

Cohomology Annihilators of Truncated Polynomial Rings

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19 October 2026 (version 1.1)

Abstract

We compute every cohomology-annihilator ideal and its least nonnegative stabilization index for $S[x]/(x^n)$, where $n \geq 2$ and S is a polynomial or formal power-series ring over any field. In base dimension zero, the eventual ideal is $(x^{\lfloor n/2 \rfloor})$ and the index is 1. In base dimension one, the ideal is (x^{n-1}) and the index is 2, in every characteristic. In base dimension $d \geq 2$, the ideal is (x^{n-1}) , with index $d + 1$, when n is invertible in the field; it is zero, with index 0, when the characteristic divides n . The dimension-one formula holds over every nonfield principal ideal domain. The proofs identify maps through projectives with an explicit transfer. In positive characteristic, a classical symmetric-power nilpotent operator yields a trace and colon-ideal obstruction; a finite-free induction then treats every divisible truncation degree. The field comparison also gives strict failures of equality under polynomial extension and localization.

1 Definitions and results

For a commutative noetherian ring R and an integer $s \geq 0$, set

$$\mathrm{ca}^s(R) = \bigcap_{M, L \in \mathrm{mod} R} \bigcap_{i \geq s} \mathrm{ann}_R \mathrm{Ext}_R^i(M, L), \quad \mathrm{ca}(R) = \bigcup_{s \geq 0} \mathrm{ca}^s(R).$$

Here $\mathrm{mod} R$ denotes the finitely generated modules. These are the conventions of [3, Definition 2.1 and Remark 2.3]. Dimension shifting shows that $\mathrm{ca}^s(R)$ is already the common annihilator in degree s . The ideals increase with s ; their chain stabilizes because R is noetherian. Also $\mathrm{ca}^0(R) = 0$, since $\mathrm{Hom}_R(R, R) = R$. We call the least *nonnegative* s with $\mathrm{ca}^s(R) = \mathrm{ca}(R)$ the stabilization index.

Throughout, $n \geq 2$, $R = S[x]/(x^n)$, and x also denotes its residue class. Our first calculation has no characteristic restriction.

Theorem 1. *If S is a nonfield principal ideal domain, then*

$$\mathrm{ca}^0(R) = \mathrm{ca}^1(R) = 0, \quad \mathrm{ca}^s(R) = (x^{n-1}) \quad (s \geq 2).$$

The stabilization index is 2. This includes $S = k[t]$, $S = k[[t]]$, and $S = \mathbb{Z}$.

Theorem 2. *Let $S = k[t_1, \dots, t_d]$ or $S = k[[t_1, \dots, t_d]]$, where k is a field and $d \geq 1$. If $n \neq 0$ in k , then*

$$\mathrm{ca}^s(R) = \begin{cases} 0 & 0 \leq s \leq d, \\ (x^{n-1}) & s \geq d + 1. \end{cases}$$

The stabilization index is $d + 1$.

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Theorem 3. *Let k have characteristic $p > 0$, let $p \mid n$, and let $d \geq 2$. For either $S = k[t_1, \dots, t_d]$ or $S = k[[t_1, \dots, t_d]]$, one has $\text{ca}(S[x]/(x^n)) = 0$. Every $\text{ca}^s(S[x]/(x^n))$ is zero, so the stabilization index is 0. No perfectness or algebraic-closure hypothesis is imposed on k .*

For comparison, over a field we will verify the classical formula

$$\text{ca}^s(k[x]/(x^n)) = (x^{\lfloor n/2 \rfloor}) \quad (s \geq 1).$$

These statements cover both base families in every dimension and characteristic. In particular, for double thickenings in characteristic 2, the eventual ideal is (x) in base dimensions 0 and 1, but zero in every base dimension at least 2. In other characteristics it is (x) in every base dimension.

The Oberwolfach discussion [4, pp. 476–478] asks about the size and stabilization of cohomology annihilators for noether algebras. It already allows examples with zero annihilator; it is not a universal nonvanishing conjecture. The present results compute explicit commutative subclasses, not that general research program. They include the known calculation $\text{ca}(k[[t, x]]/(x^2)) = (x)$ from [3, Example 2.8]. The low-degree and invertible-characteristic mechanisms are related to [3, Proposition 2.4 and Example 2.7]. We give direct proofs, including the sharp upper bounds. The classical symmetric-power operator used in Theorem 3 is credited to Friedlander and Pevtsova, and its rank-two specialization to Avramov, Buchweitz and Iyengar, where they enter.

2 The transfer and maps through projectives

Let S be any commutative noetherian ring. An R -module M which is finite projective over S is described by its underlying S -module and the endomorphism N giving the action of x . Thus $N^n = 0$. Define

$$T_N(V) = \sum_{j=0}^{n-1} N^j V N^{n-1-j}, \quad V \in \text{End}_S(M). \quad (1)$$

Lemma 4. *The R -endomorphisms of M which factor through a finite projective R -module are exactly the maps $T_N(V)$.*

Proof. First, $NT_N(V) - T_N(V)N = N^n V - V N^n = 0$, so $T_N(V)$ is R -linear. For an S -linear functional $b : M \rightarrow S$ and $v \in M$, define

$$f_b(w) = \sum_{i=0}^{n-1} x^i b(N^{n-1-i}w) : M \longrightarrow R, \quad g_v(x^i) = N^i v : R \longrightarrow M.$$

The relation $f_b(Nw) = x f_b(w)$ follows by shifting the index and using $N^n = x^n = 0$. These maps are therefore R -linear, and $g_v f_b = T_N(vb)$. A finite dual basis expresses each $V \in \text{End}_S(M)$ as a finite sum of maps vb . Consequently every $T_N(V)$ factors through a finite free R -module.

Conversely, write an R -linear map $f : M \rightarrow R$ uniquely as $f(w) = \sum_i x^i b_i(w)$. Comparing coefficients in $f(Nw) = x f(w)$ gives $b_i = b_{n-1} N^{n-1-i}$, hence $f = f_{b_{n-1}}$. Every composite $M \rightarrow R \rightarrow M$ is therefore a transfer. A factorization through R^q is a sum of these composites, and a finite projective R -module is a summand of a finite free one. $\square$

We next detect eventual annihilators on S -free modules. Choose $M = S^r$ and a basis, and put $Q = S[x]$. On Q^r set

$$\Phi = xI - N, \quad \Psi = \sum_{j=0}^{n-1} x^{n-1-j} N^j.$$

The identities $\Phi\Psi = \Psi\Phi = x^n I$ hold. The map Φ is injective because its leading coefficient is I . The map Ψ is also injective: its composite with Φ is multiplication by x^n , which is injective on

Q^r . Evaluation at N identifies $\text{coker}_Q \Phi$ with M ; division by the monic matrix $xI - N$ proves the kernel assertion.

Reduction modulo x^n gives the free resolution

$$\cdots \longrightarrow R^r \xrightarrow{\bar{\Phi}} R^r \xrightarrow{\bar{\Psi}} R^r \xrightarrow{\bar{\Phi}} R^r \longrightarrow M \longrightarrow 0. \quad (2)$$

For exactness, lift a vector in $\ker \bar{\Phi}$ to $v \in Q^r$. Write $\Phi v = x^n w = \Phi \Psi w$; injectivity gives $v = \Psi w$. The other kernel is treated using the injectivity of Ψ . In particular $\text{im } \bar{\Psi} \cong \text{coker } \bar{\Phi} \cong M$, so these syzygies satisfy $\Omega_R^2 M \cong M$.

Lemma 5. *If $c \in \text{ca}(R)$ and M is finite free over S , then multiplication by c on M belongs to $\text{im } T_N$.*

Proof. Choose s with $c \in \text{ca}^s(R)$, and $q \geq 0$ with $2q + 1 \geq s$. Set $K = \Omega_R M$. Periodicity gives the R -linear isomorphism

$$\text{Ext}_R^{2q+1}(M, K) \cong \text{Ext}_R^1(M, K),$$

so c annihilates the group on the right. Apply $\text{Hom}_R(M, -)$ to $0 \rightarrow K \rightarrow R^r \rightarrow M \rightarrow 0$. The connecting homomorphism sends cid_M to c times the extension class, hence to zero. Exactness lifts this endomorphism through R^r , and Lemma 4 applies. No vanishing assumption on $\text{Ext}_R^1(M, R)$ is needed. $\square$

3 An upper bound in positive base dimension

Proposition 6. *Suppose S is a domain with a nonzero nonunit a such that $\bigcap_{m \geq 1} a^m S = 0$. Then $\text{ca}(S[x]/(x^n)) \subseteq (x^{n-1})$.*

Proof. Let J_n be the n by n matrix with ones on the superdiagonal and zeros elsewhere. For each $m \geq 1$, take $M_m = S^n$ with x acting by $N_m = a^m J_n$. Every entry of $T_{N_m}(V)$ is divisible by $a^{m(n-1)}$. Write

$$c = \sum_{i=0}^{n-1} c_i x^i, \quad c_i \in S.$$

For $c \in \text{ca}(R)$, Lemma 5 implies $c(N_m) \in \text{im } T_{N_m}$. Its i -th superdiagonal has entries $c_i a^{mi}$. Since S is a domain, cancellation gives

$$c_i \in a^{m(n-1-i)} S \quad (0 \leq i < n-1).$$

Letting m vary forces $c_i = 0$ for every $i < n-1$. $\square$

This proposition applies to every base in the three theorems: choose a prime element of a nonfield PID, or t_1 in a polynomial or formal power-series ring of positive dimension. The required intersections vanish by divisibility or by comparison of monomials. No restriction on the characteristic is used.

4 The nonzero formulas and their indices

Proof of Theorem 1. Let L be a finite R -module and let $X = \Omega_R L$ be the kernel of a finite free cover. It is finite free over the PID S . Write N for the action of x on X . Since $\text{im } N$ is a submodule of a finite free S -module, it is free. Thus

$$0 \longrightarrow \ker N \longrightarrow X \longrightarrow \text{im } N \longrightarrow 0$$

splits over S . Choose an S -linear projection P onto $\ker N$. Then $NP = 0$ and $PN^{n-1} = N^{n-1}$, whence

$$T_N(P) = N^{n-1}.$$

By Lemma 4, multiplication by x^{n-1} on X factors through a projective R -module. It induces zero on $\text{Ext}_R^j(X, Y)$ for every finite Y and every $j \geq 1$. Dimension shifting gives $(x^{n-1}) \subseteq \text{ca}^2(R)$. Proposition 6 proves equality with $\text{ca}(R)$.

For sharpness, choose a prime element $a \in S$. Multiplication by a^m is injective on R , and the resulting two-term resolution gives

$$\text{Ext}_R^1(R/(a^m), R/(a^m)) \cong R/(a^m).$$

Hence $\text{ca}^1(R) \subseteq \bigcap_{m \geq 1} a^m R = 0$. Since $x^{n-1} \neq 0$, the least index is 2. $\square$

Proof of Theorem 2. In either case, S has global dimension d . An R -free resolution is also an S -projective resolution; therefore a d -th R -syzygy X is finite projective over S . For its endomorphism N giving the action of x , we have

$$T_N(n^{-1} \text{id}_X) = N^{n-1}.$$

Lemma 4 shows that x^{n-1} acts on X through a projective R -module. Shifting d times gives $(x^{n-1}) \subseteq \text{ca}^{d+1}(R)$, and Proposition 6 gives the reverse containment.

For $m \geq 1$, put $I_m = (t_1^m, \dots, t_d^m)R$ and $L_m = R/I_m$. These parameters are an R -regular sequence, since R is free over S . After applying $\text{Hom}_R(-, L_m)$ to their Koszul resolution, every differential is zero. In particular,

$$\text{Ext}_R^d(L_m, L_m) \cong L_m.$$

It follows that $\text{ca}^d(R) \subseteq \bigcap_m I_m = 0$. Indeed, in the S -basis $1, x, \dots, x^{n-1}$, a nonzero coefficient polynomial or power series has a nonzero monomial whose exponents are all less than m for sufficiently large m . Such an element cannot belong to $(t_1^m, \dots, t_d^m)$. The ascending property of the $\text{ca}^s(R)$ and the nonvanishing of x^{n-1} finish the proof. $\square$

5 Vanishing in every modular characteristic

Lemma 7. *Let F be a field and let $N \in M_q(F)$ have nilpotence index q . Let T_N denote the transfer with q summands. For every $V \in M_q(F)$,*

$$T_N(V) = \sum_{i=0}^{q-1} \text{tr}(VN^{q-1-i})N^i. \quad (3)$$

Consequently, if $q \geq 2$ and $T_N(V) = cN^{q-1}$, then $\text{tr}(V) = c$ and $\text{tr}(VN^{q-1}) = 0$.

Proof. Both sides commute with change of basis, so take $N = J_q$. Use indices from 0 to $q-1$. For a matrix position (a, b) with $b < a$, both sides vanish. For $b \geq a$, put $r = b - a$. The left side has entry

$$\sum_{h=0}^r V_{h+q-1-r, h} = \text{tr}(VJ_q^{q-1-r}),$$

which equals the right-side entry. The powers $I, N, \dots, N^{q-1}$ are linearly independent. Comparing their coefficients proves the last assertion. In characteristic dividing q , taking just the trace of the original transfer equation would not give this conclusion. $\square$

Fix the bases S and characteristic p in Theorem 3. Set $a = t_1$, $b = t_2$. For $m \geq 1$, put $A = a^m$, $B = b^m$ and consider the rank- p free S -module $E = S[u, v]_{p-1}$ of homogeneous polynomials of degree $p - 1$. Define

$$L = Au + Bv, \quad \delta = B\partial_u - A\partial_v, \quad N = L\delta.$$

Since $\delta(L) = 0$, one has $N^j = L^j\delta^j$, and hence $N^p = 0$. For a homogeneous polynomial f of degree $p - 1$,

$$\delta^{p-1}(f) = (p-1)! f(B, -A).$$

In the ordered basis $e_i = u^{p-1-i}v^i$, $0 \leq i \leq p-1$, the top power therefore has entries

$$\begin{aligned} (N^{p-1})_{ij} &= (p-1)! \binom{p-1}{i} (-1)^j A^{p-1-i+j} B^{p-1+i-j} \\ &= -(-1)^{i+j} A^{p-1-i+j} B^{p-1+i-j}. \end{aligned} \quad (4)$$

Here we used $(p-1)! = -1$ and $\binom{p-1}{i} = (-1)^i$ in characteristic p , including $p = 2$. The top power is nonzero, so over the fraction field of S , N is a single nilpotent block of dimension p . The off-diagonal entries in (4) lie in (A^p, B^p) , and the diagonal entries all equal $-(AB)^{p-1}$. Thus

$$N^{p-1} \equiv -(AB)^{p-1} I \pmod{(A^p, B^p)}. \quad (5)$$

Suppose $T_N(V) = cN^{p-1}$, with $V \in \text{End}_S(E)$ and $c \in S$. Lemma 7, applied over the fraction field, gives $\text{tr}(V) = c$ and $\text{tr}(VN^{p-1}) = 0$; these equalities hold in S . Equation (5) implies

$$0 = \text{tr}(VN^{p-1}) \equiv -c(AB)^{p-1} \pmod{(A^p, B^p)}.$$

Coefficientwise in either base ring,

$$((a^{mp}, b^{mp}) : a^{m(p-1)}b^{m(p-1)}) = (a^m, b^m).$$

We have proved the necessary condition

$$cN^{p-1} \in \text{im } T_N \implies c \in (a^m, b^m). \quad (6)$$

No sufficiency assertion for (6) is needed.

The representation and operator above are classical. They are the symmetric-power realization of the dimension- p Steinberg module and a parametrized nilpotent action. In the July 8, 2008 author version of [2], Example 5.7 and matrix (5.7.1), pp. 33–34, give this operator after setting their $\lambda = p-1$ and $(x, y, z) = (A^2, -B^2, AB)$; p. 42 identifies the Steinberg module. The cited construction is not asserted there to prove (6) or the cohomology-annihilator conclusion.

Proof of Theorem 3. Write $n = hp$, with $h \geq 1$, and use the injective map

$$C = S[y]/(y^p) \longrightarrow R = S[x]/(x^{hp}), \quad y \longmapsto x^h.$$

The ring R is free of rank h over C , with basis $1, x, \dots, x^{h-1}$. For each m , view the preceding E as a C -module with y acting as N , and put $M = R \otimes_C E$. It is finite free over S . As a C -module,

$$M = \bigoplus_{j=0}^{h-1} x^j \otimes E.$$

Let $\iota_0 : E \rightarrow M$ be inclusion of the first summand and $\pi_{h-1} : M \rightarrow E$ projection from the last. Both are C -linear.

Proposition 6 writes every $z \in \text{ca}(R)$ as $z = cx^{hp-1}$, with $c \in S$. Lemma 5 implies that multiplication by z on M factors through a finite projective R -module. Such a projective is also finite projective over C , since R is finite free over C . As $x^{hp-1} = x^{h-1}y^{p-1}$, the composite

$$\pi_{h-1} z \iota_0 = cN^{p-1}$$

therefore factors through a finite projective C -module. Lemma 4, now for C , and (6) yield $c \in (a^m, b^m)$. There is no factor h in this argument, so it also applies when $p \mid h$.

Finally, $\bigcap_{m \geq 1} (a^m, b^m) = 0$ in either base ring: a nonzero polynomial or power series has a nonzero monomial with finite a - and b -exponents, preventing membership when m exceeds both. Thus $c = 0$, proving $\text{ca}(R) = 0$. $\square$

5.1 The rank-two specialization

The case $p = n = 2$ has an exact obstruction and a direct positive-degree Ext interpretation.

Direct calculation for the double thickening. Write $a = t_1$, $b = t_2$, and retain any other variables in the polynomial or formal power-series base S . Proposition 6 implies that a cohomology annihilator has the form cx , with $c \in S$. For each $m \geq 1$, set $A = a^m$, $B = b^m$, and define

$$N_m = \begin{pmatrix} AB & A^2 \\ B^2 & AB \end{pmatrix}.$$

This is the characteristic-two specialization, after power substitution, of the square-zero matrix in Avramov, Buchweitz and Iyengar [1, Example 5.6]; the matrix construction itself is classical. Here $N_m = (A, B)^T(B, A)$ and $(B, A)(A, B)^T = 2AB = 0$, so $N_m^2 = 0$. It defines an R -module $M_m = S^2$.

By Lemma 5, if $cx \in \text{ca}(R)$, there is a matrix $V = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ over S with

$$N_m V + V N_m = cN_m.$$

Comparison of off-diagonal entries, followed by cancellation of A^2 or B^2 , gives $p + s = c$. The diagonal entries then give

$$B^2 q + A^2 r = cAB.$$

Consequently

$$c \in ((a^{2m}, b^{2m}) : a^m b^m) = (a^m, b^m). \quad (7)$$

The colon equality follows term by term from the monomial expansion, with any additional polynomial variables, or for power series. This is also the exact obstruction for the matrix equation: if $c = Au + Bv$, choose $p = c$, $s = 0$, $r = Bu$, and $q = Av$.

Since (7) holds for every m , $c \in \bigcap_m (a^m, b^m) = 0$. A nonzero polynomial or power series has a monomial with finite a - and b -exponents, excluding membership when m exceeds both. Thus $c = 0$, and $\text{ca}(R) = 0$. $\square$

The obstruction also has a direct positive-degree Ext description. In characteristic 2 with $n = 2$, the two matrices in (2) coincide: $\Phi = \Psi = xI + N_m$. Applying $\text{Hom}_R(-, M_m)$ gives the one-periodic complex of S -matrices with differential

$$\Delta(U) = N_m U + U N_m.$$

The identity matrix is a cycle. Multiplication by cx on its class is represented by cN_m , which is a boundary precisely when $c \in (a^m, b^m)$, by the same equations. If that condition fails, the witness survives in every positive cohomological degree.

There is no contradiction with the PID case. Here $\text{im } N_m \cong (a^m, b^m)$, a rank-one ideal which requires two generators after localizing at (a, b) . It is not projective, so the projection used in the proof of Theorem 1 need not exist. Under a convention restricting stabilization indices to positive integers, the all-zero chain in Theorem 3 has index 1, rather than 0.

6 The field case and change of base

Proposition 8. *For a field k and $A = k[x]/(x^n)$, $\text{ca}^s(A) = (x^{\lfloor n/2 \rfloor})$ for every $s \geq 1$. The stabilization index is 1.*

Proof. Every finite A -module is a direct sum of $A/(x^a)$, $1 \leq a \leq n$. Its standard resolution alternates multiplication by x^a and x^{n-a} . For $A/(x^b)$ as target, either positive parity of Ext is a cyclic module of length

$$\ell = \min(a, b, n - a, n - b).$$

For example, the odd group is

$$\frac{(x^{\max(0, a+b-n)})}{(x^{\min(a, b)})} \quad \text{inside } A/(x^b),$$

and the even group has the same length. Length zero gives the zero module and annihilator A . Intersecting all annihilators takes the largest possible ℓ , namely $\lfloor n/2 \rfloor$. The ideal is nonzero whereas $\text{ca}^0(A) = 0$. $\square$

Corollary 9. *For every field k , if $n \geq 3$ and $A = k[x]/(x^n)$, then*

$$\text{ca}(A[t]) = (x^{n-1}) \subsetneq \text{ca}(A)A[t] = (x^{\lfloor n/2 \rfloor}).$$

If $\text{char } k = p > 0$ and $p \mid n$, then

$$\text{ca}(k[a, b, x]/(x^n)) = 0, \quad \text{ca}(k(a, b)[x]/(x^n)) = (x^{\lfloor n/2 \rfloor}).$$

The first assertion follows from Theorem 1 and Proposition 8; the second follows from Theorem 3 and the field case. Thus cohomology annihilators need not commute with a smooth faithfully flat polynomial extension, and the localization inclusion of [3, Lemma 2.10] can be strict. These consequences are not claims that either general phenomenon is new. The calculations cover all $n \geq 2$, all dimensions, and all characteristics for the two stated base families, while the general noether-algebra program remains outside this scope.

The supplementary source archive contains the written algebraic argument and exact finite regression checks for the matrix identities, colon obstructions, induced-block constructions, and field-case lengths. Those computations are diagnostics, not a substitute for the universal proofs above. This unrefereed preprint was developed with AI assistance and self-audited; it has not undergone independent review or proof-assistant verification. No absolute priority claim is made.

References

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