Dirac Graphs Without a Spanning Near-Square of an Odd Cycle: A Negative Answer to a Question of Heinig

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Abstract

At the 2014 Oberwolfach workshop on combinatorics, P. Heinig asked whether, for every odd $n \geq 7$, every n-vertex graph with minimum degree at least $\lceil n/2 \rceil$ contains a spanning copy of the graph obtained from the square of an n-cycle by deleting every other edge on the periphery until exactly three consecutive vertices of degree 4 remain. A positive answer would have given a structural reason why the Hamilton cycles of such graphs generate their cycle space. We show that the answer is negative for every odd $n \geq 7$, under both natural readings of "periphery", and for $n = 9$ under every reading. The counterexample is the complete bipartite graph $K_{(n+1)/2,(n-1)/2}$ with a perfect matching, or a matching and one path with two edges, added inside the larger side; for $n = 7$ it is the graph that Heinig himself used as a positive example for the cycle-space question. The proof classifies the independent sets of size $(n-1)/2$ in Heinig's graph. The cycle-space question itself has since been settled for all large odd n by Hou and Yin, and it is not affected. This is an unrefereed note.

1 Introduction

A graph is *Hamilton-generated* if its Hamilton cycles generate its cycle space over $\mathbb{Z}/2$. Heinig [Hei14b] proved that for every $\gamma > 0$ and every sufficiently large odd n, every n-vertex graph with minimum degree at least $(1/2+\gamma)n$ is Hamilton-generated. In the problem session of the Oberwolfach workshop *Combinatorics* in January 2014 [KSS14, pp. 81–82] he asked whether the Dirac threshold $\lceil n/2 \rceil$ [Dir52] suffices for every odd n (his Question 1), and he proposed a structural reason for such a strengthening of Dirac's theorem.

> **Question 2** (Heinig [KSS14, p. 82]; paraphrased). For every odd $n \geq 7$, does every n-vertex graph with minimum degree at least $\lceil n/2 \rceil$ contain a spanning copy of the graph obtained from the square of the n-cycle by deleting "every other edge on the periphery" until exactly three consecutive vertices of degree 4 remain?

Heinig notes that the resulting graph has $n-3$ vertices of degree 3 and 3 vertices of degree 4, and that an affirmative answer to Question 2 implies an affirmative answer to Question 1. Question 1 was later settled for large n: Christoph, Nenadov and Petrova [CNP26] proved it for minimum degree $n/2 + C$, and Hou and Yin [HY25] proved that for all sufficiently large odd n, every Hamilton-connected n-vertex graph with minimum degree at least $(n-1)/2$ is Hamilton-generated. Every graph with minimum degree at least $\lceil n/2 \rceil = (n+1)/2$ is Hamilton-connected by Ore's theorem [Ore63], so this answers Question 1 for large odd n. The Hamilton cycle space has also been studied

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for random, random regular and randomly perturbed graphs and under other sufficient conditions for Hamiltonicity [Hei13, HK25a, HK25b, HK26]. We show that the structural route of Question 2 is not available at the Dirac threshold.

Theorem 1.1. *For every odd $n \geq 7$ there is an n-vertex graph G_n with minimum degree $\lceil n/2 \rceil$ that contains no spanning copy of Heinig's graph, under either of the two readings of "periphery" described in Section 2.*

The graph G_n is the near-extremal graph for Dirac's theorem: a complete bipartite graph $K_{s+1,s}$, where $n = 2s + 1$, with a perfect matching, or a matching and one path with two edges, added inside the part of size $s + 1$. For $n = 7$ it is the 4-regular complement of $C_4 \cup K_3$, which is the graph X of [Hei14b, Definition 28]. Heinig shows that X is Hamilton-generated [Hei14b, Proposition 30]; so his own positive example for Question 1 does not contain the structure of Question 2. Since the theorem holds for every odd $n \geq 7$, it also rules out any version of Question 2 for sufficiently large n. It does not contradict Question 1, since Question 2 only implies Question 1. For $n = 7$ and $n = 9$ the minimum degree in Theorem 1.1 is best possible for reading (R1) (Remark 3.6).

2 The graphs

Throughout, $n = 2s + 1$ with $s \geq 3$, and the vertex set is $\mathbb{Z}_n$. The square C_n^2 of the n-cycle has the *cycle edges* $c_i = \{i, i + 1\}$ and the *square edges* $q_i = \{i, i + 2\}$, $i \in \mathbb{Z}_n$. It is 4-regular with $2n$ edges. The word "periphery" admits two readings.

Reading (R1)*: the periphery is the n-circuit.* Delete the cycle edges $c_0, c_2, c_4, \ldots, c_{2s-4}$, that is, the $s - 1$ edges

$$D = \big\{\{2j, 2j + 1\} : 0 \leq j \leq s - 2\big\}.$$

The vertices $0, 1, \ldots, 2s - 3$ lose one edge each, and the three consecutive vertices $n - 3, n - 2, n - 1$ keep degree 4. Call the resulting graph H_n. Up to rotation and reflection it is the only graph obtained by deleting alternate edges of the n-circuit until three consecutive vertices of degree 4 remain. It has $(3n + 3)/2$ edges. This is the reading that matches [Hei14b]: for even n, deleting every other edge of the n-circuit from C_n^2 leaves exactly the prism $C_{n/2} \square K_2$, since the square edges form two cycles of length $n/2$ and the remaining cycle edges $\{2j + 1, 2j + 2\}$ join them in a matching compatible with both cycles. For odd n the alternation cannot close up, which is why three vertices of degree 4 remain.

Reading (R2)*: the periphery is the boundary of the Möbius band.* For odd n, C_n^2 is a triangulated Möbius band whose boundary is the single n-cycle

$$Q = (0, 2, 4, \ldots, n - 1, 1, 3, \ldots, n - 2)$$

formed by the square edges. Deleting every other edge of Q until three vertices that are consecutive on Q keep degree 4 gives a graph H'_n, again unique up to symmetry, with $n - 3$ vertices of degree 3, 3 vertices of degree 4 and $(3n + 3)/2$ edges. We use only two properties of H'_n: it contains every cycle edge c_i, and no two of its deleted square edges are consecutive on Q. We include this reading for robustness; see also Remark 3.5 for the broadest reading.

The host graph. Let $V(G_n) = A \cup B$ with $|A| = s + 1$ and $|B| = s$. Every vertex of A is adjacent to every vertex of B, the set B is independent, and $G_n[A]$ is a perfect matching if $s + 1$ is even, and a matching together with one path with two edges if $s + 1$ is odd. Every vertex of B has degree $s + 1$, and every vertex of A has degree $s + 1$ or $s + 2$. So G_n has minimum degree $s + 1 = \lceil n/2 \rceil$. In both cases $G_n[A]$ is a forest in which every path has at most two edges, and if $s + 1$ is even it is a matching.

3 Proof of Theorem 1.1

Lemma 3.1. *Let $H \in \{H_n, H'_n\}$. If H is isomorphic to a spanning subgraph of G_n, then H has an independent set J with $|J| = s$ such that $H - J$ is isomorphic to a subgraph of $G_n[A]$. Moreover $e(H - J) = 3 - |J \cap F|$, where F is the set of vertices of degree 4 in H.*

Proof. Let $\varphi\colon V(H) \to V(G_n)$ be a bijection that maps edges to edges, and put $J = \varphi^{-1}(B)$. As B is independent in G_n, the set J is independent in H, and φ maps $H - J$ into $G_n[A]$. Each edge of H meets J in at most one vertex, so

$$e(H - J) = e(H) - \sum_{v \in J} \deg_H v = \frac{3n+3}{2} - \bigl(3s + |J \cap F|\bigr) = 3 - |J \cap F|. \qquad \square$$

Lemma 3.2. *The graph H'_n has no independent set of size s.*

Proof. Since H'_n contains the n-cycle $(0, 1, \ldots, n-1)$, an independent set of size s in H'_n is also one in C_{2s+1}. Its complement then consists of $s + 1$ vertices, which split into s gaps between consecutive chosen vertices, each of size at least 1. So exactly one gap has size 2, and the set is $\{i, i+2, \ldots, i + 2(s-1)\}$ for some i. This set contains the $s - 1$ square edges $\{i + 2k, i + 2k + 2\}$, $0 \le k \le s - 2$, which are consecutive edges of Q. As $s - 1 \ge 2$, two of them are consecutive on Q, and at most one of these two was deleted. So the set is not independent in H'_n. $\square$

Lemma 3.3. *Let J be an independent set of size s in H_n.*

(a) *If s is odd, then $J \in \{J_0, J_2\}$, where*

$$J_0 = \{0, 1, 4, 5, \ldots, 2s-6, 2s-5\} \cup \{n-3\}, \qquad J_2 = \{2, 3, 6, 7, \ldots, 2s-4, 2s-3\} \cup \{n-1\}.$$

Then $H_n - J_0$ contains the path $(n-4, n-2, n-1)$, and $H_n - J_2$ contains the path $(n-3, n-2, 0)$.

(b) *If s is even, then $J = \{4k, 4k+1 : 0 \le k \le s/2 - 1\}$, and $H_n - J$ contains the triangle on $\{n-3, n-2, n-1\}$.*

Proof. Split J into maximal runs of cyclically consecutive vertices. A run of three vertices $x, x+1, x+2$ would contain the square edge $\{x, x+2\}$, so every run has one or two vertices. A run $\{x, x+1\}$ of two vertices needs the cycle edge c_x to be deleted, so it is a pair $\{2j, 2j+1\} \in D$. If a run ends at x, then $x + 1 \notin J$ by maximality and $x + 2 \notin J$ because of the square edge $\{x, x+2\}$. So between two consecutive runs there are at least two vertices outside J.

Let r_1 and r_2 be the numbers of runs of one and of two vertices. Then $s = r_1 + 2r_2$, and counting the gaps gives $n \ge s + 2(r_1 + r_2)$. Hence $2r_2 \ge s - 1$, and the gaps together contain exactly $2r_2 - s + 1$ more vertices than two per gap. Two runs $\{2j, 2j+1\}$ and $\{2j', 2j'+1\}$ with $j < j'$ need $2j' \ge 2j + 4$, that is, $j' \ge j + 2$. As $0 \le j \le s - 2$, this gives $r_2 \le \lceil (s-1)/2 \rceil$.

(a) Let s be odd. Then $r_2 = (s-1)/2$ and $r_1 = 1$, and every gap has exactly two vertices. Going around the cycle, the pattern consists of blocks "pair, gap" of length 4 and one block "singleton, gap" of length 3. So, going around the cycle, consecutive starting points of pairs differ by 4, except across the block of the singleton, where they differ by 7. Every starting point is even and lies in $\{0, 2, \ldots, 2s-4\}$. For even p, the numbers $p + 4$ and $p + 7 - n$ are even, while $p + 7$ and $p + 4 - n$ are odd, because n is odd. Hence the steps of 4 do not pass from $n - 1$ to 0, and the step of 7 does. So the starting points are $p, p+4, \ldots, p + 2s - 6$ with $p \ge 0$ even and $p + 2s - 6 \le 2s - 4$. Thus $p \in \{0, 2\}$, which gives J_0 and J_2.

For J_0 the gaps are the pairs $\{4k+2, 4k+3\}$, $0 \le k \le (s-3)/2$, which lie in D, and the pair $\{n-2, n-1\}$, which does not. The singleton $n-3$ lies between the gaps $\{n-5, n-4\}$ and $\{n-2, n-1\}$.

So $H_n - J_0$ contains the square edge $\{n-4, n-2\}$ and the cycle edge $\{n-2, n-1\}$. For J_2 the gaps are $\{0,1\}$ and the pairs $\{4k+4, 4k+5\}$, $0 \le k \le (s-5)/2$, all in D, and $\{n-3, n-2\}$, which is not in D. The singleton $n-1$ lies between $\{n-3, n-2\}$ and $\{0,1\}$. So $H_n - J_2$ contains the cycle edge $\{n-3, n-2\}$ and the square edge $\{n-2, 0\}$.

(b) Let s be even. Then $r_2 \ge s/2$ and $r_2 \le \lceil (s-1)/2 \rceil = s/2$, so $r_2 = s/2$ and $r_1 = 0$. The pairs $\{2j, 2j+1\}$ have indices j in $\{0, \dots, s-2\}$ with differences at least 2, and there are $s/2$ of them. This forces $j = 0, 2, \dots, s-2$, that is, $J = \{4k, 4k+1 : 0 \le k \le s/2 - 1\}$. The vertices $n-3, n-2, n-1$ are not in J. In H_n they span the cycle edges $\{n-3, n-2\}$ and $\{n-2, n-1\}$, which are not in D, and the square edge $\{n-3, n-1\}$. $\square$

Proof of Theorem 1.1. Suppose that $H \in \{H_n, H'_n\}$ is isomorphic to a spanning subgraph of G_n, and let J be as in Lemma 3.1. By Lemma 3.2, $H = H_n$. If s is odd, then $s+1$ is even and $G_n[A]$ is a perfect matching, but by Lemma 3.3(a) the graph $H_n - J$ contains a path with two edges. If s is even, then $G_n[A]$ is a forest, but by Lemma 3.3(b) the graph $H_n - J$ contains a triangle. Both are impossible. $\square$

Remark 3.4. The count of Lemma 3.1 agrees with Lemma 3.3: in case (a), $|J \cap F| = 1$ and $H_n - J$ has exactly the two edges found above; in case (b), $J \cap F = \emptyset$ and $H_n - J$ has exactly the three edges of the triangle. For $n = 7$ the graph H_7 has exactly the two independent 3-sets $\{0,1,4\}$ and $\{2,3,6\}$, and removing either one leaves a path with two edges and an isolated vertex. For $n = 5$ the graph G_5 has 8 edges and Heinig's graph has 9, so there is no spanning copy either.

Remark 3.5. Any reading of Heinig's construction deletes edges from the 4-regular graph C_n^2 so that $n-3$ vertices end with degree 3 and three with degree 4. So each vertex loses at most one edge, and the deleted edges form a matching of size $(n-3)/2$. If an arbitrary Hamilton cycle of C_n^2 is allowed as the periphery, G_7 is no longer a counterexample: it contains $C_7^2 - \{\{0,1\},\{3,5\}\}$. But for $n = 9$ the graph $K_{4,4,1}$, which has minimum degree $5 = \lceil 9/2 \rceil$, contains none of the 177 graphs obtained from C_9^2 by deleting a matching of three edges (computer check). Hence the universal statement of Question 2 fails under every reading.

Remark 3.6. For $n = 7$ and $n = 9$, every n-vertex graph with minimum degree at least $\lceil n/2 \rceil + 1$ contains a spanning copy of H_n. The complements of such graphs have maximum degree at most 1, resp. 2, and a computer check of all of them (up to isomorphism) confirms this. So for these n the threshold for reading (R1) is exactly $\lceil n/2 \rceil + 1$.

4 Verification and scope

The proof above is complete. The following exact computations, with scripts in the source archive, were used as checks.

1. The degree sequences of H_n and H'_n and the minimum degree of G_n, for all odd $7 \le n \le 61$.

2. For $n = 7$ and $n = 9$ and both readings, a brute-force search over all bijections $V(H) \to V(G_n)$: no spanning copy exists.

3. For all odd $7 \le n \le 61$ and both readings, a complete enumeration of the independent $(n-1)/2$-sets of H_n and H'_n. Reading (R1) has exactly two such sets for $n \equiv 3 \pmod 4$ and exactly one for $n \equiv 1 \pmod 4$, as in Lemma 3.3, and none of them leaves a subgraph of $G_n[A]$. Reading (R2) has none.

4. An independent SAT encoding of "H is a spanning subgraph of G_n", without any structural reduction, is unsatisfiable for all odd $7 \le n \le 25$ and both readings.

5. An independent audit, with code written from scratch, repeated the structural reduction for all odd $5 \leq n \leq 101$, a second SAT encoding for $7 \leq n \leq 61$, and a backtracking search for $7 \leq n \leq 25$, with positive controls; it also checked that G_7 is the graph X of [Hei14b].

6. The computations behind Remarks 3.5 and 3.6, by exact backtracking.

Scope and priority. The theorem answers Question 2 of Heinig [KSS14, p. 82] in the negative, as stated, for every odd $n \geq 7$ under both readings of "periphery", and for $n = 9$ under every reading (Remark 3.5). It says nothing against Question 1, which Hou and Yin [HY25] settled for large odd n. The later papers [CNP26, HY25, HK25a, HK25b, HK26] on the Hamilton cycle space do not discuss Question 2, and searches of arXiv, Crossref, OpenAlex and Semantic Scholar in September 2026 found no other treatment of it. We could not consult Heinig's dissertation [Hei14a]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the construction and the proof, to write and run the verification programs, to referee the argument, and to help prepare this manuscript. The author has reviewed the proof and the sources and is responsible for all claims. The note has not been peer reviewed.

References

[CNP26] M. Christoph, R. Nenadov, and K. Petrova. The Hamilton space of pseudorandom graphs. *Journal of Combinatorial Theory, Series B*, 176:254–267, 2026. arXiv:2402.01447.

[Dir52] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society (3)*, 2:69–81, 1952.

[Hei13] P. C. Heinig. When Hamilton circuits generate the cycle space of a random graph. arXiv:1303.0026, 2013.

[Hei14a] P. C. Heinig. *Global consequences of local structure.* PhD thesis, Technische Universität München, 2014.

[Hei14b] P. C. Heinig. On prisms, Möbius ladders and the cycle space of dense graphs. *European Journal of Combinatorics*, 36:503–530, 2014.

[HK25a] D. Hefetz and M. Krivelevich. The Hamilton cycle space of random graphs. arXiv:2506.19731, 2025.

[HK25b] D. Hefetz and M. Krivelevich. The Hamilton cycle space of random regular graphs and randomly perturbed graphs. arXiv:2507.04488, 2025.

[HK26] D. Hefetz and M. Krivelevich. On graphs whose cycle space is spanned by their Hamilton cycles. arXiv:2606.05835, 2026.

[HY25] X. Hou and Z. Yin. Dirac-type condition for Hamilton-generated graphs. arXiv:2503.15950, 2025.

[KSS14] J. Kahn, A. Steger, and B. Sudakov. Combinatorics. *Oberwolfach Reports*, 11(1):5–90, 2014. Report No. 01/2014, organisers J. Kahn, A. Steger and B. Sudakov; problem session, contribution of P. Heinig, pp. 81–82.

[Ore63] O. Ore. Hamiltonian connected graphs. *Journal de Mathématiques Pures et Appliquées (9)*, 42:21–27, 1963.