_a(V, H) \geq 0$ for each pair of Dyck paths and $-1 < a < 0$, which would allow class-by-class comparisons. This fails. For the pair of Dyck paths $V = \text{NSNSNSNNSNSS}$, $H = \text{EEEEWEWWWWWWWW}$ one finds

$$S_{-13/20}(V, H) = -\frac{124660942032543}{256000000000} \approx -486.96,$$

and for the pair of bilateral Dyck paths

$$V = \text{SSNSNNSNSNNNNSSNS}, \quad H = \text{EEEEEEWWEEWWWWWWW},$$

in which V is not a Dyck path (so that its shuffle class contains no quadrant loop), $S_{-2/5}(V, H) = -63526557090248/1220703125 \approx -52040.96$. At $a = -1$ both sums equal $\binom{i+j}{i}$, as (P_1) predicts. This is in line with the observation (N_2) of [ABM15, Section 6.3] that $(a+1)$ -positivity fails inside a shuffle class. It is an obstruction to one approach only, and it says nothing about the sign or the growth of $Q_n(a)$.

6 Computations and reproducibility

No proof in this note depends on a computation. The programs and their recorded outputs are in the folder `reproducibility/` of the source archive.

The exact program of this note, `check_note.py` (Python 3, standard library only, deterministic, a few seconds), checks the finite and exact parts of the statements as printed: the identities of Lemmas 2.1, 2.2, 2.5 and 2.6, the junction identity of Lemma 2.3 and the diagonal symmetry on all short words (of length at most 8); the inequalities of Lemmas 2.1, 2.3, 2.4 and 2.6 exactly on small cases (for Lemma 2.4, the exact weights $B_n^x(k) = B_n^y(k)$ for $n \leq 9$, $k \leq 3$); the polynomials $Q_n(a)$ for $n \leq 30$ and $L_n(a)$ for $n \leq 12$, with the closed forms at $a = \pm 1$ [ABM15, Propositions 15 and 20] and $Q_n(a) \leq G(a)^n$ exactly at $a = 0, \frac{1}{2}, 1, 7$; the inequalities (4), (5) and those of the proof of Proposition 3.1 numerically; the constants of Section 5; the six-step identity of Remark 4.1(a) for $n \leq 10$; and the shuffle-class sums of Section 5 exactly. It also checks the recorded data described next against its own transfer matrix for $n \leq 30$, against the closed forms at $a = \pm 1$ for $n \leq 200$, and against $Q_n(a) \leq G(a)^n$ (exactly) for $n \leq 200$ at $a = 0, 1, 2$.

The programs with which the results were first obtained (original/): an integer transfer matrix for $Q_n(p/q)$, $n \leq 200$, at the seven values of Table 1 (it asserts the closed forms at $a = \pm 1$), the ratio analysis of Table 1, brute-force checks of the lemmas, and exploratory searches for negative shuffle-class sums. The random seeds of the two searches were not recorded; the two examples they found are recomputed exactly and deterministically by `check_note.py`.

Programs of the independent verification runs, each written from the definitions, with their recorded outputs, in the folders `verification_run_A/` and `verification_run_B/` (the first two runs) and `independent_run_2/` (the further run on the text of the note).

Verification. The result was first obtained, with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first run checked the statement against [BM14] and [ABM15] (the latter in the arXiv source, version v3, whose numbering we use), re-derived Lemmas 2.1 to 2.6 and the proof of Theorem 1.1 line by line, computed $Q_n(a)$ for $n \leq 36$ and $L_n(a)$ for $n \leq 12$ with its own transfer matrix (equal to the original data at all seven values of a), checked the lemmas on all short words and the bound of Lemma 2.4 exactly for $n \leq 9$, and searched for theorems on Markov-additive walks in cones that might imply the result. The second run re-derived the proof, recomputed all 7×200 original values with its own program (identical), tested every lemma on short words together with deliberately wrong variants that its tests detect, computed $Q_n(a)$ in floating point up to $n = 400$ for $0 \leq a \leq 100$, checked the six-step identity of Remark 4.1(a) for $n \leq 14$, recomputed both negative shuffle-class sums, re-ran the original programs with identical outputs, checked the bibliography with Crossref, and found the routes of Remark 4.1(a) and (b). Neither run found a gap. Their required corrections were made: the formulation of the y -case of Lemma 2.4, the remark that the bound there holds for any cut point, the citations of [DZ25, Pad26, Oza21, DW15, GR16], the qualification of what is known about [Kus17], the proof of Proposition 3.1, and the description of the close routes.

A further independent verification run, also AI-assisted, then checked the text of this note. It read the question again in [BM14] and in the arXiv source of [ABM15], recounted the shared theorem counter of [ABM15] and checked every number cited from it, and re-derived Lemmas 2.1 to 2.6, the proof of Theorem 1.1 and that of Proposition 3.1 line by line. With its own programs (folder `independent_run_2/`) it compared a brute force over all words for $n \leq 5$ with its own polynomials $Q_n(a)$, $n \leq 20$, and $L_n(a)$, $n \leq 12$, checked the closed forms at $a = \pm 1$, the bound $Q_n(a) \leq G(a)^n$ exactly for $n \leq 20$ at five values of a , the bound of Lemma 2.4 exactly for $n \leq 12$ and $k \leq 3$, the six-step identity of Remark 4.1(a) for $n \leq 10$ and the two shuffle-class sums, and recomputed all 7×200 original values and Table 1 with its own transfer matrix (identical). It re-ran all programs of

the archive from an extracted copy (identical outputs), repeated the literature searches and checked every DOI with Crossref. It found no mathematical error; its corrections concern the wording of Remark 3.2, the second shuffle-class example of Section 5, the attributions to [EG17], and the credit for the close routes in the next paragraph.

Scope and priority. The note answers the question of [BM14] and proves [ABM15, Conjecture 11] for $a \geq 0$ only; with [ABM15, Propositions 15 and 17] for $a = -1$ the question is settled on $\{-1\} \cup [0, \infty)$ and remains open for $-1 < a < 0$, including the range $[-1/3, 0)$ needed for two stacks in parallel. The question and the conjecture are due to Bousquet-Mélou [BM14] and to Albert and Bousquet-Mélou [ABM15]. From [ABM15] come the functional equation [ABM15, Theorem 7], the cases $a = 1$ and $a = -1$ [ABM15, Propositions 15 and 17], the inequality $\rho_Q(a) \geq (2 + \sqrt{2 + 2a})^{-2}$ for $a \geq 0$ [ABM15, Proposition 17], the radii for unconfined and half-plane loops for all $a \geq -1$ [ABM15, Propositions 21 and 22], of which our proof re-proves the first only for $a \geq 0$, the involution of Bernardi and Courtiél [ABM15, Proposition 14], and the properties (P_1) , (P_2) , (N_2) and [ABM15, Conjecture 10] discussed in Section 5. The prediction of the radius by Raschel, described in [ABM15] as not yet rigorous, and his conjectured formula for the exponent, reported in [EG17], are his; the numerical work of Elvey Price and Guttmann [EG17] and the result of Elvey Price [Elv19] are theirs. The tools are standard. Two close routes, found during the verification, are described in Remark 4.1: for $a \geq 1$, the reading of [ABM15, Theorem 7, Eq. (1)] as a weighted six-step walk together with the results of Denisov and Wachtel [DW15] (a sketch, not in print as far as we found), and the general theory of Ozawa [Oza21], whose irreducibility condition fails for this model; the results of Denisov and Zhang [DZ25] and of Padhi [Pad26] do not apply directly. Only the proof of the case $a \geq 0$, other than $a = 1$, is to our knowledge not in print; we claim no priority.

What was read: [BM14] on p. 711 of the report; [ABM15], [EG17] and [Elv19] in their arXiv sources; of [Kus17] the abstract; of [Oza21] the statement of the lemma used in Remark 4.1(b) and its conditions; of [Pad26] the statement of its excursion theorem and the surrounding text; of [DZ25] the abstract; of [DW15, GR16, Dur14] the bibliographic records only, and what we say about [DW15] in Remark 4.1(a) is a sketch. Not read in full: the thesis of Elvey Price [Elv18], which predates his statement in [Elv19] that the conjectures are unproved; the thesis of Trotignon [Tro19]; and the papers of Kenyon, Miller, Sheffield and Wilson [KMSW17, KMSW19], of which the abstracts were read and, for [KMSW17], the arXiv source by one verification run (Peano curves of six-vertex configurations; no lattice walks with corner weights).

The searches were made on 20 October 2026: the list of 22 works citing [ABM15] in Semantic Scholar (titles, and abstracts where relevant, among them arXiv:1807.08853 on quarter-plane walks with interacting boundaries and arXiv:2309.15209 on heat kernels in the quadrant), the 2 citing works in zbMATH, the 16 citing works in OpenAlex (this list was retrieved once; later requests were refused with HTTP status 429), queries to arXiv and Crossref (two stacks in parallel, quarter-plane or quadrant loops, corners and quadrant walks, Markov-additive and Markov-modulated walks in cones), and a few web searches; the further verification run repeated the queries to arXiv, Semantic Scholar, OpenAlex, Crossref and zbMATH and one web search, with the same outcome. No printed answer to the question was found in these sources; [ABM15, EG17, Elv19] treat the case as open. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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