

On Bartholdi's Conjecture on the Closures of the Aleshin and Bellaterra Groups

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Abstract

In an Oberwolfach report of 2006, L. Bartholdi conjectured that the closures of the Aleshin automaton group $G(\mathcal{A})$ and of the Bellaterra automaton group $G(\mathcal{B})$ in the automorphism group of the binary rooted tree are $\mathbb{Z}_2$ -analytic, and that the closure of $G(\mathcal{B})$ is a pro-2 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$. We show that the conjecture is false: the closure of the Bellaterra group is 2-adic analytic but is not isomorphic to a pro-2 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ (nor of $\mathrm{PSL}_2(\mathbb{Z}_2)$ or $\mathrm{PGL}_2(\mathbb{Z}_2)$), and the closure of the Aleshin group is not 2-adic analytic. The literal Sylow statement fails for an elementary reason: the closure of $G(\mathcal{B})$ contains three distinct involutions, $\mathrm{SL}_2(\mathbb{Z}_2)$ only one. The substance is in two theorems. The closure of $G(\mathcal{B})$ is isomorphic to $\{g \in \mathrm{GL}_2(\mathbb{Z}_2) : g \equiv 1 \pmod{2}, \det g = \pm 1\} / \{\pm 1\}$, an open subgroup of index 12 of $\mathrm{PGL}_2(\mathbb{Z}_2)$; the proof uses a Möbius model of the Bellaterra automaton due to Kharlampovich and Vdovina (an unrefereed 2026 preprint whose acknowledgement declares AI assistance; all identities taken from it are verified again here), a lifting lemma for congruence subgroups, and a computation in groups of orders 32 and 256. The closure of $G(\mathcal{A})$ has index 2 in the closure of the group generated by $G(\mathcal{B})$ and the complement map, which in the Möbius coordinate is not differentiable at the fixed point of a prefix contraction; p -adic Lie theory then excludes analyticity. No printed answer to the question was found in the sources searched; two works citing the report could not be read. This is an unrefereed note.

1 Introduction

1.1 The conjecture

At the Oberwolfach workshop *Pro- p Extensions of Global Fields and pro- p Groups* (21–27 May 2006), Bartholdi proposed to study the closures of groups generated by automata in the automorphism group of a rooted tree [Bar07]. For $p = 2$ the setting is as follows. An automaton is a finite directed graph with an element $n_v \in \mathbb{F}_2$ at every vertex v and two outgoing edges at every vertex, labelled 0 and 1. A vertex v acts on the set $\partial T = \{0, 1\}^{\mathbb{N}}$ of infinite words by

$$v(x_1 x_2 \dots) = (x_1 + n_1)(x_2 + n_2) \dots,$$

where $v_1 = v$, v_{i+1} is the end of the edge labelled x_i at v_i , and $n_i = n_{v_i}$ [Bar07, (1)]. These maps are automorphisms of the binary rooted tree $T = \{0, 1\}^*$. The group $W = \mathrm{Aut}(T)$ is a pro-2 group, the stabilizers $\mathrm{St}_W(n)$ of the levels $\{0, 1\}^n$ forming a basis of neighbourhoods of 1, and $G(\mathcal{A}) \leq W$ denotes the group generated by the vertices of an automaton $\mathcal{A}$. Section 1 of [Bar07] concerns the “Aleshin” automaton $\mathcal{A}$ and the “Baby-Aleshin” automaton $\mathcal{B}$, recalls that $G(\mathcal{A})$ is free of rank 3 and that $G(\mathcal{B})$ is isomorphic to $C_2 * C_2 * C_2$, and states on p. 1489, for the closures in W :

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Conjecture 1.1 (Bartholdi [Bar07, p. 1489]). (i) *The closures $\overline{G(\mathcal{A})}$ and $\overline{G(\mathcal{B})}$ are $\mathbb{Z}_2$ -analytic groups.* (ii) *“More precisely, I conjecture that $\overline{G(\mathcal{B})}$ is a pro-2-Sylow of $\mathrm{SL}_2(\mathbb{Z}_2)$.”*

The conjecture is record OWR-1274-007 of the UnsolvedMath collection [ula26], where it is listed as open. “ $\mathbb{Z}_2$ -analytic” means 2-adic analytic in the sense of Lazard [Laz65]: a topological group with a structure of analytic manifold over $\mathbb{Q}_2$, compatible with its topology, for which multiplication and inversion are analytic [Ser92, Part II, Ch. IV], [DdMS99, Ch. 8]. Every closed subgroup of W is a pro-2 group, and a pro-2 group is 2-adic analytic if and only if it has finite rank [Laz65], [DdMS99, Ch. 3 and 8]. Since $\overline{G(\mathcal{B})}$ is a subgroup of W , part (ii) can only mean that $\overline{G(\mathcal{B})}$ is isomorphic, as a topological group, to a pro-2 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$; these are the conjugates of $S = \{g \in \mathrm{SL}_2(\mathbb{Z}_2) : g_{21} \in 2\mathbb{Z}_2\}$, which has index 3 (the reduction $\mathrm{SL}_2(\mathbb{Z}_2) \rightarrow \mathrm{SL}_2(\mathbb{F}_2) \cong S_3$ is onto and has a pro-2 kernel).

1.2 The two automata

The figure on p. 1488 of [Bar07] shows two automata with the same graph. We call its three vertices a (bottom right), b (left) and c (top right). Both pictures have the five edges

$$a \xrightarrow{0} b, \quad a \xrightarrow{1} c, \quad b \xrightarrow{0} c, \quad b \xrightarrow{1} b \text{ (a loop)}, \quad c \xrightarrow{0,1} a,$$

and the vertex labels are $(n_a, n_b, n_c) = (1, 1, 0)$ in the left picture and $(0, 0, 1)$ in the right one. Write $g = (g_0, g_1)\sigma^e$ for the automorphism with $g(xw) = (x + e)g_x(w)$ ($x \in \{0, 1\}$, $w \in \{0, 1\}^*$). By the convention above, the left picture defines automorphisms $\hat{a}, \hat{b}, \hat{c}$ and the right picture automorphisms a, b, c with

$$\hat{a} = (\hat{b}, \hat{c})\sigma, \quad \hat{b} = (\hat{c}, \hat{b})\sigma, \quad \hat{c} = (\hat{a}, \hat{a}); \quad a = (b, c), \quad b = (c, b), \quad c = (a, a)\sigma. \quad (1)$$

The text of [Bar07] says that $\mathcal{A}$ and $\mathcal{B}$ are “respectively the right and left automata”. This cannot be meant: a, b, c are involutions (Lemma 2.1(a)), so they do not generate a free group of rank 3. The facts recalled in [Bar07] and the names force $\mathcal{A}$ to be the left automaton and $\mathcal{B}$ the right one. The first fact, that $G(\mathcal{A})$ is free of rank 3, was claimed by Aleshin [Ale83], who introduced the automaton, and proved by M. and Y. Vorobets [VV07]; the left automaton is the conjugate by the complement map τ (Section 2.1) of the Aleshin automaton $a = (c, b)\sigma$, $b = (b, c)\sigma$, $c = (a, a)$ of [VV10, Section 3]. The second, $G(\mathcal{B}) \cong C_2 * C_2 * C_2$, is due to Muntyan and Savchuk, at the 2004 summer school on automata groups at the Autonomous University of Barcelona, after whose location the Bellaterra automaton is named ([VV10, Theorem 1.4 and the remark after it]; [Bar07] cites [Nek05, Theorem 1.10.2]); the right automaton is literally the Bellaterra automaton in the convention [KV26, (2.1)], which differs from that of [VV10, Section 6] by complementing the letters [KV26, Section 2]. Neither fact is used in our proofs. So $G(\mathcal{A}) = \langle \hat{a}, \hat{b}, \hat{c} \rangle$ and $G(\mathcal{B}) = \langle a, b, c \rangle$. Part (i) is decided below for both automata, so it does not depend on this choice; under the assignment of the text, part (ii) would say that the closure of the Aleshin group is a Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$, which is false by Theorem A.

1.3 Results

Theorem B. *The closure $\overline{G(\mathcal{B})}$ of the Bellaterra group in W is isomorphic, as a topological group, to*

$$\Gamma' = \{g \in \mathrm{GL}_2(\mathbb{Z}_2) : g \equiv 1 \pmod{2}, \det g = \pm 1\} / \{\pm 1\},$$

which is an open subgroup of index 12 of $\mathrm{PGL}_2(\mathbb{Z}_2)$. In particular, $\overline{G(\mathcal{B})}$ is a 2-adic analytic group of dimension 3.

Corollary B. (a) $\overline{G(\mathcal{B})}$ is not isomorphic, even as an abstract group, to any subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$; in particular it is not isomorphic to a pro-2 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$. (b) $\overline{G(\mathcal{B})}$ is not isomorphic, as a topological group, to a pro-2 Sylow subgroup of $\mathrm{PSL}_2(\mathbb{Z}_2)$. (c) $\overline{G(\mathcal{B})}$ is not isomorphic, as a topological group, to a pro-2 Sylow subgroup of $\mathrm{PGL}_2(\mathbb{Z}_2)$.

Theorem A. The closure $\overline{G(\mathcal{A})}$ of the Aleshin group in W is not a 2-adic analytic group; equivalently, it has infinite rank. More precisely, $\overline{G(\mathcal{A})}$ has index 2 in the closure J of $\langle G(\mathcal{B}), \tau \rangle$, and no closed subgroup of W that contains $\overline{G(\mathcal{B})}$ and τ is 2-adic analytic.

Consequently, the conjecture is false: the closure of the Bellaterra group is 2-adic analytic but is not isomorphic to a pro-2 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ (nor of $\mathrm{PSL}_2(\mathbb{Z}_2)$ or $\mathrm{PGL}_2(\mathbb{Z}_2)$), and the closure of the Aleshin group is not 2-adic analytic.

1.4 Comments, credits and scope

Read literally, part (ii) fails for an elementary reason, Corollary B(a). The substance of the note is in Theorems B and A, and Corollary B(b), (c) decide the two natural corrected readings of (ii): the Möbius action of $\mathrm{SL}_2(\mathbb{Z}_2)$ on $\mathbb{P}^1(\mathbb{Q}_2)$ factors through $\mathrm{PSL}_2(\mathbb{Z}_2)$, and Γ' lies in $\mathrm{PGL}_2(\mathbb{Z}_2)$. The group $\overline{G(\mathcal{B})}$ is commensurable with $\mathrm{SL}_2(\mathbb{Z}_2)$, and its level quotients have the same orders 2^{3n-5} as the permutation groups induced by the Möbius action of the Sylow subgroup of $\mathrm{PSL}_2(\mathbb{Z}_2)$ (Remark 5.2); this may be where the conjecture came from, but we do not know.

The Möbius model of Section 3 is due to Kharlampovich and Vdovina [KV26], who use it to prove the Ramanujan property of the Schreier graphs of $G(\mathcal{B})$, conjectured in the same report. Their preprint is unrefereed, its acknowledgement states that it was written with the editing assistance of an AI system, which also helped to find known results used for its Theorem 8.1, and it makes no statement about closures. We use only explicit data from it (five quaternions, a splitting over $\mathbb{Q}_2$ and the coding of the tree by 2-adic balls); every identity is verified again by exact computation (Lemmas 3.1 and 3.2), and the coding is proved in full (Proposition 3.3). Corollary B does not depend on the preprint (Remark 5.1).

No printed answer to the question was found in the sources searched (listed under “Scope and priority” in Section 8); two works citing the report, the thesis of Siegenthaler [Sie10] and the survey of Grigorchuk [Gri11], could not be read. A search that finds nothing is not a proof of novelty, and no priority is claimed.

2 Preliminaries

2.1 The tree and the complement map

For $\xi \neq \eta$ in ∂T let $|\xi \wedge \eta|$ be the length of their longest common prefix. The metric $2^{-|\xi \wedge \eta|}$ is an ultrametric on ∂T , and W is exactly its group of isometries (an isometry maps cylinders of length n onto cylinders of length n); the topology of W is that of uniform convergence. For $G \leq W$ we write G_n for the group of permutations of $\{0, 1\}^n$ induced by G ; since $\mathrm{St}_W(n)$ is open, $(\overline{G})_n = G_n$. The complement map is $\tau = (\tau, \tau)\sigma$, that is $\tau(x_1 x_2 \dots) = (1 + x_1)(1 + x_2) \dots$.

Lemma 2.1. (a) a, b, c are three distinct non-trivial involutions. (b) $\hat{s} = \tau \circ s$ and $\tau \hat{s} \tau = \hat{s}^{-1}$ for $s \in \{a, b, c\}$. (c) $\langle G(\mathcal{A}), \tau \rangle = \langle G(\mathcal{B}), \tau \rangle = G(\mathcal{A}) \cup G(\mathcal{A})\tau$. Hence, if J is the closure of $\langle G(\mathcal{B}), \tau \rangle$ in W , then $J = G(\mathcal{A}) \cup \overline{G(\mathcal{A})}\tau$, and $\overline{G(\mathcal{A})}$ is an open subgroup of index 2 of J .

Proof. (a) By (1), $a^2 = (b^2, c^2)$, $b^2 = (c^2, b^2)$ and $c^2 = (a, a)\sigma(a, a)\sigma = (a^2, a^2)$, so by induction on n the elements a^2, b^2, c^2 act trivially on $\{0, 1\}^n$ for every n . The element c moves the first letter and

a, b do not; $a \neq 1$ and $b \neq 1$ because their sections at 1 and at 0 are $c \neq 1$; and $a \neq b$, since otherwise their sections at 0 would give $b = c$.

(b) If $s = (s_0, s_1)\sigma^e$, then $(\tau \circ s)(xw) = \tau((x+e)s_x(w)) = (x+e+1)(\tau \circ s_x)(w)$, so $\tau \circ s = (\tau \circ s_0, \tau \circ s_1)\sigma^{e+1}$. Hence $\tau \circ a, \tau \circ b, \tau \circ c$ satisfy the recursion of $\hat{a}, \hat{b}, \hat{c}$ in (1), and since such a recursion determines the action on every level, $\hat{s} = \tau \circ s$. As s and τ are involutions, $\tau \hat{s} \tau = s \tau = (\tau s)^{-1} = \hat{s}^{-1}$.

(c) By (b), τ normalizes $G(\mathcal{A})$, so $\langle G(\mathcal{A}), \tau \rangle = G(\mathcal{A}) \cup G(\mathcal{A})\tau$; this group contains $s = \tau \hat{s}$, and $\langle G(\mathcal{B}), \tau \rangle$ contains $\hat{s} = \tau s$. The closure of $G(\mathcal{A}) \cup G(\mathcal{A})\tau$ is $\overline{G(\mathcal{A})} \cup \overline{G(\mathcal{A})}\tau$, so $\overline{G(\mathcal{A})}$ is a closed subgroup of index at most 2 of J , hence open. The index is 2: the permutation of $\{0, 1\}^4$ induced by τ does not belong to $G(\mathcal{A})_4 = (\overline{G(\mathcal{A})})_4$, a group of order 2^7 (a finite computation, Section 8, part [8]). $\square$

2.2 $\mathrm{PGL}_2(\mathbb{Z}_2)$ and its congruence subgroups

The group $\mathrm{PGL}_2(\mathbb{Z}_2) = \mathrm{GL}_2(\mathbb{Z}_2)/\mathbb{Z}_2^\times$ acts on $\mathbb{P}^1(\mathbb{Q}_2)$ by Möbius maps $g(x) = (ax+b)/(cx+d)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $k \geq 1$ let $\mathrm{PGL}_2(\mathbb{Z}/2^k) = \mathrm{GL}_2(\mathbb{Z}/2^k)/(\mathbb{Z}/2^k)^\times$, a group of order $6 \cdot 16^{k-1}/2^{k-1} = 6 \cdot 8^{k-1}$. Reduction $\mathrm{PGL}_2(\mathbb{Z}_2) \rightarrow \mathrm{PGL}_2(\mathbb{Z}/2^k)$ is onto, and its kernel is $K(k) = \{g : g \equiv \text{scalar mod } 2^k\}$; so $|K(k) : K(k+1)| = 8$, and the $K(k)$ are open normal subgroups forming a basis of neighbourhoods of 1. For $N \leq \mathrm{PGL}_2(\mathbb{Z}_2)$ let N_k be its image in $\mathrm{PGL}_2(\mathbb{Z}/2^k)$. The homomorphism $\chi : \mathrm{PGL}_2(\mathbb{Z}_2) \rightarrow \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$, $\chi(g) = \det g \bmod (\mathbb{Z}_2^\times)^2$, is well defined, the target being the group of order 4 of the classes of 1, 3, 5, 7 modulo 8; its kernel is the image $\mathrm{PSL}_2(\mathbb{Z}_2) = \mathrm{SL}_2(\mathbb{Z}_2)/\{\pm 1\}$ of $\mathrm{SL}_2(\mathbb{Z}_2)$. The *Iwahori subgroup* I consists of the classes with c even (then a and d are odd); every element of I has a unique representative $\begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$, and modulo 2^k such elements are given by triples (a, b, c) of residues, a odd and c even.

Lemma 2.2. (a) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_2)$ with c even. Then g maps $\mathbb{Z}_2$ onto $\mathbb{Z}_2$ and $|g(x) - g(y)| = |x - y|$ for $x, y \in \mathbb{Z}_2$; so g permutes the balls $u + 2^n\mathbb{Z}_2$, and this permutation of $\mathbb{Z}/2^n$ depends only on $g \bmod 2^n$. (b) Let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_2)$ with $a \equiv 2 \bmod 4$, $b \equiv s \bmod 2$ for some $s \in \{0, 1\}$, $c \equiv 0 \bmod 4$, d odd and $v_2(\det h) = 1$. Then h maps $\mathbb{Z}_2$ bijectively onto $s + 2\mathbb{Z}_2$, and $|h(x) - h(y)| = \frac{1}{2}|x - y|$.

Proof. (a) $cx + d$ is odd, so $g(x) \in \mathbb{Z}_2$, and $g(x) - g(y) = \det g (x - y)/((cx + d)(cy + d))$ with three units; the inverse $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ has the same form. (b) Again $cx + d$ is odd, and $h(x) \equiv b \equiv s \bmod 2$; the same difference formula gives the factor $\frac{1}{2}$. For $y \in s + 2\mathbb{Z}_2$ put $x = (dy - b)/(a - cy)$: the numerator is even and the denominator is $\equiv 2 \bmod 4$, so $x \in \mathbb{Z}_2$, and $h(x) = y$ because $a(dy - b) + b(a - cy) = y \det h$ and $c(dy - b) + d(a - cy) = \det h$. $\square$

Lemma 2.3. (a) For $i, j \geq 1$, $[K(i), K(j)] \leq K(i + j)$ and $k^2 \in K(j + 1)$ for $k \in K(j)$. For $j \geq 3$, every element of $K(j)$ is congruent modulo $K(j + 1)$ to the square of an element of $K(j - 1)$. (b) Let $N \leq \mathrm{PGL}_2(\mathbb{Z}_2)$ be closed and $m \geq 2$, and suppose that $|N_{m+1}| = 8|N_m|$. Then $K(m) \leq N$, so N is the full preimage of N_m .

Proof. (a) Represent elements of $K(i), K(j)$ by $X = 1 + 2^i u, Y = 1 + 2^j v$ with $u, v \in M_2(\mathbb{Z}_2)$. Then $XY - YX = 2^{i+j}(uv - vu)$ and $XYX^{-1}Y^{-1} = 1 + (XY - YX)X^{-1}Y^{-1}$, which gives the first claim; $Y^2 = 1 + 2^{j+1}v + 2^{2j}v^2$ gives the second. For $j \geq 3$ and $y = 1 + 2^j v$ put $x = 1 + 2^{j-1}v \in K(j - 1)$; then $x^2 = 1 + 2^j v + 2^{2j-2}v^2 \equiv y \bmod 2^{j+1}$, as $2j - 2 \geq j + 1$.

(b) The hypothesis says that N_{m+1} contains the kernel of $\mathrm{PGL}_2(\mathbb{Z}/2^{m+1}) \rightarrow \mathrm{PGL}_2(\mathbb{Z}/2^m)$, which has order 8; that is, $K(m) \leq N K(m + 1)$. Suppose $K(m) \leq N K(t)$ for some $t \geq m + 1$, and let $y \in K(t)$. As $t \geq 3$, (a) gives $y \equiv x^2 \bmod K(t + 1)$ with $x \in K(t - 1) \leq K(m)$; write $x = nk$ with $n \in N, k \in K(t)$, so that $n = xk^{-1} \in K(m)$. Then $x^2 = n^2 k[k, n]k$ with $[k, n] \in K(t + m) \leq K(t + 1)$ and $k^2 \in K(t + 1)$, so $x^2 \in n^2 K(t + 1)$, as $K(t + 1)$ is normal. Hence $K(t) \leq N K(t + 1)$ and $K(m) \leq N K(t + 1)$. By induction $K(m) \leq \bigcap_t N K(t) = \overline{N} = N$. $\square$

3 The Möbius model of the Bellaterra automaton

The data of this section are taken from [KV26, Sections 4 and 6]; all identities are verified again here. Let $F = \mathbb{Q}(\alpha)$ with $\alpha^2 + 3\alpha - 2 = 0$, and $\rho = 1 - \alpha$. Let $D = (-1, -3)_F$ be the quaternion algebra with basis $1, i, j, k = ij$, where $i^2 = -1$, $j^2 = -3$ and $ij = -ji$; for $X = x_0 + x_1i + x_2j + x_3k$ its reduced trace and norm are $\text{trd } X = 2x_0$ and $\text{nrd } X = x_0^2 + x_1^2 + 3x_2^2 + 3x_3^2$. Following [KV26, (4.4), (6.1)], where these elements are called A, B, C, H_0, H_1 , put

$$\begin{aligned}\xi_a &= \tfrac{1}{3}(j + \alpha k), & \xi_b &= \tfrac{1}{3}j - \tfrac{1}{2}\alpha i - \tfrac{1}{6}\alpha k, & \xi_c &= \tfrac{1}{3}j + \tfrac{1}{2}\alpha i - \tfrac{1}{6}\alpha k, \\ \eta_0 &= -3\alpha + 6i + \alpha j + 2k, & \eta_1 &= -6 - 3\alpha i + 2j - \alpha k.\end{aligned}$$

Lemma 3.1 ([KV26, (4.6), (6.2), (6.3)]). *In D ,*

$$\xi_a\eta_0 = -\eta_0\xi_b, \quad \xi_b\eta_0 = -\eta_0\xi_c, \quad \xi_c\eta_0 = \eta_1\xi_a, \quad \xi_a\eta_1 = \eta_1\xi_c, \quad \xi_b\eta_1 = \eta_1\xi_b, \quad \xi_c\eta_1 = \eta_0\xi_a. \quad (2)$$

Moreover $\text{trd } \xi_s = 0$ and $\text{nrd } \xi_s = \rho$ for $s \in \{a, b, c\}$, $\text{trd } \eta_0 = -6\alpha$, $\text{trd } \eta_1 = -12$ and $\text{nrd } \eta_0 = \text{nrd } \eta_1 = 36(2 - \alpha)$; hence

$$\frac{\text{trd}(\eta_0)^2}{\text{nrd}(\eta_0)} = \frac{1 - \alpha}{2} \neq \frac{5 + \alpha}{2} = \frac{\text{trd}(\eta_1)^2}{\text{nrd}(\eta_1)}. \quad (3)$$

Proof. The products are computed with the multiplication table of D and $\alpha^2 = 2 - 3\alpha$. For example, $\eta_1 = 6(\xi_b - 1)$, so ξ_b and η_1 commute; and $\text{nrd } \xi_a = \frac{1}{9}(3 + 3\alpha^2) = 1 - \alpha$. For (3), which is not stated in [KV26]: $36\alpha^2 = 18(2 - \alpha)(1 - \alpha)$ and $144 = 18(2 - \alpha)(5 + \alpha)$, because $(2 - \alpha)(1 - \alpha) = 2\alpha^2$ and $(2 - \alpha)(5 + \alpha) = 8$; equality of the two values would give $\alpha = -2$, which is not a root of $X^2 + 3X - 2$. The program `check_note.py` verifies all these identities in exact rational arithmetic (part [1] in Section 8). $\square$

The polynomials $X^2 + 3X - 2$ and $X^2 + X + 2$ are congruent to $X(X + 1)$ modulo 2, so each has a unique root in $2\mathbb{Z}_2$ (Hensel's lemma). We let α and z be these roots, so that $\alpha \equiv 10 \pmod{16}$ and $z \equiv 2 \pmod{8}$, and regard F as a subfield of $\mathbb{Q}_2$. The F -linear map $\psi : D \rightarrow M_2(\mathbb{Q}_2)$ with

$$\psi(i) = \begin{pmatrix} 1 & 1+z \\ z & -1 \end{pmatrix}, \quad \psi(j) = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$$

[KV26, (4.9)] is a homomorphism of algebras, since $\psi(i)^2 = (1 + z + z^2) \cdot 1 = -1$, $\psi(j)^2 = -3$ and $\psi(i)\psi(j) = -\psi(j)\psi(i)$; it carries trd and nrd to trace and determinant. For $g \in \text{GL}_2(\mathbb{Q}_2)$ let $\tilde{g} = \delta^{-1}g\delta$ with $\delta = \text{diag}(2, 1)$, that is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sim = \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}$; as Möbius maps $\tilde{g}(t) = g(2t)/2$ [KV26, (4.11)]. Put

$$M_s = \widetilde{\psi(\xi_s)} \quad (s \in \{a, b, c\}), \quad h_x = \widetilde{\psi(\eta_x)}/4 \quad (x \in \{0, 1\}).$$

Lemma 3.2. *The matrices M_s and h_x have entries in $\mathbb{Z}_2$, and*

$$\begin{aligned}M_a &\equiv \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}, & M_b &\equiv \begin{pmatrix} 5 & 2 \\ 0 & 3 \end{pmatrix}, & M_c &\equiv \begin{pmatrix} 7 & 1 \\ 0 & 1 \end{pmatrix} \pmod{8}, \\ h_0 &\equiv \begin{pmatrix} 2 & 8 \\ 12 & 15 \end{pmatrix}, & h_1 &\equiv \begin{pmatrix} 2 & 11 \\ 8 & 11 \end{pmatrix} \pmod{16},\end{aligned}$$

with $\det M_s = \rho \equiv 7 \pmod{8}$ and $v_2(\det h_x) = 1$. In particular $M_s \in I$ has lower left entry $\equiv 0 \pmod{4}$, and h_x satisfies the hypotheses of Lemma 2.2(b) with $s = x$. As Möbius maps,

$$\begin{aligned}M_a h_0 &= h_0 M_b, & M_b h_0 &= h_0 M_c, & M_c h_0 &= h_1 M_a, \\ M_a h_1 &= h_1 M_c, & M_b h_1 &= h_1 M_b, & M_c h_1 &= h_0 M_a.\end{aligned} \quad (4)$$

Proof. The entries are polynomials in α and z with rational coefficients; their integrality and residues follow from α and z modulo a power of 2, a finite computation (part [3] of `check_note.py`, which uses α, z modulo 2^{200}). [KV26, (4.8), (6.6)] state the residues of $\psi(\xi_s)$ modulo 2 and of $\psi(\eta_x)/4$ modulo 8. The identities (4) (see [KV26, (6.13)]) follow from (2), because ψ and $g \mapsto \tilde{g}$ are multiplicative and scalar matrices act trivially. $\square$

Proposition 3.3 (compare [KV26, Proposition 6.2]). *For a word $w = w_1 \dots w_n$ let $h_w = h_{w_1} \circ \dots \circ h_{w_n}$ and $B(w) = h_w(\mathbb{Z}_2)$. (i) $B(w)$ is a residue class modulo 2^n , the map $w \mapsto B(w)$ is a bijection from $\{0, 1\}^n$ onto $\mathbb{Z}/2^n$, and $B(wx) \subseteq B(w)$. (ii) $M_s(B(w)) = B(s(w))$ for $s \in \{a, b, c\}$ and all words w . Consequently, $\kappa(x_1 x_2 \dots) =$ the point of $\bigcap_n B(x_1 \dots x_n)$ defines a bijective isometry $\kappa : \partial T \rightarrow \mathbb{Z}_2$, with $\kappa \circ s = M_s \circ \kappa$ for $s \in \{a, b, c\}$ and $\kappa(x\xi) = h_x(\kappa(\xi))$.*

Proof. (i) Induction on n , the case $n = 0$ being $B(\emptyset) = \mathbb{Z}_2$. By Lemma 2.2(b), h_x maps $\mathbb{Z}_2$ bijectively onto $x + 2\mathbb{Z}_2$ and halves distances, so it maps each of the 2^n residue classes modulo 2^n into a residue class modulo 2^{n+1} contained in $x + 2\mathbb{Z}_2$. The images are disjoint and cover $x + 2\mathbb{Z}_2$, which consists of exactly 2^n classes modulo 2^{n+1} ; so each image is one of these classes. As $B(xw) = h_x(B(w))$, this proves (i) for words of length $n + 1$, and $B(wx) \subseteq B(w)$ because $h_x(\mathbb{Z}_2) \subseteq \mathbb{Z}_2$. (ii) If $s(xw') = y s'(w')$, then $M_s h_x = h_y M_{s'}$ by (4): the six identities are exactly the six transitions $a(0w) = 0b(w)$, $a(1w) = 1c(w)$, $b(0w) = 0c(w)$, $b(1w) = 1b(w)$, $c(0w) = 1a(w)$, $c(1w) = 0a(w)$ of (1). By induction on the length, $M_s(B(xw')) = h_y(M_{s'}(B(w')))) = h_y(B(s'(w')))) = B(s(xw'))$, and $M_s(\mathbb{Z}_2) = \mathbb{Z}_2$ by Lemma 2.2(a). The last statement follows: two infinite words with a common prefix of length exactly m have images in one class modulo 2^m and in different classes modulo 2^{m+1} . $\square$

Corollary 3.4. *Let H be the closure in $\mathrm{PGL}_2(\mathbb{Z}_2)$ of the group generated by M_a, M_b, M_c . The map $\Psi : I \rightarrow W$, $\Psi(g) = \kappa^{-1} \circ g \circ \kappa$, is an isomorphism of topological groups onto a closed subgroup of W , and $\Psi(H) = \overline{G(\mathcal{B})}$. In particular, $\overline{G(\mathcal{B})} \cong H$.*

Proof. By Lemma 2.2(a) and Proposition 3.3, $\Psi(g)$ is an isometry of ∂T , hence an element of W . Ψ is a homomorphism; it is continuous, as $\Psi(K(n)) \leq \mathrm{St}_W(n)$ by Lemma 2.2(a); and it is injective, since a Möbius map fixing every point of $\mathbb{Z}_2$ satisfies $cx^2 + (d-a)x - b = 0$ for all $x \in \mathbb{Z}_2$, so it is the identity. As I is compact, Ψ is a homeomorphism onto its closed image. By Proposition 3.3, $\Psi(M_s) = s$, so Ψ maps $\langle M_a, M_b, M_c \rangle$ onto $G(\mathcal{B})$ and H onto $\overline{G(\mathcal{B})}$. $\square$

4 The closure of the Bellaterra group

Let $\Gamma \leq I$ be the set of classes g with a representative $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod{4}$ and $\chi(g) \in \{\pm 1\}$, that is, $\det g \equiv \pm 1 \pmod{8}$ up to squares of units. It is a subgroup (the lower left entry of a product is $ca' + dc'$, and $\{\pm 1\}$ is a subgroup of $\mathbb{Z}_2^\times / (\mathbb{Z}_2^\times)^2$), and it contains $K(3)$, as $1 + 8u$ has lower left entry $\equiv 0 \pmod{8}$ and determinant $\equiv 1 \pmod{8}$. So Γ is open, and it is the full preimage of Γ_3 .

Theorem 4.1. *$H = \Gamma$. Moreover $\delta\Gamma\delta^{-1} = \Gamma'$, where Γ' is the group of Theorem B, identified with the subgroup $\{g \in \mathrm{PGL}_2(\mathbb{Z}_2) : g \equiv \text{scalar} \pmod{2}, \chi(g) = \pm 1\}$ of $\mathrm{PGL}_2(\mathbb{Z}_2)$, and $[\mathrm{PGL}_2(\mathbb{Z}_2) : \Gamma] = [\mathrm{PGL}_2(\mathbb{Z}_2) : \Gamma'] = 12$.*

Proof. By Lemma 3.2, each M_s has lower left entry $\equiv 0 \pmod{4}$ and $\det M_s = \rho \equiv -1 \pmod{8}$; so $M_s \in \Gamma$ and $H \leq \Gamma$. In normalized form $\begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$, the elements of Γ_k ($k \geq 3$) are the triples (a, b, c) modulo 2^k with $c \equiv 0 \pmod{4}$ and $a - bc \equiv \pm 1 \pmod{8}$; counting gives $|\Gamma_3| = 2 \cdot 8 \cdot 2 = 32$ and $|\Gamma_4| = 4 \cdot 16 \cdot 4 = 256$. The residues of M_a, M_b, M_c modulo 8 generate a group of order 32, and modulo 16 a group of order 256 (a finite computation in groups of these orders, part [5] of `check_note.py`). Hence $H_3 = \Gamma_3$ and $|H_4| = 8|H_3|$, so $K(3) \leq H$ by Lemma 2.3(b) with $m = 3$. As H and Γ both contain $K(3)$ and have the same image modulo 8, $H = \Gamma$. The index is $|\mathrm{PGL}_2(\mathbb{Z}/8)|/|\Gamma_3| = 384/32 = 12$.

Conjugation by δ maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$; for a, d odd this matrix is $\equiv 1 \pmod{2}$ if and only if $c \equiv 0 \pmod{4}$. As the determinant does not change, $\delta\Gamma\delta^{-1} = \{g : g \equiv \text{scalar} \pmod{2}, \chi(g) = \pm 1\}$. Its index is $|\text{PGL}_2(\mathbb{F}_2)| \cdot |\chi(K(1)) : \{\pm 1\}| = 6 \cdot 2$, because $\chi(K(1))$ contains the classes of $\det \text{diag}(u, 1) = u$ for all units u . Finally, if $g \equiv 1 \pmod{2}$ and $\det g = \pm u^2$ with $u \in \mathbb{Z}_2^\times$, then $g/u \equiv 1 \pmod{2}$ and $\det(g/u) = \pm 1$. So the continuous homomorphism from $\{g \in \text{GL}_2(\mathbb{Z}_2) : g \equiv 1 \pmod{2}, \det g = \pm 1\}$ to $\text{PGL}_2(\mathbb{Z}_2)$ has image $\delta\Gamma\delta^{-1}$, and its kernel consists of the scalars λ with $\lambda^2 = \pm 1$, that is $\lambda = \pm 1$ (-1 is not a square in $\mathbb{Q}_2$). This is the identification with Γ' , as a topological group since the groups are compact. $\square$

Proof of Theorem B. By Corollary 3.4 and Theorem 4.1, $\overline{G(\mathcal{B})} \cong H = \Gamma \cong \Gamma'$, an open subgroup of index 12 of $\text{PGL}_2(\mathbb{Z}_2)$. As $\text{PGL}_2(\mathbb{Q}_2)$ is a 2-adic Lie group of dimension 3, so is its open subgroup Γ' . $\square$

Remark 4.2 (Level orders). An element $\begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$ of I acts trivially on $\mathbb{Z}/2^n$ if and only if $(a-1)t + b - ct^2 \equiv 0 \pmod{2^n}$ for all t . The values $t = 0, 1, -1$ give $b \equiv 0$, $a - 1 \equiv c$ and $2c \equiv 0$, and these imply the condition, as $t(t-1)$ is even. So the kernel of the action of I_n on $\mathbb{Z}/2^n$ is $\{1, z_n\}$ with $z_n = \begin{pmatrix} 1+2^{n-1} & 0 \\ 2^{n-1} & 1 \end{pmatrix}$, and $z_n \in \Gamma_n$ if and only if $n \geq 4$. Since Ψ carries the action of Γ on $\mathbb{Z}/2^n$ to the action of $\overline{G(\mathcal{B})}$ on $\{0, 1\}^n$, Theorem B gives $|G(\mathcal{B})_n| = |\Gamma_n|/2 = 2^{3n-5}$ for $n \geq 4$ (and $|G(\mathcal{B})_3| = 2^5$). `check_note.py` confirms, for $3 \leq n \leq 6$, that $G(\mathcal{B})_n$, computed from the automaton, is the group induced by Γ_n through κ ; in particular $G(\mathcal{B})_4, G(\mathcal{B})_5, G(\mathcal{B})_6$ contain 43, 167 and 647 involutions. The orders 2^{3n-5} were also computed from the automaton alone for $n \leq 20$ (Section 8).

5 The Sylow statement

For a profinite group G let $d(G)$ be the minimal number of topological generators. For a pro-2 group, $d(G) = \dim_{\mathbb{F}_2} G/\Phi(G)$, where the Frattini subgroup $\Phi(G)$ is the closed subgroup generated by squares and commutators, and a subset generates G topologically if and only if its image generates $G/\Phi(G)$ (the Burnside basis theorem, [DdMS99, Ch. 1]). In normalized form, the product of two elements of I is

$$\begin{pmatrix} a & b \\ c & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & 1 \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + b \\ ca' + c' & cb' + 1 \end{pmatrix}, \quad cb' + 1 \in \mathbb{Z}_2^\times. \quad (5)$$

Proof of Corollary B. (a) By Lemma 2.1(a), $\overline{G(\mathcal{B})}$ contains three distinct involutions. If K is a field of characteristic $\neq 2$ and $g \in \text{SL}_2(K)$ satisfies $g^2 = 1$, then g is diagonalizable with eigenvalues in $\{\pm 1\}$ (its minimal polynomial divides $(X-1)(X+1)$) and $\det g = 1$, so $g = \pm 1$. Hence $\text{SL}_2(\mathbb{Z}_2) \subset \text{SL}_2(\mathbb{Q}_2)$ and each of its subgroups contains at most one involution.

(b) Let S be as in Section 1 and $P = S/\{\pm 1\}$ its image in $\text{PSL}_2(\mathbb{Z}_2)$. As S has index 3 in $\text{SL}_2(\mathbb{Z}_2)$, P is a pro-2 Sylow subgroup of $\text{PSL}_2(\mathbb{Z}_2)$, and by the Sylow theorems for profinite groups all of them are conjugate to P . A class in I with determinant u^2 has the representative $g/u \in S$, so $P = \{g \in I : \chi(g) = 1\}$; in particular $P \supseteq K(3)$. We give two arguments.

Abelianizations. $\overline{G(\mathcal{B})}$ is topologically generated by the involutions a, b, c , so its abelianization is topologically generated by three commuting elements of order at most 2. In an abelian topological group the set $\{x : x^2 = 1\}$ is a closed subgroup; so the abelianization of $\overline{G(\mathcal{B})}$ is elementary abelian (of order at most 8). On the other hand, P maps onto P_4 , whose abelianization has order 16 and exponent 8 (part [7] of `check_note.py`); so the abelianization of P has an element of order at least 8.

Generators. The elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ of S generate P_4 , a group of order 2^8 (part [7]). For $j \geq 4$, Lemma 2.3(a) shows that every element of $K(j)$ is congruent modulo $K(j+1)$ to the square of an element of $K(j-1) \leq K(3) \leq P$, so $K(j) \leq \Phi(P)K(j+1)$; as in the proof of

Lemma 2.3(b) this gives $K(4) \leq \Phi(P)$. So $P/\Phi(P)$ is a quotient of $P/K(4) = P_4$, the images of T, w generate it, and $d(P) \leq 2$. But $d(\Gamma) \geq 3$: by (5), the maps

$$\varepsilon_1(g) = [\chi(g) = -1], \quad \varepsilon_2(g) = b \bmod 2, \quad \varepsilon_3(g) = c/4 \bmod 2$$

(for the normalized representative) are homomorphisms $\Gamma \rightarrow \mathbb{F}_2$: the new b is $\equiv b + b' \bmod 2$, and for $c \equiv c' \equiv 0 \bmod 4$ the new c is $\equiv c + c' \bmod 8$, as $ca' \equiv c \bmod 8$ and $cb' + 1 \equiv 1 \bmod 4$. The normalized residues of M_a, M_b, M_c modulo 16 are $(15, 4, 12), (15, 6, 0), (15, 1, 8)$, on which $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ takes the values $(1, 0, 1), (1, 0, 0), (1, 1, 0)$, an invertible matrix over $\mathbb{F}_2$. So Γ maps onto $\mathbb{F}_2^3$, and $d(\overline{G(\mathcal{B})}) = d(\Gamma) = 3$.

(c) I is a pro-2 Sylow subgroup of $\mathrm{PGL}_2(\mathbb{Z}_2)$: its image in $\mathrm{PGL}_2(\mathbb{F}_2) \cong S_3$ has order 2, and the kernel $K(1)$ is pro-2. The homomorphism χ (with values in a group isomorphic to $\mathbb{F}_2^2$) and the maps $b \bmod 2$ and $c/2 \bmod 2$ (normalized; for c, c' even the new c is $\equiv c + c' \bmod 4$ by (5)) map I onto $\mathbb{F}_2^4$: on $\mathrm{diag}(-1, 1)$, $\mathrm{diag}(5, 1)$, T and $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ they take the four unit vectors. So $d(I) \geq 4 > 3 \geq d(\overline{G(\mathcal{B})})$. $\square$

Remark 5.1. Corollary B does not depend on the Möbius model of Section 3: (a) uses only Lemma 2.1(a), and in (b) and (c) the facts about $\overline{G(\mathcal{B})}$ that are used, that it is generated by three involutions and that $d(\overline{G(\mathcal{B})}) \leq 3$, are clear. The second argument for (b) needs $d(\overline{G(\mathcal{B})}) \geq 3$, which also follows without the model from $d(\overline{G(\mathcal{B})}) \geq d(G(\mathcal{B})_3) = 3$, a computation in a group of order 32 (part [7]).

Remark 5.2. The group $\overline{G(\mathcal{B})} \cong \Gamma'$ contains, as an open subgroup, the image of $\ker(\mathrm{SL}_2(\mathbb{Z}_2) \rightarrow \mathrm{SL}_2(\mathbb{Z}/4))$, which meets $\{\pm 1\}$ trivially; so $\overline{G(\mathcal{B})}$ is commensurable with $\mathrm{SL}_2(\mathbb{Z}_2)$ (both have the Lie algebra $\mathfrak{sl}_2(\mathbb{Q}_2)$). Orders of finite quotients do not tell P from Γ : $|P_k| = |\Gamma_k| = 2^{3k-4}$ for $k \geq 3$, and as $z_n \in P_n$ for $n \geq 4$ (Remark 4.2), the Möbius action of P on $\mathbb{Z}/2^n$ induces a group of order 2^{3n-5} , as $G(\mathcal{B})$ does on $\{0, 1\}^n$. This may have suggested the conjecture; we do not know. The invariants d and the abelianization separate the two groups.

6 The closure of the Aleshin group

6.1 The complement map in the Möbius coordinate

Let $\tau' = \kappa \circ \tau \circ \kappa^{-1} : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, and let p_x be the fixed point of h_x in $\mathbb{Z}_2$; it is unique, as h_x is a contraction of $\mathbb{Z}_2$ (Lemma 2.2(b), Lemma 3.2).

Lemma 6.1. (i) τ' is an isometry of $\mathbb{Z}_2$. (ii) $\tau' \circ h_0 = h_1 \circ \tau'$ and $\tau' \circ h_1 = h_0 \circ \tau'$. (iii) $p_x = \kappa(xxx\dots)$, and $\tau'(p_0) = p_1$.

Proof. (i) κ is an isometry (Proposition 3.3) and τ preserves the length of common prefixes. (ii) $\kappa(x\xi) = h_x(\kappa(\xi))$ and $\tau(x\xi) = (1+x)\tau(\xi)$. (iii) $h_x(\kappa(xx\dots)) = \kappa(xx\dots)$, and $\tau(00\dots) = 11\dots$. $\square$

Lemma 6.2. τ' is not differentiable at p_0 : the difference quotient $(\tau'(y) - p_1)/(y - p_0)$ has no limit as $y \rightarrow p_0$.

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_2)$ have a fixed point $p \in \mathbb{Q}_2$ with $e = cp + d \neq 0$. Then $(p, 1)^\top$ is an eigenvector of M with eigenvalue e , the other eigenvalue is $e' = \det M/e$, and $M'(p) = \det M/e^2 = e'/e$; hence

$$\frac{\mathrm{tr}(M)^2}{\det M} = \frac{(e + e')^2}{ee'} = M'(p) + 2 + M'(p)^{-1}.$$

For h_x and p_x , $cp_x + d$ is odd by Lemma 3.2, and $\mathrm{tr}(h_x)^2/\det(h_x) = \mathrm{trd}(\eta_x)^2/\mathrm{nrd}(\eta_x)$, since scalars and conjugation do not change $\mathrm{tr}^2/\det$. By (3), the multipliers $\lambda_x = h'_x(p_x)$ satisfy $\lambda_0 + \lambda_0^{-1} \neq \lambda_1 + \lambda_1^{-1}$,

so $\lambda_0 \neq \lambda_1$; both are non-zero, with $|\lambda_x| = \frac{1}{2}$. Suppose that $L = \lim_{y \rightarrow p_0} (\tau'(y) - p_1)/(y - p_0)$ exists. By Lemma 6.1(i), $|\tau'(y) - p_1| = |y - p_0|$, so $|L| = 1$. For $y \neq p_0$, by Lemma 6.1(ii) and (iii),

$$\frac{\tau'(h_0 y) - p_1}{h_0 y - p_0} = \frac{h_1(\tau' y) - h_1(p_1)}{\tau' y - p_1} \cdot \frac{\tau' y - p_1}{y - p_0} \cdot \frac{y - p_0}{h_0 y - h_0 p_0},$$

where no denominator vanishes, the maps being injective. As $y \rightarrow p_0$, also $h_0 y \rightarrow p_0$ and $\tau' y \rightarrow p_1$, so the left side tends to L and the right side to $\lambda_1 L \lambda_0^{-1}$. As $L \neq 0$, this gives $\lambda_0 = \lambda_1$, a contradiction. $\square$

Numerically, $\lambda_1/\lambda_0 \equiv 377 \pmod{2^{10}}$, and at the points $y_j = h_0^j(1) \rightarrow p_0$ the difference quotients of τ' are, for $j \geq 2$, alternately $\equiv 3$ and $\equiv 11 \pmod{16}$ (for $j = 1$ and $j = 2$ both are $\equiv 3$; part [9] of `check_note.py`; not used).

6.2 Analytic groups

We use two standard facts of p -adic Lie theory, for which we refer to [Laz65], [Bou89, Ch. III], [Ser92, Part II, Ch. IV and V], [DdMS99, Ch. 8 and 9] and [Sch11, Ch. II and III].

(F1) If S is a closed subgroup of a p -adic Lie group G , then S is a Lie subgroup (Cartan's theorem over $\mathbb{Q}_p$), and G/S carries a structure of analytic manifold for which the projection $\pi : G \rightarrow G/S$ is analytic and every left translation $xS \mapsto gxS$ is analytic.

(F2) Every continuous homomorphism between p -adic Lie groups is analytic.

Two consequences are used. (U1) A topological group carries at most one structure of p -adic Lie group compatible with its topology (apply (F2) to the identity map), so being 2-adic analytic is a property of the topological group. (U2) Let U be an open normal subgroup of index 2 of a topological group $G = U \cup U\tau$. If U is a p -adic Lie group, so is G : give $U\tau$ the charts of U moved by right multiplication by τ ; conjugation by τ is a continuous automorphism of U , hence analytic by (F2), and then the group operations, such as $(u\tau)(v\tau) = u(\tau v \tau^{-1})\tau^2$, are analytic (a special case of a general fact on groups with an open Lie subgroup [Bou89, Ch. III]).

Lemma 6.3. *Let $\mathcal{J}$ be a closed subgroup of W containing $\Psi(\Gamma)$. If $\mathcal{J}$ is a 2-adic analytic group, then for every $j \in \mathcal{J}$ the map $j' = \kappa \circ j \circ \kappa^{-1}$ of $\mathbb{Z}_2$ is differentiable at every point.*

Proof. $\mathcal{J}$ is compact and acts continuously on $\mathbb{Z}_2$ by $j \mapsto j'$. Fix $x_0 \in \mathbb{Z}_2$, let S be its stabilizer in $\mathcal{J}$ and $\pi : \mathcal{J} \rightarrow \mathcal{J}/S$ the projection. Γ contains the translations $u_v = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$, $v \in \mathbb{Z}_2$, so $\mathcal{J}$ is transitive, and $\iota : \mathcal{J}/S \rightarrow \mathbb{Z}_2$, $jS \mapsto j'(x_0)$, is a continuous bijection from a compact space onto a Hausdorff space, hence a homeomorphism. Let $f = \iota^{-1}$. As $u_{y-x_0}(x_0) = y$, we have $f(y) = \pi(\Psi(u_{y-x_0}))$. The map $v \mapsto u_v$ from $\mathbb{Z}_2$ to Γ is analytic; $\Psi : \Gamma \rightarrow \mathcal{J}$ is a continuous homomorphism of 2-adic Lie groups (Γ is open in $\mathrm{PGL}_2(\mathbb{Z}_2)$), hence analytic by (F2); and π is analytic by (F1). So f is analytic.

Let e be the dimension of $\mathcal{J}/S$ (constant, as $\mathcal{J}$ acts transitively by analytic maps). As $\mathcal{J}/S \cong \mathbb{Z}_2$ has no isolated points, $e \geq 1$. Suppose $e \geq 2$. Cover $\mathbb{Z}_2$ by finitely many balls B_i of radii r_i on each of which f is given, in a chart, by convergent power series; there f is Lipschitz, say with constant L_i (by the ultrametric inequality). Splitting B_i into 2^k balls of radius $2^{-k}r_i$ shows that $f(B_i)$ is covered by 2^k sets of diameter at most $L_i 2^{-k}r_i$, of total Haar measure at most $2^k(L_i r_i 2^{-k})^e \rightarrow 0$. Analytic changes of charts are locally Lipschitz and preserve null sets, so the manifold $\mathcal{J}/S = \bigcup_i f(B_i)$ would be a null set in every chart, which is absurd. Hence $e = 1$, and f has a derivative f' in charts.

For $h \in \Gamma$ we have $f(h(y)) = \Psi(h) \cdot f(y)$, both sides corresponding to $h(y)$ under ι . Γ acts on $\mathbb{Z}_2$ by Möbius maps, which are analytic with non-vanishing derivative, and on $\mathcal{J}/S$ by analytic diffeomorphisms (F1); by the chain rule the zero set of f' is Γ -invariant, hence empty or all of $\mathbb{Z}_2$. If $f' = 0$ on $\mathbb{Z}_2$, then f is locally constant (a convergent power series over $\mathbb{Q}_2$ with zero derivative is

constant), contradicting injectivity. So f' never vanishes. Then $\iota = f^{-1}$ is differentiable in charts: if $u = f(y) \rightarrow u_0 = f(y_0)$, then $y \rightarrow y_0$ by continuity of ι , and $(\iota(u) - \iota(u_0))/(u - u_0) = (y - y_0)/(f(y) - f(y_0)) \rightarrow 1/f'(y_0)$. Finally $j' = \iota \circ L_j \circ f$, where $L_j(kS) = jkS$ is analytic by (F1), because $\iota(jkS) = (jk)'(x_0) = j'(k'(x_0))$. By the chain rule, j' is differentiable at every point. $\square$

Proof of Theorem A. Let $\mathcal{J} \leq W$ be closed with $\overline{G(\mathcal{B})} \leq \mathcal{J}$ and $\tau \in \mathcal{J}$. By Corollary 3.4 and Theorem 4.1, $\Psi(\Gamma) = \overline{G(\mathcal{B})} \leq \mathcal{J}$. If $\mathcal{J}$ were 2-adic analytic, then by Lemma 6.3 the map $\tau' = \kappa \circ \tau \circ \kappa^{-1}$ would be differentiable at p_0 , contrary to Lemma 6.2. So no such $\mathcal{J}$ is 2-adic analytic; in particular J is not. By Lemma 2.1(c), $\overline{G(\mathcal{A})}$ is an open normal subgroup of index 2 of $J = \overline{G(\mathcal{A})} \cup \overline{G(\mathcal{A})}\tau$ (τ normalizes $G(\mathcal{A})$, hence $\overline{G(\mathcal{A})}$). If $\overline{G(\mathcal{A})}$ were 2-adic analytic, then so would be J by (U2). Hence $\overline{G(\mathcal{A})}$ is not 2-adic analytic, and by Lazard's theorem it has infinite rank [Laz65], [DdMS99, Ch. 3 and 8]. $\square$

Remark 6.4 (Computational evidence). Table 1 lists the layer ranks $d_n = \log_2(|G_n|/|G_{n-1}|)$; as $2^{d_n} = |\text{St}_G(n-1) : \text{St}_G(n)|$ is the order of an elementary abelian subquotient, a closed subgroup of finite rank has bounded d_n . For $G(\mathcal{A})$ they grow roughly by a factor 2 per level, consistent with Theorem A (and $\log_2 |G(\mathcal{A})_{14}|/(2^{14} - 1) \approx 0.085$ suggests a positive Hausdorff dimension, which we have not proved); for $G(\mathcal{B})$, $d_n = 3$ for $n \geq 5$ (Remark 4.2). The table is not used in any proof.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d_n for $G(\mathcal{A})$	1	1	2	3	3	4	7	13	23	45	87	174	344	686
$\log_2 G(\mathcal{A})_n $	1	2	4	7	10	14	21	34	57	102	189	363	707	1393
d_n for $G(\mathcal{B})$	1	2	2	2	3	3	3	3	3	3	3	3	3	3

Table 1: Layer ranks d_n of the level quotients of the Aleshin group $G(\mathcal{A})$ and of the Bellaterra group $G(\mathcal{B})$, computed from the automata (Section 8). For $G(\mathcal{B})$ the values are proved for all n (Remark 4.2) and were computed up to $n = 20$.

7 Conclusion

The conjecture is false: the closure of the Bellaterra group is 2-adic analytic but is not isomorphic to a pro-2 Sylow subgroup of $\text{SL}_2(\mathbb{Z}_2)$ (nor of $\text{PSL}_2(\mathbb{Z}_2)$ or $\text{PGL}_2(\mathbb{Z}_2)$), and the closure of the Aleshin group is not 2-adic analytic. Precisely, $\overline{G(\mathcal{B})} \cong \Gamma'$ (Theorem B), which is commensurable with $\text{SL}_2(\mathbb{Z}_2)$ but is not a Sylow subgroup of SL_2 , PSL_2 or $\text{PGL}_2(\mathbb{Z}_2)$ (Corollary B), and no closed subgroup of W containing $\overline{G(\mathcal{B})}$ and τ , in particular neither $\overline{G(\mathcal{A})}$ nor J , is 2-adic analytic (Theorem A). Whether $\overline{G(\mathcal{A})}$ has positive Hausdorff dimension, as Table 1 suggests, is not decided here.

8 Computations and reproducibility

The proofs use four finite computations: the residues of Lemma 3.2; $|H_3| = 32$ and $|H_4| = 256$ (Theorem 4.1); $\tau \notin G(\mathcal{A})_4$ (for the index 2 in Lemma 2.1(c)); and, for Corollary B(b), $\langle T, w \rangle = P_4$ or the exponent 8 of the abelianization of P_4 . Everything else is proved without computation. *The exact program of this note, check_note.py* (standard library only; about one second), checks Lemmas 3.1 and 3.2 in exact arithmetic (with α, z modulo 2^{200}), Proposition 3.3 for all words of length at most 10 with a negative control, the computations of Theorem 4.1, Remark 4.2 for $3 \leq n \leq 6$ ($G(\mathcal{B})_n$, computed from the automaton, is the group induced by Γ_n through κ), the data of Corollary B and Remark 5.1, Lemma 2.1 on level 6, and the multipliers and difference quotients of Lemma 6.2; all 60 checks pass. *The programs with which the results were first obtained (original/, eight programs*

with their recorded outputs) check the same statements, some on larger ranges (the coding for words of length at most 14, $H_k = \Gamma_k$ for $k \leq 7$, d and abelianizations of Γ_k , P_k and I_k for $k \leq 6$), and compute Table 1 (about 15 minutes for $G(\mathcal{A})_{14}$); *the programs of the verification runs* are in `independent_runs/` (the two runs on the first version) and `independent_run_2/` (the run on this note, seven programs). All programs of the package were run again from a copy, when this note was written (except the two runs for $G(\mathcal{A})_{14}$) and by the verification run of this note from the archive (all of them, the two runs for $G(\mathcal{A})_{14}$ included); the outputs agree with the recorded ones up to running times.

Verification. The results were first obtained with written proofs and the programs in `original/`. Two independent verification runs, both AI-assisted, examined this first written version; each read the source [Bar07] in the publisher’s open PDF and decoded its figure (one from the drawing commands of the PDF, the other from a rendering of the page), re-derived all proofs line by line and wrote its own programs. Both confirmed Theorems A and B and Corollary B and found no gap. The first run recomputed in exact arithmetic the data of Lemmas 3.1 and 3.2, the coding for words of length at most 12, $H_k = \Gamma_k$ for $k \leq 6$, the level orders for $n \leq 8$ (with SymPy), and the data of Corollary B and Lemma 6.2. The second run computed, in the conventions of [Bar07, VV10, KV26], the level orders of $G(\mathcal{B})$ for $n \leq 20$ and of $G(\mathcal{A})$ for $n \leq 14$ (validated against breadth-first enumeration, SymPy and 100 random groups), the coding for words of length at most 16 with negative controls, $H_k = \Gamma_k$ for $k \leq 64$, and d and the abelianizations of Γ_k and of the images of the Sylow subgroups of SL_2 , PSL_2 and $\mathrm{PGL}_2(\mathbb{Z}_2)$; it re-ran the programs of the first version, reproducing their outputs, and repeated the literature search. The runs required corrections, all of which were made: the references and (U1), (U2) in Section 6; the decoded figure in Section 1.2; two points of wording and of proof in Theorem B and Corollary B(b); the framing of part (ii) and the caveats on novelty; the year of the report; and Remarks 5.1 and 5.2.

A further independent verification run, also AI-assisted, examined the text of this note before its final revision. It read [Bar07] again in the publisher’s open PDF and decoded the figure once more from the drawing commands of the PDF, with a rendering of the page as a check; compared every item taken from [KV26], [VV10] and [VV07] with these texts; re-derived all proofs line by line; and wrote seven programs of its own (`independent_run_2/`). They check Lemma 3.1 in exact arithmetic, the data of Lemma 3.2 and the identities (4) modulo 2^{204} , the coding of Proposition 3.3 for all words of length at most 14 with negative controls, $H_k = \Gamma_k$ for $k \leq 7$ and the identification of Theorem 4.1 for $k \leq 6$, the squaring step of Lemma 2.3(a) and its failure for $j = 2$, the invariants d and the abelianizations of Γ_k , P_k and I_k for $k \leq 6$ with the other data of Corollary B and Remarks 4.2, 5.1 and 5.2, the multipliers and difference quotients of Lemma 6.2, and the level orders of $G(\mathcal{A})$ and $G(\mathcal{B})$ for $n \leq 10$ with SymPy. It ran all programs of the package again from the archive and repeated the literature search. It found no mathematical error. It required seven corrections, all of which were made: a section number of [VV10], the range $j \geq 2$ in the numerical remark after Lemma 6.2, the equation numbers in Lemma 3.1, a wording and the description of the acknowledgement of [KV26] in Section 1, and the record of this run and of its searches.

Scope and priority. The note decides both parts of Conjecture 1.1, and part (ii) also in its two corrected readings. The conjecture is due to Bartholdi [Bar07]; the automata, the freeness of $G(\mathcal{A})$ and $G(\mathcal{B}) \cong C_2 * C_2 * C_2$ to Aleshin [Ale83], M. and Y. Vorobets [VV07] and Muntyan and Savchuk (see [VV10, Nek05]); the Möbius model of Section 3 to Kharlampovich and Vdovina [KV26], an unrefereed preprint whose acknowledgement declares AI assistance and whose identities we verify again; the facts of p -adic Lie theory to Lazard [Laz65] and the textbooks [Bou89, Ser92, DdMS99, Sch11]. Corollary B(a) is elementary, and Theorem B is a short consequence of the model of [KV26]

and Lemma 2.3; we found neither Theorem B nor Theorem A in [KV26] or elsewhere. No printed answer to the question was found in the sources searched: the report [Bar07]; [KV26], [VV07], [VV10] and the arXiv papers 0902.3220 (which cites [Bar07] only for the Hanoi towers group), 1212.0605, 2312.05427, 2203.12963, 1407.4600, 2602.22356, 0806.4801, 1707.00215 and 1012.1531 (automaton groups) and 1812.09921, 2002.02053, 2012.00919 and 2303.14852 (self-similar p -adic analytic groups), in full text or in the abstract; queries to the arXiv (Aleshin, Bellaterra, Siegenthaler, bireversible automata, automaton groups with “closure”, “analytic”, “Sylow” or “PGL”), to OpenAlex (including the five works citing [Bar07]), zbMATH Open, Semantic Scholar and Crossref; and six web searches, all on 22 October 2026. Two works citing the report could not be read: the thesis of Siegenthaler [Sie10] (the repository refused automated access, which was not circumvented) and the survey of Grigorchuk [Gri11] (closed access); if either determines these closures, the statement on novelty must be revised. A search that finds nothing is not a proof of novelty, and no priority is claimed. [Bar07] was read in the publisher’s open PDF (pp. 1485–1489, with the figure), [KV26] in version 1 (the only version on 22 October 2026), [VV10] in the arXiv version 1 (Sections 1, 3 and 6; our section numbers for [VV10] refer to this version) and [VV07] in the arXiv version 2 (Sections 1 and 3). [Ale83], [Nek05], [Laz65] and the textbooks were not consulted: (F1), (F2), Lazard’s theorem and the Burnside basis theorem are quoted in their standard form and cited by chapter, since we could not check theorem numbers.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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