

Components and Cohomology of Fock's Baby Teichmüller Space

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Abstract

At an Oberwolfach problem session in 2006, V. Fock asked for the cohomology and the number of components of the space B_n of maps $f: \mathbb{Z}/n \rightarrow \mathbb{RP}^1$ with $f(i) \neq f(i+1)$ and at least three values, modulo $\mathrm{SL}(2, \mathbb{R})$. We answer both questions for all $n \geq 3$. The space B_n is a real-analytic $(n-3)$ -manifold with exactly $n-1$ components $B_{n,k}$, indexed by the winding number k . For $k \neq n/2$ they are Hausdorff and contractible, and for $k=1$ and $k=n-1$ they are the Teichmüller space of the ideal n -gon. For even n the middle component is not Hausdorff: two distinct points have no disjoint neighbourhoods exactly when one is represented by a map constant on the even positions and the other by a map constant on the odd positions, and both loci are spheres $S^{n/2-2}$. The middle component is weakly homotopy equivalent to S^{n-3} . Hence $H^0(B_n; \mathbb{Z}) \cong \mathbb{Z}^{n-1}$, $H^{n-3}(B_n; \mathbb{Z}) \cong \mathbb{Z}$ for even n , and all other singular cohomology vanishes; the cohomology of the constant sheaf agrees, and so does its Čech cohomology on the Hausdorff components and, on the middle component, in degrees at most 2. For $n=4$ the de Rham cohomology of smooth forms differs, and $B_{4,2}$ is weakly homotopy equivalent to a circle but not homotopy equivalent to it. Modulo $\mathrm{PGL}(2, \mathbb{R})$ there are $\lfloor n/2 \rfloor$ components, and for even n the middle one is weakly homotopy equivalent to $\mathbb{RP}^{n-3}$; for odd n the space of real tame frieze patterns of width $n-3$ has $(n-1)/2$ components, all contractible. This is an unrefereed note.

1 Introduction

Let $n \geq 3$. Let X_n be the set of maps $f: \mathbb{Z}/n \rightarrow \mathbb{RP}^1$ with $f(i) \neq f(i+1)$ for all $i \in \mathbb{Z}/n$, an open subset of $(\mathbb{RP}^1)^n$, and let $X_n^* \subset X_n$ be the open subset of maps with at least three values. The group $\mathrm{SL}(2, \mathbb{R})$ acts on X_n^* by $g \cdot f = g \circ f$, through $G = \mathrm{PSL}(2, \mathbb{R})$. At the problem session of the Oberwolfach workshop *Teichmüller Space (Classical and Quantum)* in June 2006, V. Fock called $B_n = X_n^*/\mathrm{SL}(2, \mathbb{R})$ the *baby Teichmüller space*. He asked “What is the cohomology of this space?”, and in particular how many components it has [MPP06, p. 1607, Problem 6]. This is record OWR-1275-010 of the UnsolvedMath dataset [ula26]. We give B_n the quotient topology, with quotient map $q: X_n^* \rightarrow B_n$.

Orient $\mathbb{RP}^1$. For $f \in X_n$ let $\theta_i(f) \in (0, 2\pi)$ be the length of the positively oriented arc from $f(i)$ to $f(i+1)$, for the rotation-invariant length of total length 2π . Then $\sum_i \theta_i(f) = 2\pi k(f)$ for an integer $1 \leq k(f) \leq n-1$, the *winding number* of f . It is G -invariant, and we put $B_{n,k} = q(\{f \in X_n^* : k(f) = k\})$. For even n let $\mathcal{E}_n$ and $\mathcal{O}_n$ be the sets of classes of maps that are constant on the even positions $0, 2, \dots, n-2$ and on the odd positions, respectively.

Theorem 1.1. *Let $n \geq 3$.*

- (i) *B_n is a second countable real-analytic manifold of dimension $n-3$; it is Hausdorff if and only if n is odd. Its components are the $n-1$ nonempty, open, path-connected sets $B_{n,k}$, $1 \leq k \leq n-1$, and a reflection of $\mathbb{RP}^1$ induces homeomorphisms $B_{n,k} \cong B_{n,n-k}$.*

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- (ii) If $2k \neq n$, then $B_{n,k}$ is Hausdorff and contractible. $B_{n,1}$ and $B_{n,n-1}$ consist of the classes of maps whose n values are distinct and in cyclic order, and they are homeomorphic to $\mathbb{R}^{n-3}$, the Teichmüller space of the ideal n -gon.
- (iii) Let n be even. Then $\mathcal{E}_n$ and $\mathcal{O}_n$ are disjoint closed subsets of $B_{n,n/2}$, homeomorphic to $S^{n/2-2}$. Two distinct points of B_n have no disjoint neighbourhoods if and only if one lies in $\mathcal{E}_n$ and the other in $\mathcal{O}_n$. The component $B_{n,n/2}$ is weakly homotopy equivalent to S^{n-3} .
- (iv) Let A be an abelian group. Then $H^0(B_n; A) \cong A^{n-1}$, $H^{n-3}(B_n; A) \cong A$ if n is even, and all other singular cohomology groups with coefficients in A vanish; the same holds for singular homology. The cohomology of the constant sheaf A on B_n is isomorphic to singular cohomology. The Čech cohomology of the constant sheaf A agrees with it on each Hausdorff component and, on $B_{n,n/2}$, in degrees at most 2; in every degree the natural map from Čech to sheaf cohomology is surjective.
- (v) The de Rham cohomology of smooth forms on $B_{4,2}$ vanishes in degree 1, whereas $H^1(B_{4,2}; \mathbb{R}) \cong \mathbb{R}$. The space $B_{4,2}$ is weakly homotopy equivalent to S^1 but not homotopy equivalent to S^1 .

Corollary 1.2. (a) $\mathrm{PGL}(2, \mathbb{R})$ acts freely on X_n^* , and $B_n^{\mathrm{PGL}} = X_n^*/\mathrm{PGL}(2, \mathbb{R})$ has $\lfloor n/2 \rfloor$ components. For $1 \leq k < n/2$ the image of $B_{n,k} \cup B_{n,n-k}$ is a component homeomorphic to $B_{n,k}$. For even n the image of $B_{n,n/2}$ is a non-Hausdorff component that is weakly homotopy equivalent to $\mathbb{RP}^{n-3}$.

- (b) Let n be odd, and for $c \in \mathbb{R}^n$ let $M(c) = \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix}$. The space $\mathcal{F}_n = \{c \in \mathbb{R}^n : M(c) = -\mathrm{Id}\}$ of real tame frieze patterns of width $n-3$ [MGOST14, Thm. 3.4.1(i)] is homeomorphic to B_n^{PGL} and to the union of the $B_{n,k}$ with k odd (orientation as in Section 2). It has $(n-1)/2$ components, each a contractible $(n-3)$ -manifold, and the Conway–Coxeter friezes lie in the one with $k=1$.

Context and credits. Fock–Goncharov coordinates identify $B_{n,1}$ with the positive real points of a cluster Poisson variety [FG06, FG16], and B_n drops positivity; compare Fock’s abstract in the same report [MPP06, pp. 1550–1551]. The coordinate charts of the triangulations cover B_n^{PGL} and are glued in a non-separated way (Section 6.1); non-separated gluings of coordinate tori also occur for positive atlases [FG16, §1.2]. The decomposition by winding number is elementary. For solutions of $M(c) = \pm \mathrm{Id}$ in positive integers, Ovsienko [Ovs18, §5] introduced a rotation index, which can be read as the homotopy class of the associated n -gon in $\mathbb{RP}^1$; it is an integer or a half-integer according to the sign of the monodromy. For odd n , Morier-Genoud, Ovsienko, Schwartz and Tabachnikov proved that tame friezes correspond to n -gons in $\mathbb{RP}^1$ up to projective equivalence [MGOST14, Thm. 3.4.1(ii)], that is, to B_n^{PGL} . Positive integral friezes are due to Coxeter and to Conway and Coxeter [Cox71, CC73a, CC73b].

2 Coordinates and components

A point of $\mathbb{RP}^1$ is a line $[v_1 : v_2] \subset \mathbb{R}^2$, with affine coordinate $x = v_1/v_2 \in \mathbb{R} \cup \{\infty\}$; so $0 = [0 : 1]$, $\infty = [1 : 0]$, and $\mathrm{SL}(2, \mathbb{R})$ acts by Möbius transformations. We orient $\mathbb{RP}^1$ by the counterclockwise rotation of lines and use the orientation-preserving homeomorphism $\omega : \mathbb{RP}^1 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$, $\omega([\cos \varphi : \sin \varphi]) = 2\varphi$. The group $K = \mathrm{PSO}(2) \subset G$ acts by the translations $\omega \mapsto \omega + 2\alpha$ (α the rotation angle), simply transitively; G is connected, so it preserves the orientation. Thus $\theta_i(f) \in (0, 2\pi)$ is the representative of $\omega(f(i+1)) - \omega(f(i))$. Vectors in $\mathbb{R}^n$ are indexed by $\mathbb{Z}/n$. For $1 \leq k \leq n-1$ let

$$\Delta_k = \left\{ \theta \in (0, 2\pi)^n : \sum_i \theta_i = 2\pi k \right\},$$

a nonempty open convex subset of the hyperplane $H_k = \{\sum_i \theta_i = 2\pi k\}$; it contains the constant vector $2\pi k/n$. For even n put $\theta^\circ = (\pi, \dots, \pi)$ and $\ell = (1, -1, \dots, 1, -1)$, and let $L = \Delta_{n/2} \cap (\theta^\circ + \mathbb{R}\ell)$, the open segment of the points $(t, 2\pi - t, \dots, t, 2\pi - t)$, $0 < t < 2\pi$; it is closed in $\Delta_{n/2}$. Let $\Delta'_k = \Delta_k$ if $2k \neq n$, and $\Delta'_{n/2} = \Delta_{n/2} \setminus L$.

Lemma 2.1. (a) $\Phi: X_n \rightarrow (\mathbb{R}/2\pi\mathbb{Z}) \times \bigsqcup_{k=1}^{n-1} \Delta_k$, $\Phi(f) = (\omega(f(0)), \theta(f))$, is a homeomorphism, and K acts only on the first factor, by translations.

(b) The winding number is locally constant on X_n and G -invariant.

(c) $f \in X_n$ has at most two values if and only if n is even and $\theta(f) \in L$. So Φ maps $X_{n,k}^* := \{f \in X_n^* : k(f) = k\}$ homeomorphically onto $(\mathbb{R}/2\pi\mathbb{Z}) \times \Delta'_k$.

(d) The sets $B_{n,k} = q(X_{n,k}^*)$ are nonempty, open, closed and path-connected, and $B_n = \bigsqcup_k B_{n,k}$. A reflection $\rho \in \text{PGL}(2, \mathbb{R})$, such as $x \mapsto -x$, induces a homeomorphism of B_n mapping $B_{n,k}$ onto $B_{n,n-k}$.

Proof. (a) The inverse sends (s, θ) to the map with $\omega(f(i)) = s + \theta_0 + \dots + \theta_{i-1}$, $0 \leq i \leq n-1$, which lies in X_n because $\sum_i \theta_i \in 2\pi\mathbb{Z}$ and $\theta_i \notin 2\pi\mathbb{Z}$. (b) k is continuous and integer-valued, and $g \mapsto k(gf)$ is constant on the connected group G . (c) A map with at most two values alternates between two points, since adjacent values differ; then n is even and $\theta(f) = (t, 2\pi - t, \dots)$. Conversely, $\theta(f) \in L$ gives $f(i+2) = f(i)$ for all i . (d) The sets $X_{n,k}^*$ are open, disjoint, G -invariant and cover X_n^* , and quotient maps of group actions are open. By (c), $B_{n,k}$ is an image of $(\mathbb{R}/2\pi\mathbb{Z}) \times \Delta'_k$, which is nonempty as $\dim \Delta_k = n-1 > 1$ and path-connected because Δ_k is convex and $\Delta'_{n/2} \simeq S^{n-3}$ (Proposition 3.4(b)). The reflection ρ normalizes G and reverses the orientation, so $\theta_i(\rho \circ f) = 2\pi - \theta_i(f)$. $\square$

3 Local structure and two bundles

Lemma 3.1. For a triple $\tau = (t_1, t_2, t_3)$ of distinct elements of $\mathbb{Z}/n$ let $U_\tau \subset X_n^*$ consist of the maps with $f(t_1), f(t_2), f(t_3)$ pairwise distinct, and $S_\tau \subset U_\tau$ of those with $(f(t_1), f(t_2), f(t_3)) = (0, \pm 1, \infty)$. Then G acts freely on X_n^* , and $G \times S_\tau \rightarrow U_\tau$, $(g, s) \mapsto g \cdot s$, is a G -equivariant homeomorphism. Hence $q(U_\tau) \cong S_\tau$, an open subset of $\{\pm 1\} \times (\mathbb{RP}^1)^{n-3}$; with these charts B_n is a second countable real-analytic manifold of dimension $n-3$, and q is a principal G -bundle, trivial over each $q(U_\tau)$.

Proof. G acts simply transitively on each of the two orbits of ordered triples of distinct points (the two cyclic orientations), and $(0, 1, \infty), (0, -1, \infty)$ lie in different orbits. For $f \in U_\tau$ let $\gamma(f) \in G$ satisfy $\gamma(f) \cdot (0, \pm 1, \infty) = (f(t_1), f(t_2), f(t_3))$; it is given by a rational formula, so it is continuous, and $\gamma(gf) = g\gamma(f)$. Hence $f \mapsto (\gamma(f), \gamma(f)^{-1}f)$ is the inverse map. A Möbius transformation fixing three points is the identity, and the sets U_τ cover X_n^* ; this gives freeness. As q is open, $q(U_\tau) \cong U_\tau/G \cong S_\tau$, where only the open conditions $f(i) \neq f(i+1)$ remain; chart changes $s \mapsto \gamma'(s)^{-1}s$ are rational. $\square$

Lemma 3.2. Let $p: Y \rightarrow Z$ be a fibre bundle with contractible fibre F , trivial over the sets of an open cover of Z ; no separation axiom is assumed. Then p is a weak homotopy equivalence and induces isomorphisms in singular homology and cohomology with arbitrary coefficients.

Proof. By [Hat02, Prop. 4.48], p has the homotopy lifting property with respect to disks; the proof subdivides a cube into small cubes mapped into trivializing sets and uses no property of Z . As the fibres are contractible, the exact sequence of [Hat02, Thm. 4.41] shows that $p_*: \pi_j(Y, y) \rightarrow \pi_j(Z, p(y))$ is bijective for $j \geq 1$, and path lifting shows that p induces a bijection of path components. Now apply [Hat02, Prop. 4.21]. $\square$

Proposition 3.3. (a) Let $\widehat{B}_n = X_n^*/K$. The induced map $\pi: \widehat{B}_n \rightarrow B_n$ is a fibre bundle with fibre $K \backslash G \cong \mathbb{R}^2$, trivial over each $q(U_\tau)$, and Φ induces a homeomorphism $\widehat{B}_n \cong \bigsqcup_k \Delta'_k$ with $\pi^{-1}(B_{n,k}) = \Delta'_k$.

(b) Let $C_n = \{f \in X_n^* : f(0) = \infty, f(1) = 0\}$, and let $D = \{x \mapsto \lambda x : \lambda > 0\} \cong \mathbb{R}$ be the stabilizer of $(\infty, 0)$ in G . Then $C_n \rightarrow C_n/D$ is a principal D -bundle, the inclusion induces a homeomorphism $C_n/D \cong B_n$, and Φ maps C_n homeomorphically onto $\bigsqcup_k Y'_k$, where $Y_k = \{\theta \in \Delta_k : \theta_0 = \pi\}$, $Y'_k = Y_k$ for $2k \neq n$, and $Y'_{n/2} = Y_{n/2} \setminus \{\theta^\circ\}$.

Proof. (a) Under $U_\tau \cong G \times S_\tau$, K acts by left multiplication on G , as $\gamma(\kappa f) = \kappa\gamma(f)$. So $\pi^{-1}(q(U_\tau)) \cong (K \backslash G) \times S_\tau$, and $K \backslash G$ is the hyperbolic plane. By Lemma 2.1, $\widehat{B}_n \cong \bigsqcup_k \Delta'_k$. (b) As $\omega(\infty) = 0$ and $\omega(0) = \pi$, $\Phi(C_n)$ is the set of $(0, \theta)$ with $\theta \in \Delta'_k$ and $\theta_0 = \pi$, and $L \cap \{\theta_0 = \pi\} = \{\theta^\circ\}$. For $2 \leq i \leq n-1$ the D -invariant open sets $C_{n,i} = \{f \in C_n : f(i) \notin \{0, \infty\}\}$ cover C_n (every $f \in C_n$ has a third value), and $f \mapsto (|x(f(i))|, |x(f(i))|^{-1} \cdot f)$ identifies $C_{n,i}$ with $\mathbb{R}_{>0} \times \{f \in C_n : x(f(i)) = \pm 1\}$. The map $C_n/D \rightarrow B_n$ is continuous and bijective, since G acts transitively on ordered pairs of distinct points with stabilizer D . For the inverse, fix $z \in \mathbb{RP}^1$; for distinct $y_0, y_1 \neq z$ the element $s(y_0, y_1) \in G$ mapping $(\infty, 0, \pm 1)$ to (y_0, y_1, z) depends continuously on (y_0, y_1) , and $f \mapsto D \cdot s(f(0), f(1))^{-1}f$, defined where $f(0), f(1) \neq z$, is continuous, G -invariant and independent of z . $\square$

Proposition 3.4. (a) For $2k \neq n$ the component $B_{n,k}$ is weakly contractible.

(b) Let n be even, $T = \{v \in \mathbb{R}^n : \sum_i v_i = 0\}$, $U \subset T$ a linear complement of $\mathbb{R}\ell$ with a norm, and write $\theta - \theta^\circ = \lambda(\theta)\ell + u(\theta)$ with $u(\theta) \in U$. Choose $\varepsilon > 0$ with $\theta^\circ + \varepsilon u \in \Delta_{n/2}$ whenever $|u| = 1$. Then $r(\theta) = u(\theta)/|u(\theta)|$ is a homotopy equivalence from $\Delta'_{n/2}$ to the unit sphere $S(U) \cong S^{n-3}$, with homotopy inverse $s(u) = \theta^\circ + \varepsilon u$. Also $Y_{n/2} \setminus \{\theta^\circ\} \simeq S^{n-3}$, and $B_{n,n/2}$ is weakly homotopy equivalent to S^{n-3} .

Proof. By Lemma 3.2, applied to the restrictions of the two bundles of Proposition 3.3, $B_{n,k}$ is weakly homotopy equivalent to Δ'_k and to Y'_k . The sets Δ_k and Y_k are convex and nonempty; Y_k contains the point with $\theta_0 = \pi$ and $\theta_i = (2\pi k - \pi)/(n-1)$ for $i \neq 0$. This gives (a) twice. In (b), $\theta \in L$ if and only if $u(\theta) = 0$. We have $r \circ s = \text{id}$, and $H_t(\theta) = (1-t)\theta + t s(r(\theta))$ lies in $\Delta_{n/2}$ by convexity and has $u(H_t(\theta)) = (1-t + t\varepsilon/|u(\theta)|)u(\theta) \neq 0$; so H_t is a homotopy in $\Delta'_{n/2}$ from the identity to $s \circ r$. The point θ° lies in the relative interior of the $(n-2)$ -dimensional convex set $Y_{n/2}$, so radial projection deforms $Y_{n/2} \setminus \{\theta^\circ\}$ onto a small sphere. The two bundles thus give two proofs of (a) and (b), which share only Lemmas 2.1, 3.1 and 3.2. $\square$

4 Separation properties

Let n be even, and let $\widetilde{\mathcal{E}}_n$ and $\widetilde{\mathcal{O}}_n$ be the sets of $f \in X_n^*$ that are constant on the even, respectively odd, positions. They are closed, G -invariant and disjoint, and $f \in \widetilde{\mathcal{E}}_n$ (resp. $\widetilde{\mathcal{O}}_n$) if and only if $\theta_{2j} + \theta_{2j+1} = 2\pi$ (resp. $\theta_{2j-1} + \theta_{2j} = 2\pi$) for all j ; then $k(f) = n/2$. So $\mathcal{E}_n = q(\widetilde{\mathcal{E}}_n)$ and $\mathcal{O}_n = q(\widetilde{\mathcal{O}}_n)$ are disjoint closed subsets of $B_{n,n/2}$.

Lemma 4.1. Two distinct points of B_n have no disjoint neighbourhoods if and only if n is even and one lies in $\mathcal{E}_n$ and the other in $\mathcal{O}_n$.

Proof. Suppose that $[f] \neq [f']$ have no disjoint neighbourhoods. As X_n^* is first countable and q is open, there are $f_m \rightarrow f$ and $g_m \in G$ with $g_m f_m \rightarrow f'$. No subsequence of (g_m) converges, for a limit g would give $f' = gf$. Write $g_m = \kappa_m \delta_m \kappa'_m$ with $\kappa_m, \kappa'_m \in K$ and $\delta_m = \text{diag}(\lambda_m, \lambda_m^{-1})$, $\lambda_m \geq 1$. After passing to a subsequence, $\kappa_m \rightarrow \kappa$, $\kappa'_m \rightarrow \kappa'$ and $\lambda_m \rightarrow \infty$. Since δ_m acts by $x \mapsto \lambda_m^2 x$,

which tends to ∞ uniformly on compact subsets of $\mathbb{RP}^1 \setminus \{0\}$, we get $g_m x_m \rightarrow p := \kappa(\infty)$ whenever $x_m \rightarrow x \neq r := \kappa'^{-1}(0)$. Hence $f'(i) = p$ whenever $f(i) \neq r$. The sets $I = \{i : f(i) = r\}$ and $J = \{i : f'(i) = p\}$ contain no two adjacent elements of $\mathbb{Z}/n$ and cover $\mathbb{Z}/n$. Then $I \cap J = \emptyset$ (for $i \in I \cap J$, $i+1 \notin I \cup J$), so membership in I alternates: n is even, I and J are the two parity classes, and f, f' are constant on complementary parity classes.

Conversely, let $f \in \tilde{\mathcal{E}}_n$ and $f' \in \tilde{\mathcal{O}}_n$; we may assume $f(0) = \infty$ and $f'(1) = 0$. Then $f = (\infty, a_1, \infty, a_2, \dots, \infty, a_{n/2})$ with $a_j \in \mathbb{R}$, and $f' = (b_1, 0, b_2, 0, \dots, b_{n/2}, 0)$ with $b_j \in \mathbb{RP}^1 \setminus \{0\}$; here $b_j = \infty$ is allowed, with the convention $\lambda \cdot \infty = \infty$. Put $f_m = (m^2 b_1, a_1, m^2 b_2, a_2, \dots)$ and let $g_m \in G$ act by $x \mapsto x/m^2$. Then $f_m \rightarrow f$ (so $f_m \in X_n^*$ for large m) and $g_m f_m = (b_1, a_1/m^2, b_2, a_2/m^2, \dots) \rightarrow f'$, so every neighbourhood of $[f]$ meets every neighbourhood of $[f']$. Finally $[f] \neq [f']$, since $\tilde{\mathcal{E}}_n$ is G -invariant and $f' \notin \tilde{\mathcal{E}}_n$. $\square$

Proposition 4.2. (a) For $2k \neq n$ the component $B_{n,k}$ is Hausdorff and contractible.

(b) $B_{n,1}$ and $B_{n,n-1}$ consist of the classes of maps whose values are distinct and in positive, resp. negative, cyclic order, and they are homeomorphic to $\mathbb{R}^{n-3}$.

(c) For even n the sets $\mathcal{E}_n$ and $\mathcal{O}_n$ are homeomorphic to $S^{n/2-2}$.

Proof. (a) $B_{n,k}$ is Hausdorff by Lemma 4.1, as $\mathcal{E}_n \cup \mathcal{O}_n \subset B_{n,n/2}$. So it is a second countable Hausdorff manifold, hence metrizable and separable, and it has the homotopy type of a CW complex [Mil59, Cor. 1]. Being weakly contractible (Proposition 3.4), it is contractible [Hat02, Thm. 4.5].

(b) $k(f) = 1$ means that the partial sums $\theta_0 + \dots + \theta_{i-1}$, $1 \leq i \leq n-1$, increase and stay below 2π , that is, $f(0), \dots, f(n-1)$ are distinct and in positive cyclic order. Normalizing $(f(0), f(1), f(2))$ to the positively oriented triple among $(0, \pm 1, \infty)$ (Lemma 3.1) identifies $B_{n,1}$ with the set of $(f(3), \dots, f(n-1))$ lying in increasing order on the positive arc from $f(2)$ to $f(0)$, that is, with the convex set $\{t_3 < \dots < t_{n-1}\} \subset \mathbb{R}^{n-3}$. Reflection gives $k = n-1$.

(c) By Proposition 3.3(b), $B_n \cong C_n/D$, and $\tilde{\mathcal{E}}_n \cap C_n$ is closed and D -invariant, so $\mathcal{E}_n \cong (\tilde{\mathcal{E}}_n \cap C_n)/D$. Its elements are the maps with $f(2j) = \infty$ and $f(2j+1) = a_j \in \mathbb{R}$, where $a_0 = 0$ and $(a_1, \dots, a_{n/2-1}) \neq 0$; D acts by scalar multiplication, so $\mathcal{E}_n \cong (\mathbb{R}^{n/2-1} \setminus 0)/\mathbb{R}_{>0} \cong S^{n/2-2}$. Likewise $\tilde{\mathcal{O}}_n \cap C_n$ consists of the maps with $f(2j+1) = 0$ and $f(2j) = b_j$, where $b_0 = \infty$ and $b_1, \dots, b_{n/2-1} \in \mathbb{RP}^1 \setminus \{0\}$ are not all ∞ ; in the coordinates $u_j = 1/b_j$ the group D acts by $u \mapsto \lambda^{-1}u$. $\square$

Example 4.3 ($n = 4$). Write $f = (\infty, 0, x, y)$ for the map in C_4 with $f(2) = x$, $f(3) = y$; here $x \neq 0$, $y \neq \infty$, $x \neq y$, $(x, y) \neq (\infty, 0)$, and D acts by $(x, y) \mapsto (\lambda x, \lambda y)$. The D -orbits are $E_{\pm}$ ($x = \infty$, $\pm y > 0$), which form $\mathcal{E}_4$; $O_{\pm}$ ($y = 0$, $\pm x > 0$), which form $\mathcal{O}_4$; and, for $x, y \in \mathbb{R}^*$, the six open arcs of directions (a) $0 < x < y$, (b) $0 < y < x$, (c) $y < 0 < x$, (d) $x < 0 < y$, (e) $x < y < 0$, (f) $y < x < 0$. The arcs (a) and (f) consist of cyclically ordered configurations and form $B_{4,1} \cup B_{4,3}$. So $B_{4,2}$ consists of $E_{\pm}$, $O_{\pm}$ and the arcs (b)–(e). The charts $u \mapsto [\infty, 0, 1/u, \pm 1]$ at $E_{\pm}$ and $v \mapsto [\infty, 0, \pm 1, v]$ at $O_{\pm}$ ($|u|, |v| < 1$) show that each of these arcs has an end at which it converges both to an E -point and to an O -point. The incidences form the cycle $E_+, (b), O_+, (c), E_-, (e), O_-, (d), E_+$.

5 Strata and cohomology theories

For $f \in X_n^*$ let $p_0 = f(0), p_1, \dots, p_{m-1}$ be its $m \geq 3$ distinct values in positive cyclic order, and define its *type* $\sigma_f: \mathbb{Z}/n \rightarrow \mathbb{Z}/m$ by $f(i) = p_{\sigma_f(i)}$. Types are the surjections $\sigma: \mathbb{Z}/n \rightarrow \mathbb{Z}/m$ with $m \geq 3$, $\sigma(0) = 0$ and $\sigma(i) \neq \sigma(i+1)$, and $\sigma_{gf} = \sigma_f$. Let $F_{\sigma} \subset \hat{B}_n$ be the set of $\theta(f)$ with $\sigma_f = \sigma$, the image of the open simplex of gap vectors (the lengths of the arcs from p_l to p_{l+1}) under an injective affine map. With $s_i = \theta_0 + \dots + \theta_{i-1}$ we have $\omega(f(i)) = \omega(f(0)) + s_i$, and the signs of the functions $s_j - s_i - 2\pi l$ ($0 \leq i < j \leq n-1$, $l \in \mathbb{Z}$) at $\theta(f)$ determine the type, and conversely. So the F_{σ} are the faces in

Δ'_k of the decomposition of H_k by the locally finite arrangement $\mathcal{A}$ of the hyperplanes $s_j - s_i = 2\pi l$, which contains the facets $\theta_i \in \{0, 2\pi\}$ of the cube. The closure of F_σ in Δ'_k is the union of the $F_{\sigma'}$ for the types σ' obtained by merging cyclically consecutive values, never $f(i)$ with $f(i+1)$; we write $\sigma' \leq \sigma$. The strata $S_\sigma = q(\{f : \sigma_f = \sigma\})$ satisfy $\pi^{-1}(S_\sigma) = F_\sigma$.

Lemma 5.1. *For a type σ let $\text{St}(\sigma) = \bigcup_{\sigma' \geq \sigma} S_{\sigma'}$. Then $\text{St}(\sigma)$ is open, and $\pi^{-1}(\text{St}(\sigma))$ is convex. Hence every nonempty finite intersection W of such sets has convex preimage $\pi^{-1}(W)$.*

Proof. $\pi^{-1}(\text{St}(\sigma))$ is the open star of the face F_σ . For a hyperplane arrangement this is the set of points of Δ_k lying strictly on the same side as F_σ of every hyperplane of $\mathcal{A}$ not containing F_σ ; as only finitely many hyperplanes of $\mathcal{A}$ meet $\overline{\Delta}_k$, it is an open convex set. It lies in Δ'_k , as its faces have at least as many values as σ . As π is open, $\text{St}(\sigma)$ is open. $\square$

Remark 5.2 (Finite models). Let $P_{n,k}$ be the set of types of winding number k , ordered by $\leq$ and topologized by its up-sets. The type map $B_{n,k} \rightarrow P_{n,k}$ is continuous, and by Lemmas 3.2 and 5.1 the preimage of each minimal open set $\{\sigma' \geq \sigma\}$ is weakly contractible. By McCord's theorem on basis-like covers [McC66, Thm. 6], $B_{n,k}$ is weakly homotopy equivalent to $P_{n,k}$, hence to its order complex [McC66]. Also, the faces of $\mathcal{A}$ form a regular cell structure on the closed polytope $\overline{\Delta}_k$ in which $\overline{\Delta}_k \setminus \Delta'_k$ is a subcomplex. Hence the relative cellular complex, with one generator of degree $m-1$ for each type, computes the Borel–Moore homology of Δ'_k , and $H^i(\Delta'_k) \cong H_{n-1-i}^{\text{BM}}(\Delta'_k)$ by Poincaré duality. Both models, evaluated by computer below, reach B_n only through π ; so they check the combinatorics of Δ'_k , not the bundle step.

Proposition 5.3 (Sheaf cohomology). *Let A be an abelian group and $W \subset B_n$ open. The cohomology of the constant sheaf A on W is isomorphic to $H_{\text{sing}}^*(W; A)$, and it vanishes in positive degrees if $\pi^{-1}(W)$ is contractible.*

Proof. Let $\widehat{W} = \pi^{-1}(W)$, an open subset of $\bigsqcup_k \Delta'_k$, and $p = \pi|_{\widehat{W}}$. The fibres of p are connected and p is open, so locally constant functions on $p^{-1}(V)$ descend to V , and $p_*A = A$. By [Har77, Prop. III.8.1], valid for any continuous map, the stalk of $R^j p_*A$ at b is the colimit of $H^j(p^{-1}(V); A)$ over the neighbourhoods V of b . Balls V in the charts of Lemma 3.1 are cofinal, and $p^{-1}(V) \cong V \times \mathbb{R}^2$ is a contractible Hausdorff manifold. On paracompact Hausdorff locally contractible spaces sheaf cohomology with constant coefficients agrees with singular cohomology [Bre97, §III.1]; by [Sel16] this even holds on all locally contractible spaces. Hence $R^j p_*A = 0$ for $j > 0$, and [Har77, Ex. III.8.1] gives $H^j(W; A) \cong H^j(\widehat{W}; A) \cong H_{\text{sing}}^j(\widehat{W}; A)$, which is $H_{\text{sing}}^j(W; A)$ by Lemma 3.2. $\square$

Proposition 5.4 (Čech cohomology). *Let A be an abelian group, and consider Čech cohomology with coefficients in the constant sheaf A . On a Hausdorff component $B_{n,k}$ it agrees with sheaf cohomology. On $B_{n,n/2}$ the natural map $\check{H}^j(B_{n,n/2}; A) \rightarrow H^j(B_{n,n/2}; A)$ to sheaf cohomology is surjective for all j and bijective for $j \leq 2$.*

Proof. $B_{n,k}$ is locally compact, Hausdorff and second countable, hence paracompact, and Čech and sheaf cohomology agree on paracompact spaces [God58, Chap. II, §5.10]. On $B_{n,n/2}$, the finite cover $\mathcal{U}$ by the sets $\text{St}(\sigma)$ is a Leray cover by Lemma 5.1 and Proposition 5.3, so $\check{H}^j(\mathcal{U}; A) \rightarrow H^j(B_{n,n/2}; A)$ is an isomorphism [Har77, Ex. III.4.11]. (The nonempty finite intersections of sets of $\mathcal{U}$ are connected, so $\check{H}^j(\mathcal{U}; A)$ is the simplicial cohomology of the nerve of $\mathcal{U}$.) This isomorphism factors through $\check{H}^j(B_{n,n/2}; A)$. On every space $\check{H}^j \rightarrow H^j$ is bijective for $j \leq 1$ and injective for $j = 2$ [God58, Chap. II, §5.9]. $\square$

Proposition 5.5 ($n = 4$). *(a) Every smooth 1-form on $B_{4,2}$ is the differential of a smooth function; so the de Rham cohomology of smooth forms is $\mathbb{R}$ in degree 0 and 0 in degree 1, while $H^1(B_{4,2}; \mathbb{R}) \cong \mathbb{R}$.*

(b) Every map from $B_{4,2}$ to a Hausdorff space factors through a contractible space. So every map $B_{4,2} \rightarrow S^1$ is null-homotopic, and $B_{4,2}$ is not homotopy equivalent to S^1 .

Proof. A map to a Hausdorff space takes the same value at points without disjoint neighbourhoods, hence at the points $E_{\pm}, O_{\pm}$ of Example 4.3. (a) Let α be a smooth 1-form. On each arc (b)–(e) let $F(p)$ be the limit of $\int_{p'}^p \alpha$ as p' tends to the end at which the arc meets the charts of the special points, and let $F = 0$ at $E_{\pm}, O_{\pm}$. In the chart at E_+ , $\alpha = a(u) du$ with a smooth, the limit exists, and $F(u) = \int_0^u a(t) dt$ for $|u| < 1$ on both arcs; likewise in the other charts. So F is smooth with $dF = \alpha$. As $B_{4,2}$ is connected, closed 0-forms are constant, and $H^1(B_{4,2}; \mathbb{R}) \cong \mathbb{R}$ by Proposition 3.4. (b) A map to a Hausdorff space factors through the quotient Q in which the four points are identified. By the charts of Example 4.3, Q is homeomorphic to the union of four half-open segments in $\mathbb{R}^2$ with a common closed end, which is contractible. If $h: B_{4,2} \rightarrow S^1$ and $g: S^1 \rightarrow B_{4,2}$ were inverse homotopy equivalences, $\text{id}_{S^1} \simeq h \circ g$ would be null-homotopic. $\square$

Proof of Theorem 1.1. (i) follows from Lemmas 2.1, 3.1 and 4.1 and Proposition 4.2(c); (ii) is Proposition 4.2(a),(b); (iii) follows from Lemma 4.1 and Propositions 3.4 and 4.2; (iv) from Propositions 3.4, 5.3 and 5.4 and [Hat02, Prop. 4.21]; and (v) is Proposition 5.5. $\square$

6 Variants

Proof of Corollary 1.2(a). An element of $\text{PGL}(2, \mathbb{R}) \setminus \text{PSL}(2, \mathbb{R})$ is represented by a matrix g with $\det g < 0$, and $(\text{tr } g)^2 - 4 \det g > 0$, so it has exactly two fixed points. Hence $\text{PGL}(2, \mathbb{R})$ acts freely, and $B_n^{\text{PGL}} = B_n / \langle \rho \rangle$ with ρ as in Lemma 2.1(d), which exchanges $B_{n,k}$ and $B_{n,n-k}$. Let n be even. The proofs of Lemma 3.1 and Proposition 3.3(a) apply to $\text{PGL}(2, \mathbb{R})$, which is simply transitive on ordered triples of distinct points, and to $\text{PO}(2) = K \rtimes \langle \rho \rangle$ with $\rho(x) = -x$. So the middle component of B_n^{PGL} is weakly homotopy equivalent to $\Delta'_{n/2} / \rho$, where ρ acts by $\theta - \theta^\circ \mapsto -(\theta - \theta^\circ)$. This involution is free on $\Delta'_{n/2}$, and in Proposition 3.4(b), $r(\rho\theta) = -r(\theta)$, $s(-u) = \rho(s(u))$, and H_t commutes with ρ . Hence $\Delta'_{n/2} / \rho \simeq S(U) / \{\pm 1\} \cong \mathbb{RP}^{n-3}$. As $B_n \rightarrow B_n^{\text{PGL}}$ is open and ρ preserves $\mathcal{E}_n$ and $\mathcal{O}_n$, the middle component is not Hausdorff (Lemma 4.1). $\square$

Proof of Corollary 1.2(b). Let n be odd and $f \in X_n$, extended n -periodically. For $0 \neq V_0 \in f(0)$ there are unique $V_i \in f(i)$, $i \in \mathbb{Z}$, with $\det(V_{i-1}, V_i) = 1$. Then $V_n = \mu V_0$, and replacing V_0 by tV_0 replaces V_i by $t^{(-1)^i} V_i$ and μ by μ/t^2 . So there is a lift with $V_{i+n} = \varepsilon V_i$, $\varepsilon = \pm 1$, unique up to sign. Put $c_i = \det(V_{i-1}, V_{i+1})$; then $V_{i+1} = c_i V_i - V_{i-1}$ and $c_{i+n} = c_i$. For a linear form ϕ the numbers $u_i = \phi(V_i)$ satisfy $(u_{i+1}, u_i)^T = \begin{pmatrix} c_i & -1 \\ 1 & 0 \end{pmatrix} (u_i, u_{i-1})^T$, so $M(c)(u_1, u_0)^T = \varepsilon(u_1, u_0)^T$ for all (u_1, u_0) and $M(c) = \varepsilon \text{Id}$. As $\det(V_i, V_{i+1}) > 0$, V_{i+1} is a positive multiple of V_i rotated counterclockwise by $\theta_i(f)/2$, so V_n is V_0 rotated by $\pi k(f)$ and $\varepsilon = (-1)^{k(f)}$. The map $f \mapsto c$ is continuous and $\text{SL}(2, \mathbb{R})$ -invariant. Conversely, if $M(c) = \varepsilon \text{Id}$, the solution of $V_{i+1} = c_i V_i - V_{i-1}$ with $(V_0, V_1) = (e_1, e_2)$ has $\det(V_{i-1}, V_i) = 1$ and $V_{i+n} = \varepsilon V_i$, so $f_c(i) = [V_i]$ defines $f_c \in X_n$ with invariant c , continuously in c . Maps with the same invariant have normalized lifts satisfying the same recurrence, so they differ by an element of $\text{SL}(2, \mathbb{R})$. As every $f \in X_n$ has at least three values (n odd), $f \mapsto c$ induces a homeomorphism from B_n onto $\{M(c) = \text{Id}\} \sqcup \{M(c) = -\text{Id}\}$, and $\mathcal{F}_n$ corresponds to the $B_{n,k}$ with k odd. Since $k \mapsto n - k$ exchanges odd and even k , this union is homeomorphic to B_n^{PGL} , and it has $(n - 1)/2$ contractible components. The Conway–Coxeter friezes are the solutions of index $1/2$ in the sense of [Ovs18, §5.4], for which V_i turns by π over a period; so $k = 1$. $\square$

6.1 Fock–Goncharov coordinates

Lemma 6.1. *Let $f: \mathbb{Z}/n \rightarrow S$ be a map to a set with $f(i) \neq f(i+1)$ for all i and with at least three values. Then the n -gon with vertices $0, \dots, n-1$ has a triangulation all of whose edges $\{i, j\}$ (sides*

and diagonals) satisfy $f(i) \neq f(j)$.

Proof. We use induction on n , the case $n = 3$ being clear. Let $n \geq 4$. Call v an *ear* if $f(v-1) \neq f(v+1)$, and a *good ear* if moreover f has at least four values or $f(v)$ is taken at least twice. If v is a good ear, f restricted to the other $n-1$ vertices, in their cyclic order, satisfies the hypotheses, and a triangulation for it plus the triangle $\{v-1, v, v+1\}$ is as required. An ear exists, for otherwise $f(v-1) = f(v+1)$ for all v and f has at most two values; if f has at least four values, it is good. Suppose that f has exactly three values and no good ear. Then some value is taken at only one vertex w , since otherwise every ear would be good. The neighbours $w \pm 1$ are ears, as $f(w \pm 2) \neq f(w)$; they are not good, so their values are taken only once, and these values are distinct since $w+1 \neq w-1$. So $f(w-1), f(w), f(w+1)$ are three values taken once each, and $n = 3$, a contradiction. $\square$

Proposition 6.2. *Let $n \geq 4$. For a triangulation T of the n -gon let $U_T \subset X_n^*$ consist of the maps with $f(i) \neq f(j)$ for all edges $\{i, j\}$ of T . For a diagonal $d = \{i, j\}$ let i, k, j, l be the vertices of the two triangles of T containing d , in cyclic order, and let $x_d(f) = r(f(i), f(k), f(j), f(l))$, where $r(y_1, y_2, y_3, y_4) = \frac{(y_1 - y_2)(y_3 - y_4)}{(y_2 - y_3)(y_1 - y_4)}$ as in [FG16, §1.2]. Then the U_T are open, $\mathrm{PGL}(2, \mathbb{R})$ -invariant and cover X_n^* , and $(x_d)_{d \in T}$ induces a homeomorphism $U_T / \mathrm{PGL}(2, \mathbb{R}) \cong (\mathbb{R}^*)^{n-3}$. All coordinates are positive exactly on the image of $B_{n,1} \cup B_{n,n-1}$, and $x_d(f) = -1$ if and only if $f(k) = f(l)$.*

Proof. The covering is Lemma 6.1. For $f \in U_T$ the points $f(i), f(k), f(j)$ are distinct and $f(l) \notin \{f(i), f(j)\}$, so $x_d(f) \in \mathbb{R}^*$; and $y \mapsto r(y_1, y_2, y_3, y)$ is a Möbius map sending y_3, y_1, y_2 to $0, \infty, -1$. So $x_d = -1$ if and only if $f(l) = f(k)$, and $x_d > 0$ if and only if $\{f(i), f(j)\}$ separates $f(k)$ from $f(l)$. Normalize f on one triangle by $\mathrm{PGL}(2, \mathbb{R})$ and add the other triangles along the dual tree; each new vertex is determined by the coordinate of the diagonal just crossed, which ranges over $\mathbb{R}^*$. This gives the homeomorphism, and the same induction shows that all coordinates are positive exactly for cyclically ordered f . $\square$

These are the coordinates of the positive atlas on $\mathcal{M}_{0,n}$ in [FG16, §1.2], where Fock and Goncharov note that such an atlas determines a possibly non-separated scheme glued from coordinate tori. In B_n^{PGL} the tori are glued only along the sets $U_T \cap U_{T'}$. For $n = 4$ the coordinates of the two triangulations satisfy $x_{\{1,3\}} = 1/x_{\{0,2\}}$, and the overlap is where both differ from -1 ; so B_4^{PGL} is the line $\mathbb{R}^*$ with the point -1 doubled, although $x \mapsto 1/x$ is defined on all of $\mathbb{R}^*$.

Open questions. (1) For $2 \leq k \leq n-2$, $2k \neq n$, $n \geq 6$, is the contractible manifold $B_{n,k}$ homeomorphic to $\mathbb{R}^{n-3}$? (For $n \leq 5$ it is.) (2) Does the Čech cohomology of the constant sheaf on $B_{n,n/2}$, as a colimit over all covers, agree with singular cohomology in degrees ≥ 3 ? (3) For $n \geq 6$, what is the de Rham cohomology of smooth forms on $B_{n,n/2}$, and is $B_{n,n/2}$ homotopy equivalent to S^{n-3} ? Proposition 5.5(a) is an instance of a known phenomenon: O’Connell [O’C23] proves a de Rham theorem for non-Hausdorff manifolds glued from Hausdorff ones under assumptions on the gluing sets, among them regular openness [O’C23, Thm. 4.4]; they fail for the line with two origins, where the two cohomologies differ [O’C23, §4].

Verification. The proofs above do not depend on computation. The following checks were made; they are checks, not proofs. (1) Four programs by the author, in standard library Python, run in about seventy seconds. The first, `sanity_checks.py`, tests in exact arithmetic the invariance of the winding number under random elements of $\mathrm{SL}(2, \mathbb{Z})$ (3200 configurations, $n \leq 10$) and the sequences in the proof of Lemma 4.1; it verifies Lemma 6.1 on all 22,077 coincidence patterns with $n \leq 10$, and, in floating point, $\varepsilon = (-1)^k$ and $M(c) = \varepsilon \mathrm{Id}$ on 1200 configurations. `types_poset.py` enumerates the types for $n \leq 8$ (43,086 for $n = 8$), checks that merging preserves k , and computes

the homology of the order complexes of the $P_{n,k}$ over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_{1000003}$ after removing beat points [Sto66] and weak points [BM08]: a point for $k \neq n/2$, and S^1 , S^3 for $(n, k) = (4, 2), (6, 3)$ (the case $(8, 4)$ was too large). `bm_cellular.py` computes the integral Borel–Moore homology of Remark 5.2 by Smith normal form for all 35 components with $n \leq 9$: no torsion, and cohomology $\mathbb{Z}$ in degree 0, plus $\mathbb{Z}$ in degree $n - 3$ exactly when $k = n/2$. `paper_checks.py`, written for this release, checks that the reflection is a free involution on the types of winding $n/2$, and finds over $\mathbb{F}_2$ the cohomology of $\mathbb{RP}^{n-3}$ for $\Delta'_{n/2}/\rho$ ($n = 4, 6, 8$) and of $S^{n/2-2}$ for $\mathcal{E}_n$, from the cells of $\pi^{-1}(\mathcal{E}_n)$ ($n \leq 14$). (2) A first independent verification run, which was AI-assisted, checked every proof step by step, re-derived the main homotopy statement by the slice argument of Proposition 3.3(b), and re-ran the programs above with identical outputs up to timings. Its own code, written from scratch, reproduced the numbers of types ($n \leq 8$) and computed the $\mathbb{F}_2$ Borel–Moore homology of all 27 components with $n \leq 8$ (a sign-free check of the incidences), the unreduced order complexes ($n \leq 6$) and the Euler characteristics of the strata in $\mathcal{E}_n$, $\mathcal{O}_n$ ($n = 4, 6, 8$); a convexity test for Lemma 5.1 found no failure in 53,700 samples. (3) A second independent verification run, also AI-assisted, read the source and the cited results again, re-checked the proofs, among them Proposition 5.4 and Lemma 6.1, and re-ran the release package from its archive with identical outputs up to timings. Its own code, again written from scratch, computed the Borel–Moore homology of all 35 components with $n \leq 9$ over $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_{10007}$, with incidence numbers obtained from determinants of tangent frames, so that the signs are derived independently; the cohomology of $\pi^{-1}(\mathcal{E}_n)$ and $\pi^{-1}(\mathcal{O}_n)$ ($n \leq 8$, and $n = 10, 12$ for $\mathcal{E}_n$); and the integral cohomology of $\Delta'_{n/2}/\rho$, which is that of $\mathbb{RP}^{n-3}$ for $n = 4, 6, 8$. In exact arithmetic it checked Lemma 6.1 by dynamic programming for $n \leq 11$, the relations of Corollary 1.2(b) and $M(c) = -\text{Id}$, $k = 1$ for all Conway–Coxeter friezes with odd $n \leq 11$, the sequences of Lemma 4.1, the description of the open stars in the proof of Lemma 5.1, and Proposition 6.2. Neither run found a mathematical error; their required corrections, on credit, wording, precision and presentation, have been applied, and their code is part of the release. (4) As said in Remark 5.2, the finite models check the combinatorics of Δ'_k , not the bundle step.

Scope and priority. Theorem 1.1 answers both parts of Fock’s question [MPP06, p. 1607] for the space defined there and every $n \geq 3$; Corollary 1.2 treats $\text{PGL}(2, \mathbb{R})$ and friezes. The report [MPP06] records the problem without an answer; the later Oberwolfach reports on Teichmüller theory (53/2010, 7/2014, 40/2018, 33/2023) and the arXiv papers of Fock that we searched do not mention the space. In September 2026 we searched arXiv, Crossref, OpenAlex, zbMATH Open, Semantic Scholar, OpenCitations and MathOverflow and made a web search; two independent verification runs repeated these searches. Crossref records no work citing [MPP06], OpenAlex counts one, and Semantic Scholar lists one, [Bai07], which cites the report as a whole and concerns the quantization of Teichmüller spaces, not this problem; OpenCitations lists none. We found no earlier answer. The count of components is elementary and may be known to experts; we did not find the description of the middle component, its weak homotopy type, the comparison of cohomology theories, or the component count for real friezes. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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