

A Three-Vertex Counterexample to Product-Order Lifting of TF Classes

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Abstract

Does every comparison in the facial semistable order admit coordinatewise ordered representatives in the corresponding TF-equivalence classes? We give a negative answer for the path algebra of $3 \rightarrow 2 \rightarrow 1$. The classes

$$C = \{(-a, -b, a+b) : a, b > 0\}, \quad D = \{(-c, 0, c) : c > 0\}$$

satisfy $C \leq_{\text{fss}} D$, but no point of C is coordinatewise comparable with any point of D . Both classes lie on the hyperplane perpendicular to the sincere dimension vector $(1, 1, 1)$. We compute their strict and weak semistable torsion classes directly from the six interval modules. The example also shows that taking the transitive closure of the relation induced by coordinatewise comparisons does not recover the facial semistable order. This answers the representative-lifting question in Hanson's Remark 4.19, with either choice of sign convention. It does not resolve the general conjecture that g -vector cones determine the support τ -tilting order.

1 The lifting question

Let A be a basic finite-dimensional algebra over a field k , and write dimension vectors in the basis of its simple modules. For $\theta \in \mathbb{R}^n$, set

$$\begin{aligned} \mathcal{T}_\theta &= \{X : \theta \cdot \dim X' > 0 \text{ for every nonzero quotient } X' \text{ of } X\}, \\ \overline{\mathcal{T}}_\theta &= \{X : \theta \cdot \dim X' \geq 0 \text{ for every quotient } X' \text{ of } X\}. \end{aligned}$$

These are torsion classes. Two weights are *TF-equivalent* when their pairs $(\mathcal{T}_\theta, \overline{\mathcal{T}}_\theta)$ agree. Write $[\theta]$ for the equivalence class. Following the inclusion convention of [1, Definitions 2.9 and 4.6], the facial semistable order is

$$[\theta] \leq_{\text{fss}} [\eta] \iff \mathcal{T}_\theta \subseteq \mathcal{T}_\eta \text{ and } \overline{\mathcal{T}}_\theta \subseteq \overline{\mathcal{T}}_\eta.$$

Coordinatewise order implies this order: if $\theta_i \leq \eta_i$ for every i , every quotient dimension vector has nonnegative coordinates, so both inclusions follow.

Hanson asks whether an ordered pair of TF classes always has representatives ordered coordinatewise [1, Remark 4.19]. We formulate the question with the orientation dictated by the displayed definitions. The printed remark writes the difference in the opposite direction and omits primes in its final difference; the example below excludes coordinatewise comparability in *both* directions, so it is independent of these notational issues. The subsequent correction [2] concerns other statements and is compatible with the definitions used here.

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Theorem 1. *Let $A = kQ$ for $Q = (3 \rightarrow 2 \rightarrow 1)$, with modules interpreted as representations of this quiver. Put*

$$C = \{(-a, -b, a+b) : a, b > 0\}, \quad D = \{(-c, 0, c) : c > 0\}.$$

Then C and D are entire TF-equivalence classes and $C <_{\text{fss}} D$. Nevertheless, for every $x \in C$ and $y \in D$, neither $x \leq y$ nor $y \leq x$ holds coordinatewise.

This is a counterexample among proper faces of a finite g -vector fan. In particular, it does not contradict the known recovery of the support τ -tilting order from a finite fan [3, Corollary 6.13], nor does it settle the infinite case [3, Conjecture 6.14]. The latter is the question represented by OWR-12732-005 in the frozen UnsolvedMath v1.6.0 collection, originating in [4].

2 The six interval modules

For $1 \leq i \leq j \leq 3$, let M_{ij} have a copy of k at vertices $i, \dots, j$, zero elsewhere, and identity maps on arrows within its support. Write $S_i = M_{ii}$. Every finite-dimensional representation of Q is a direct sum of these six modules. One can see this by choosing bases adapted to the image and kernel in a pair of maps $V_3 \xrightarrow{g} V_2 \xrightarrow{f} V_1$. Explicitly, if r_f, r_g, r_{fg} are their ranks, the multiplicities of $M_{13}, M_{12}, M_{23}, S_1, S_2, S_3$, respectively, are

$$r_{fg}, \quad r_f - r_{fg}, \quad r_g - r_{fg}, \quad \dim V_1 - r_f, \quad \dim V_2 - r_f - r_g + r_{fg}, \quad \dim V_3 - r_g.$$

The compatible basis choice gives the claimed decomposition; in particular, it is valid over any field.

The nonzero quotients of M_{ij} are exactly $M_{\ell j}$ for $i \leq \ell \leq j$. Hence membership in $\mathcal{T}_\theta$ or $\overline{\mathcal{T}}_\theta$ is decided by the suffix sums

$$\theta_j, \theta_{j-1} + \theta_j, \dots, \theta_i + \dots + \theta_j :$$

they must all be positive or all be nonnegative, respectively. It suffices to test indecomposables. Indeed, both classes are closed under extensions and direct summands. For extension closure, any quotient of an extension is an extension of a quotient of the subobject by a quotient of the quotient object; weights add, with at least one nonzero term for strict positivity.

Let $x = (-a, -b, a+b) \in C$ and $y = (-c, 0, c) \in D$. The resulting calculation is

Module	$\mathcal{T}_x$	$\overline{\mathcal{T}}_x$	$\mathcal{T}_y$	$\overline{\mathcal{T}}_y$
S_1	no	no	no	no
S_2	no	no	no	yes
S_3	yes	yes	yes	yes
M_{12}	no	no	no	no
M_{23}	yes	yes	yes	yes
M_{13}	no	yes	no	yes

Thus, with add denoting finite direct sums and direct summands,

$$\mathcal{T}_x = \mathcal{T}_y = \text{add}(S_3 \oplus M_{23}), \tag{1}$$

$$\overline{\mathcal{T}}_x = \text{add}(S_3 \oplus M_{23} \oplus M_{13}), \tag{2}$$

$$\overline{\mathcal{T}}_y = \text{add}(S_3 \oplus M_{23} \oplus M_{13} \oplus S_2). \tag{3}$$

3 Entire classes and the obstruction

We now verify that the parameter sets in the theorem are whole equivalence classes, rather than subsets of larger ones. If θ is TF-equivalent to y , membership of S_2 in $\overline{\mathcal{T}}_\theta \setminus \mathcal{T}_\theta$ forces $\theta_2 = 0$. Membership of S_3 and M_{23} in $\mathcal{T}_\theta$ makes both proper nonzero quotient weights of M_{13} positive. Since $M_{13} \in \overline{\mathcal{T}}_\theta \setminus \mathcal{T}_\theta$, its total weight must therefore vanish. Consequently $\theta_1 + \theta_2 + \theta_3 = 0$ and $\theta_3 > 0$, giving $\theta \in D$.

If instead θ is TF-equivalent to x , exclusion of S_2 from $\overline{\mathcal{T}}_\theta$ gives $\theta_2 < 0$. Again, the proper quotients of M_{13} have positive weights and $M_{13} \in \overline{\mathcal{T}}_\theta \setminus \mathcal{T}_\theta$, so the total weight is zero. Moreover $\theta_2 + \theta_3 > 0$, whence $\theta_1 < 0$. Setting $a = -\theta_1$ and $b = -\theta_2$ gives $\theta \in C$. The table proves the converses.

Equations (1)–(3) show $C <_{\text{fss}} D$. This is a cover: a class between them must have the same strict torsion class, and its weak torsion class can contain either none or one of the sole additional indecomposables, namely S_2 . There is no third possibility. Since a TF class is determined by the two torsion classes, no intermediate class exists.

Finally, every $x \in C$ and $y \in D$ has coordinate sum zero. A coordinatewise nonnegative difference between them would also have sum zero, and so would be the zero vector. But $x_2 < 0 = y_2$, making equality impossible. This completes the proof of Theorem 1.

Lemma 2. *Let E, F be distinct TF classes. Suppose that a module of strictly positive dimension vector d is semistable for every weight in $E \cup F$. Then no point of E is coordinatewise comparable with any point of F .*

Proof. Semistability includes the equality $\theta \cdot d = 0$. If $\eta - \theta$ were coordinatewise nonnegative, then $0 = (\eta - \theta) \cdot d$ and $d_i > 0$ would force $\eta = \theta$. Disjoint equivalence classes cannot share this point. The reverse comparison is identical. $\square$

In our example the common module is M_{13} . Its proper nonzero submodules are S_1 and M_{12} ; their weights are negative on $C \cup D$, whereas its total weight is zero. Thus it is in fact stable throughout both classes.

Corollary 3. *On the set of TF classes, let $E \preceq_{\text{rep}} F$ mean that some $e \in E$ and $f \in F$ satisfy $e \leq f$ coordinatewise. Even the transitive closure of $\preceq_{\text{rep}}$ can be strictly smaller than $\leq_{\text{fss}}$.*

Proof. Coordinatewise monotonicity gives $\preceq_{\text{rep}} \subseteq \leq_{\text{fss}}$. A finite chain of such relations from the class C to the class D in Theorem 1 could contain only these two classes, since $C <_{\text{fss}} D$. Its first change from C to D would supply comparable representatives, a contradiction. $\square$

4 Location in the g -vector fan

The projective module at vertex j is $P_j = M_{1j}$. The minimal projective resolutions

$$0 \longrightarrow P_1 \longrightarrow P_3 \longrightarrow M_{23} \longrightarrow 0, \quad 0 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow S_3 \longrightarrow 0$$

give $g(M_{23}) = (-1, 0, 1)$ and $g(S_3) = (0, -1, 1)$. Moreover $\tau M_{23} = M_{12}$ and $\tau S_3 = S_2$. All four Hom spaces from M_{23} or S_3 to M_{12} or S_2 vanish: commutativity with the arrow $3 \rightarrow 2$ forces any possible component at vertex 2 to be zero. Thus $M_{23} \oplus S_3$ is τ -rigid. The class D is the relative interior of the ray for M_{23} , and C is the relative interior of the two-dimensional cone for $M_{23} \oplus S_3$, consistent with the general correspondence in [1, Section 4]. The counterexample concerns these lower-dimensional faces, not two maximal chambers.

The computation and proof use standard interval representations and the cited definitions. A bounded primary-literature search on 26 October 2026 located the explicit lifting question and the published correction, but no earlier answer with this example; no absolute-priority claim is made. The accompanying exact-arithmetic checks verify the interval table, Hom equations,

and parameter identities. They are diagnostics for the written proof, not formal verification. AI assistance was used in exploring examples, checking calculations, and drafting this unrefereed manuscript.

References

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