

The Sharp Hexagonal Constant in Steinerberger's Geometric Uncertainty Principle on the Square

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Abstract

For a partition of $[0, 1]^2$ into N measurable pieces Ω_i , let $\mathcal{F} = \sum_i |\Omega_i| \mathcal{A}(\Omega_i) + \sum_i (|\Omega_i| - \min_j |\Omega_j|)$, where $\mathcal{A}$ is the Fraenkel asymmetry with respect to discs. Steinerberger proved $\mathcal{F} \geq 1/60000$ for large N and asked, in Oberwolfach Report 49/2013, whether the extremal configuration is a hexagonal tiling, which would give the optimal constant $c \approx 0.07$. We show that this is so on the square, in an asymptotic sense, with the constant $c_H = 0.07446575455\dots$, the asymmetry of the regular hexagon. Every partition of $[0, 1]^2$, into any number of pieces, satisfies $\mathcal{F} > c_H + 0.06m$, where m is the smallest measure of a piece, while for every N a honeycomb partition with a boundary correction has $\mathcal{F} \leq c_H + O(N^{-1/2})$. Hence the infimum I_N over partitions into N pieces lies between $c_H + 0.03/N$ and $c_H + O(N^{-1/2})$, I_N tends to c_H as $N \rightarrow \infty$, and the value c_H is attained by no partition. We do not show that near-minimisers are geometrically close to a honeycomb, and we do not treat other domains. The proof bounds $\mathcal{F}$ below by a covering functional of discs of unequal sizes and uses Laguerre cells, a moment lemma of Fejes Tóth for the indicator of the complement of a disc (proved in an appendix), Euler's formula, and a tangent inequality in the number of sides with explicit constants. The same proof answers, in the same asymptotic sense, a weighted version asked by Burchard: the limit of the infimum is $\min\{\alpha c_H, 1\}$, so honeycomb partitions stop being asymptotically extremal exactly at the weight $\alpha = 1/c_H = 13.4289\dots$. This is an unrefereed note.

1 Introduction

Throughout, $|E|$ is the Lebesgue measure of a set $E \subset \mathbb{R}^2$, $S = [0, 1]^2$, and $B(c, r)$ is the closed disc with centre c and radius r . The *Fraenkel asymmetry* of a measurable set $\Omega \subset \mathbb{R}^2$ with $0 < |\Omega| < \infty$ is

$$\mathcal{A}(\Omega) = \inf_B \frac{|\Omega \triangle B|}{|\Omega|},$$

where B ranges over all discs in the plane with $|B| = |\Omega|$; the discs may be placed anywhere in $\mathbb{R}^2$. Clearly $0 \leq \mathcal{A} \leq 2$. For a partition $S = \Omega_1 \cup \dots \cup \Omega_N$ into measurable sets, with $m = \min_j |\Omega_j|$, put

$$\mathcal{F}(\Omega_1, \dots, \Omega_N) = \sum_{i=1}^N |\Omega_i| \mathcal{A}(\Omega_i) + \sum_{i=1}^N (|\Omega_i| - m). \quad (1)$$

The first sum measures how far the pieces are from discs, the second how far they are from having equal measure. Both cannot be small, since the square is not a finite union of discs of equal size.

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Steinerberger [Ste14] proved that $\mathcal{F} \geq 1/60000$ for all large N (his theorem is for every bounded open domain, with the pieces weighted by $|\Omega_i|/|\Omega|$; on S the functional is exactly (1)), with an application to Pleijel's estimate for nodal domains. He remarks that this constant is far from optimal and that the hexagonal tiling is a natural candidate for an extremiser, which would give an optimal constant $c_2 \approx 0.074465754\dots$ [Ste14, Sect. 1.3]. In the Oberwolfach workshop on uniform distribution theory he posed the question [Ste13, pp. 2891–2892]: is the extremal configuration given by a “decomposition into hexagons (with obvious modifications at the boundary)”, corresponding to an optimal constant $c \approx 0.07\dots$?

We answer this question for the square. Let $c_H = \mathcal{A}(\text{regular hexagon})$ (the asymmetry is invariant under similarities). As we show in Section 3.4,

$$c_H = \frac{12}{\pi} \arctan \frac{\lambda}{\sqrt{3}} - 2\lambda = 0.07446575455794907873\dots, \quad \lambda = \sqrt{3\left(\frac{2\sqrt{3}}{\pi} - 1\right)}. \quad (2)$$

Put $\kappa = \sqrt{2/(3\sqrt{3})} = 0.6204\dots$ and, for $N \geq 1$, $\delta_N = 1 - N(\sqrt{N} + 4\kappa)^{-2}$; then $\delta_N \leq 8\kappa N^{-1/2} + 16\kappa^2 N^{-1}$.

Theorem 1.1. *Let $N \geq 1$ be an integer.*

- (i) *Every partition of S into N measurable sets satisfies $\mathcal{F} \geq c_H + 0.06m > c_H$, where m is the smallest measure of a piece.*
- (ii) *There is a partition of S into N measurable sets, consisting of hexagons of a honeycomb with the left-over region attached to one of them, with $\mathcal{F} \leq c_H + 3\delta_N$.*

Corollary 1.2. *Let I_N be the infimum of $\mathcal{F}$ over all partitions of S into N measurable sets. Then $c_H + 0.03/N \leq I_N \leq c_H + 3\delta_N$ for every N . In particular $I_N \rightarrow c_H$, $\mathcal{F} > c_H$ for every finite partition, and the value c_H is the infimum of $\mathcal{F}$ over all N and all partitions but is attained by none. The best constant $\liminf_N I_N$ in the geometric uncertainty principle on S equals c_H .*

Call a sequence of partitions P_N of S into N sets *asymptotically extremal* if $\mathcal{F}(P_N) - I_N \rightarrow 0$; since $I_N \rightarrow c_H$, this is the same as $\mathcal{F}(P_N) \rightarrow c_H$. By Theorem 1.1(ii) and Corollary 1.2, the honeycomb partitions of Theorem 1.1(ii) are asymptotically extremal.

So the answer to the question is affirmative on the square, in the sense that the optimal constant is exactly $c_H = 0.0744657\dots$ (the value suggested in [Ste14]) and the honeycomb partitions with boundary modification of Theorem 1.1(ii) attain it asymptotically. We do not claim that every near-minimiser is geometrically close to a honeycomb; only a combinatorial form of this is proved (Proposition 4.2). The decay rate $N^{-1/2}$ in (ii) is only an upper bound; the lower bound has order $1/N$ and we do not know the true rate of $I_N - c_H$.

The proof of the lower bound is the following chain. By compactness each piece Ω_i has an optimal disc B_i with $|B_i| = |\Omega_i|$, and $|\Omega_i| \mathcal{A}(\Omega_i) = 2|\Omega_i \setminus B_i|$. The sets $\Omega_i \setminus B_i$ are disjoint and cover the part of S not covered by the discs, so $\mathcal{F} \geq 2|S \setminus \bigcup_j B_j| + \sum_j (|B_j| - m)$ (Lemma 2.1). This is a functional of a system of discs of different sizes. We pass to the Laguerre (power) cells of the discs, which split the uncovered area exactly into one contribution per cell (Lemma 3.1); bound each contribution below by that of a regular polygon with the same number of sides, using a moment lemma of Fejes Tóth (Lemma 3.2); use Euler's formula to see that the cells have on average at most six sides, up to a boundary term (Lemma 3.3); and sum over the cells with a tangent inequality in the cell area and the number of sides (Proposition 3.5), whose constants μ, ν are explicit. The size term is absorbed because every disc has area at least m .

Precedents and what is new. The toolkit is classical. The moment lemma of Fejes Tóth [Fej72] (see [Gru99] for a short analytic proof, and [BC21, Lemma 2.3] for the second-moment form) and Hales’ proof of the honeycomb conjecture [Hal01], a well-known instance of a per-cell inequality combined with Euler’s formula, are the origin of the method. Bourne, Peletier and Theil [BPT14] proved the optimality of the triangular lattice for a Wasserstein-type particle system, and Bourne and Cristofori [BC21] extended this to approximation problems with a constraint or a penalty on the sizes of the particles, which they also read as optimal partitioning problems with hexagonal optimisers; their argument uses Laguerre cells, the moment lemma [BC21, Lemma 2.3] and an Euler-formula lemma [BC21, Lemma 2.6]. The functional (1) differs from those: the kernel is the indicator function of the complement of a disc, not a quadratic function, and the size term is $-m$, the smallest area, not a power of the areas. What we add is the reduction of (1) to systems of discs of different sizes, the moment lemma with an indicator kernel and arbitrary centre (with a complete proof), the tangent inequality with explicit constants μ and $\nu = 0.03$ that absorbs the term $-m$, and the resulting sharp constant for the explicitly posed question. As far as we could find, the sharp constant is not in print (see the remarks at the end).

Scope. The theorem is for the square; the proof uses that the Laguerre cells inside S are convex polygons forming an edge-to-edge subdivision of a convex polygon. Steinerberger’s constant c_2 is a universal constant for all bounded open domains; by Corollary 1.2 it satisfies $c_2 \leq c_H$, but we do not prove $c_2 \geq c_H$ for arbitrary domains.

A weighted version. Steinerberger [Ste14, Sect. 1.3] records a question of Burchard: for $\alpha > 0$ minimise $\alpha \sum |\Omega_i| \mathcal{A}(\Omega_i) + \sum (|\Omega_i| - m)$; the hexagonal packing should minimise for $0 < \alpha \leq 1$, and there is some $\alpha_0 \geq 1$ beyond which it does not. The proof above gives:

Theorem 1.3. *Let $\mathcal{F}_\alpha = \alpha \sum_i |\Omega_i| \mathcal{A}(\Omega_i) + \sum_i (|\Omega_i| - m)$ and let $I_N(\alpha)$ be its infimum over partitions of S into N sets. Then $\lim_{N \rightarrow \infty} I_N(\alpha) = \min\{\alpha c_H, 1\}$. In particular the honeycomb partitions of Theorem 1.1(ii) are asymptotically extremal for $\mathcal{F}_\alpha$ (that is, $\mathcal{F}_\alpha(P_N) - I_N(\alpha) \rightarrow 0$) if and only if $\alpha \leq 1/c_H = 13.42899\dots$, and at $\alpha = 1/c_H$ they are asymptotically extremal together with partitions into many small discs and a remainder of small measure; for $\alpha > 1/c_H$ the limit of $I_N(\alpha)$ is 1, while $\mathcal{F}_\alpha \rightarrow \alpha c_H > 1$ for the honeycomb partitions.*

2 Conventions and reduction to discs

A partition of S into N sets is a family of measurable sets $\Omega_1, \dots, \Omega_N$ with $\bigcup_i \Omega_i = S$ and $|\Omega_i \cap \Omega_j| = 0$ for $i \neq j$. Thus partitions are considered up to null sets, and closed hexagons that overlap only along their boundaries are allowed. Pieces may be null; for a null piece we set $|\Omega| \mathcal{A}(\Omega) := 0$, so that (1) is defined for every partition. If $m = 0$ then $\mathcal{F} \geq \sum_i |\Omega_i| = 1$. For $|B| = |\Omega|$ one has $|\Omega \setminus B| = |B \setminus \Omega|$, hence

$$|\Omega| \mathcal{A}(\Omega) = 2 \inf_B |\Omega \setminus B|. \quad (3)$$

The infimum is attained. Indeed, if r is the radius with $\pi r^2 = |\Omega|$, the function $c \mapsto |\Omega \setminus B(c, r)|$ is continuous and equals $|\Omega|$ when $B(c, r) \cap \Omega$ is null, in particular when $|c|$ is large and Ω is bounded, while $\int |\Omega \cap B(c, r)| dc = |\Omega| \pi r^2 > 0$ shows that it is smaller than $|\Omega|$ somewhere. We call a disc B with $|B| = |\Omega|$ and $|\Omega \setminus B| = \frac{1}{2} |\Omega| \mathcal{A}(\Omega)$ an *optimal disc* of Ω .

For discs $B_1, \dots, B_N$ in the plane with positive radii (centres anywhere, overlaps allowed) and

total area $\sum_j |B_j| = 1$, put $s_j = |B_j|$, $m = \min_j s_j$, and

$$\Phi(B_1, \dots, B_N) = 2 \left| S \setminus \bigcup_j B_j \right| + \sum_{j=1}^N (s_j - m). \quad (4)$$

Lemma 2.1 (Reduction to discs). *Let $\Omega_1, \dots, \Omega_N$ be a partition of S with $m > 0$, and let B_i be an optimal disc of Ω_i . Then $\mathcal{F}(\Omega_1, \dots, \Omega_N) \geq \Phi(B_1, \dots, B_N)$.*

Proof. By (3), $\mathcal{F} = 2 \sum_i |\Omega_i \setminus B_i| + \sum_i (|B_i| - m)$. A point of $S \setminus \bigcup_j B_j$ lies in some Ω_i and not in B_i . As the Ω_i are disjoint up to null sets, $\sum_i |\Omega_i \setminus B_i| \geq |S \setminus \bigcup_j B_j|$. Also $\sum_i |B_i| = \sum_i |\Omega_i| = 1$. $\square$

The heart of the matter is the following inequality for disc systems, which does not refer to partitions.

Theorem 2.2. *For every system of $N \geq 1$ discs with positive radii and total area 1, $\Phi(B_1, \dots, B_N) \geq c_H + 0.06m$, where m is the smallest area.*

Proof of Theorem 1.1(i), given Theorem 2.2. If $m = 0$, then $\mathcal{F} \geq 1 > c_H$. If $m > 0$, Lemma 2.1 and Theorem 2.2 apply to the optimal discs of the pieces, which have total area 1. $\square$

3 The lower bound

3.1 Power cells

Let $B_j = B(c_j, r_j)$, $j = 1, \dots, N$, with pairwise distinct centres. Put $\text{pow}_j(x) = |x - c_j|^2 - r_j^2$ and let $P_j = \{x \in \mathbb{R}^2 : \text{pow}_j(x) \leq \text{pow}_i(x) \text{ for all } i\}$ be the *power* (Laguerre) cells. They are closed convex polygons (possibly empty or unbounded), they cover the plane, and $P_i \cap P_j$ is contained in the line $\ell_{ij} = \{\text{pow}_i = \text{pow}_j\}$ for $i \neq j$. Put $Q_j = P_j \cap S$.

Lemma 3.1. $|S \setminus \bigcup_j B_j| = \sum_j |Q_j \setminus B_j|$ and $\sum_j |Q_j| = 1$.

Proof. The cells overlap in null sets and cover S , so $\sum_j |Q_j| = 1$. If $x \in P_j$ lies outside B_j , then $\text{pow}_j(x) > 0$, hence $\text{pow}_i(x) \geq \text{pow}_j(x) > 0$ for all i , that is, x lies outside every disc. Thus $Q_j \setminus B_j \subset S \setminus \bigcup_i B_i$. Conversely every $x \in S \setminus \bigcup_i B_i$ lies in some P_j and outside B_j . $\square$

3.2 The moment lemma

For $k \geq 3$, $y > 0$, $s > 0$ let $M_k(y, s)$ be the area of the part lying outside the concentric disc of area s of the regular k -gon of area y ; put $M_k(0, s) = 0$. Then $M_k(\theta y, \theta s) = \theta M_k(y, s)$ for $\theta > 0$.

Lemma 3.2 (Moment lemma, indicator kernel). *Let Q be a convex polygon with at most k sides, $k \geq 3$, let $z \in \mathbb{R}^2$ be an arbitrary point and $\rho > 0$. Then $|Q \setminus B(z, \rho)| \geq M_k(|Q|, \pi \rho^2)$.*

This is the case $f = \mathbf{1}_{(\rho, \infty)}$ of the moment lemma of L. Fejes Tóth, which, as we understand it, says that for a non-decreasing function f the moment $\int_Q f(|x - z|) dx$ of a convex k -gon is at least that of the regular k -gon of the same area centred at z [Fej72]; see [Gru99] for a short analytic proof and [BC21, Lemma 2.3] for the second-moment case. We have not checked the exact formulation of the general statement in the primary sources (arbitrary z , at most k sides), and so we prove the case we need, completely, in Appendix A.

3.3 The Euler count

Lemma 3.3. *Let K be a convex polygon with p sides, let $c_1, \dots, c_N$ be distinct points and $r_1, \dots, r_N > 0$, and let $Q_j = P_j \cap K$ be the cells of the power diagram, k_j the number of sides of Q_j when $|Q_j| > 0$, and $J' = \{j : |Q_j| > 0\}$, $N' = |J'|$. Then $\sum_{j \in J'} k_j \leq 6N' + p - 6$. For the square, $\sum_{j \in J'} k_j \leq 6N' - 2$.*

A bound of this kind is classical for partitions into convex polygons (compare [BC21, Lemma 2.6]). Here one must check that the clipped power diagram is edge-to-edge, and handle cells of area zero and $N' < N$; this is done in Appendix B.

3.4 The functions h_k and the constants

Let $\rho = \pi^{-1/2}$, the radius of the disc of area 1. For $k \geq 3$, $t > 0$ let $h_k(t) = 2M_k(t, 1)$, and write $q = \tan(\pi/k)$ and $a = \sqrt{t/(kq)}$ for the apothem of the regular k -gon of area t ; its circumradius is $a/\cos(\pi/k)$.

Lemma 3.4. (a) $h_k(t) = 2(t - 1)$ if $a \geq \rho$; $h_k(t) = 0$ if $\rho \geq a/\cos(\pi/k)$; and otherwise

$$h_k(t) = 2\left(t - 1 + k\rho^2 \arccos \frac{a}{\rho} - k a \sqrt{\rho^2 - a^2}\right).$$

The function h_k is C^1 and convex on $(0, \infty)$; $h'_k(t) = 2$, 0 in the first two ranges and $h'_k(t) = 2(1 - \sqrt{\rho^2 - a^2}/(aq))$ in the third.

(b) For $0 < \lambda < 1$ put $\mu_\lambda = 2(1 - \lambda)$ and $\varphi_\lambda(k) = \frac{2k}{\pi} \arctan(\lambda \tan \frac{\pi}{k})$. Then $\min_{t>0} [h_k(t) - \mu_\lambda t] = \varphi_\lambda(k) - 2$, attained exactly where $\sqrt{\rho^2 - a^2} = \lambda a q$.

Proof. (a) The regular k -gon is the union of k congruent wedges of apex angle $2\pi/k$ and area t/k , so the formulas are those of (10) and (11) (the three regimes of the function ε of Appendix A) with $\tau = t/k$, $\gamma = \pi/k$, $\pi\rho^2 = 1$. In the third range, $\frac{d}{da}[a\sqrt{\rho^2 - a^2} + \rho^2(\pi/k - \arccos(a/\rho))] = 2\sqrt{\rho^2 - a^2}$ and $dt/da = 2kaq$; this gives the derivative. At $a = \rho \cos(\pi/k)$ the quotient $\sqrt{\rho^2 - a^2}/(aq)$ equals 1 and at $a = \rho$ it equals 0, so h'_k is continuous; it is non-decreasing because $\sqrt{\rho^2 - a^2}/a$ decreases in a .

(b) Since h'_k increases continuously from 0 to 2 and $0 < \mu_\lambda < 2$, the minimum is attained exactly where $h'_k = \mu_\lambda$, that is where $\sqrt{\rho^2 - a^2} = \lambda a q$, i.e. $a^2 = \rho^2/(1 + \lambda^2 q^2)$; this value lies in the third range because $\lambda < 1$. There $\omega := \arccos(a/\rho) = \arctan(\lambda q)$, $t = kqa^2$ and $a\sqrt{\rho^2 - a^2} = \lambda qa^2$, so $h_k(t) - \mu_\lambda t = 2(\lambda t - 1 + k\rho^2\omega - k\lambda qa^2) = 2k\rho^2\omega - 2 = \varphi_\lambda(k) - 2$. $\square$

(The identities of this proof and of Lemma 3.2 are verified symbolically in the supplement.) Now fix

$$\lambda = \sqrt{3(2\sqrt{3}/\pi - 1)} = 0.55495348681\dots, \quad \mu = 2(1 - \lambda) = 0.89009302636\dots, \quad \varphi = \varphi_\lambda.$$

For $k = 6$, $t = 1$ we have $a^2 = \sqrt{3}/6$ and $q = 1/\sqrt{3}$, and $(\rho^2 - a^2)/(aq)^2 = 3(\rho^2/a^2 - 1) = 3(2\sqrt{3}/\pi - 1) = \lambda^2$. So $t = 1$ is the minimiser for $k = 6$, and Lemma 3.4(b) gives $h_6(1) - \mu = \varphi(6) - 2$, that is

$$h_6(1) = \varphi(6) - 2\lambda = \frac{12}{\pi} \arctan \frac{\lambda}{\sqrt{3}} - 2\lambda. \quad (5)$$

By Lemma 3.2 with $k = 6$, a regular hexagon of area 1 has $|\text{hexagon} \setminus B| \geq \frac{1}{2}h_6(1)$ for every disc B of area 1, with equality for the concentric disc. By (3), $h_6(1) = \mathcal{A}(\text{regular hexagon}) = c_H$. This proves (2) (the numerical value is computed to 60 digits in the supplement).

Proposition 3.5 (Tangent inequality). *Let $\nu = 3/100$ and, for $k \geq 3$ and $t > 0$, $G_k(t) = h_k(t) - c_H - \mu(t - 1)$. Then*

- (a) $G_k(t) \geq \Psi(k) := \varphi(k) - \varphi(6)$ for all $t > 0$, with equality at one t ; $\Psi(6) = 0$;
(b) $G_k(t) \geq -c_H > -1$ for all $k \geq 3$, $t > 0$;
(c) $G_k(t) \geq -\nu(k-6) + s_*$ for all $k \neq 6$ and $t > 0$, and $G_6(t) \geq 0$, where $s_* = 0.0055$.

Proof. (a) By (5), $\mu - c_H = 2 - \varphi(6)$. By Lemma 3.4(b), $G_k(t) = h_k(t) - \mu t + \mu - c_H \geq \varphi(k) - 2 + 2 - \varphi(6) = \Psi(k)$. (b) The area outside a disc of area 1 of a set of area t is at least $(t-1)_+$, so $h_k(t) \geq 2(t-1)_+$. For $t \geq 1$, $G_k \geq (2-\mu)(t-1) - c_H \geq -c_H$; for $t < 1$, $G_k \geq \mu(1-t) - c_H \geq -c_H$. Also $c_H < 0.075 < 1$. (c) For $k = 3, \dots, 8$ the numbers $\Psi(k) + \nu(k-6)$ are listed in Table 1; they are at least 0.0055 for $k \neq 6$ (the smallest is 0.00558, for $k = 5$). For $k \geq 9$, (b) gives $G_k \geq -c_H = -\nu(k-6) + (\nu(k-6) - c_H) \geq -\nu(k-6) + (3\nu - c_H)$, and $3\nu - c_H > 0.0155 > s_*$. $\square$

k	$\tan(\pi/k)$	$\varphi(k)$	$\Psi(k)$	$\Psi(k) + 0.03(k-6)$
3	1.7320508	1.4622282	0.2778555	0.1878555
4	1.0000000	1.2901439	0.1057711	0.0457711
5	0.7265425	1.2199533	0.0355806	0.0055806
6	0.5773503	1.1843727	0	0
7	0.4815746	1.1637653	-0.0206074	0.0093926
8	0.4142136	1.1507244	-0.0336483	0.0263517

Table 1: The tangent inequality for $k = 3, \dots, 8$. Here $\varphi(k) = \frac{2k}{\pi} \arctan(\lambda \tan \frac{\pi}{k})$ and $\Psi(k) = \varphi(k) - \varphi(6)$. Each entry is an explicit elementary expression ($\tan \frac{\pi}{7}$ is the smallest positive root of $t^6 - 21t^4 + 35t^2 - 7$; the other tangents are $\sqrt{3}, 1, \sqrt{5-2\sqrt{5}}, 1/\sqrt{3}, \sqrt{2}-1$); the table was computed with 60 digits and re-computed in rigorous interval arithmetic (supplement), so rounding is irrelevant against the margins. For $k = 3, \dots, 8$ the inequality $\Psi(k) + \nu(k-6) \geq 0$ holds exactly for $\nu \in [0.0206074, 0.0355806]$; the argument for $k \geq 9$ below needs $\nu \geq c_H/3 = 0.02482\dots$, so the proof works for every $\nu \in [0.0249, 0.0355]$, in particular for $\nu = 0.03$.

The numbers in Table 1 are the only place where numerics enter the proof of the tangent inequality; they are values of explicit elementary functions of π and algebraic numbers. For $k \geq 9$ no computation and no monotonicity in k is needed: the trivial bound (b) and $c_H < 3\nu$ suffice.

3.5 Proof of Theorem 2.2

Let $B_j = B(c_j, r_j)$ have total area 1. Both sides of the inequality are continuous in the centres ($|S \setminus \bigcup_j B_j|$ changes by at most $\sum_j |B_j \triangle B'_j|$ when the centres move), so we may assume that the centres are pairwise distinct. Put $\sigma_j = s_j/m \geq 1$, $y_j = |Q_j|/m$, so that $\sum_j y_j = \sum_j \sigma_j = 1/m =: S_0$. By Lemma 3.1 and Lemma 3.2 applied to Q_j ($k = k_j$) and the disc B_j (any centre is allowed), for $j \in J'$

$$|Q_j \setminus B_j| \geq M_{k_j}(|Q_j|, s_j) = m M_{k_j}(y_j, \sigma_j) = \frac{1}{2} m \sigma_j h_{k_j}(t_j), \quad t_j = y_j / \sigma_j,$$

and $|Q_j \setminus B_j| \geq 0$ for $j \notin J'$. Hence

$$\frac{\Phi}{m} \geq \sum_{j \in J'} \sigma_j h_{k_j}(t_j) + \sum_{j=1}^N (\sigma_j - 1). \quad (6)$$

For $j \in J'$ write $G_j = G_{k_j}(t_j)$; then $\sigma_j h_{k_j}(t_j) = \sigma_j(G_j + c_H + \mu(t_j - 1)) = \sigma_j G_j + c_H \sigma_j + \mu(y_j - \sigma_j)$, so

$$\begin{aligned} \sigma_j h_{k_j}(t_j) + \sigma_j - 1 &= (\sigma_j - 1)(G_j + 1) + G_j + c_H \sigma_j + \mu(y_j - \sigma_j) \\ &\geq G_j + c_H \sigma_j + \mu(y_j - \sigma_j), \end{aligned}$$

because $\sigma_j \geq 1$ and $G_j + 1 > 0$ by Proposition 3.5(b). For $j \notin J'$ we have $y_j = 0$ and $\sigma_j - 1 \geq 0 = c_H \sigma_j + \mu(y_j - \sigma_j) + (\mu - c_H) \sigma_j$. Summing over j and using $\sum_j (y_j - \sigma_j) = 0$,

$$\frac{\Phi}{m} \geq c_H S_0 + \sum_{j \in J'} G_j + \sum_{j \notin J'} (\mu - c_H) \sigma_j. \quad (7)$$

By Proposition 3.5(c), $G_j \geq -\nu(k_j - 6)$, with the extra term s_* if $k_j \neq 6$, and by Lemma 3.3 (with $p = 4$) $-\nu \sum_{j \in J'} (k_j - 6) \geq 2\nu$. Since $\mu - c_H > s_*$ and $\sigma_j \geq 1$,

$$\frac{\Phi}{m} \geq c_H S_0 + 2\nu + s_* \#\{j : j \notin J' \text{ or } k_j \neq 6\}.$$

Multiplying by m and using $mS_0 = 1$ gives

$$\Phi \geq c_H + 0.06m + 0.0055m \#\{j : |Q_j| = 0 \text{ or } k_j \neq 6\}. \quad \square \quad (8)$$

4 The upper bound and the main results

Let $\kappa = \sqrt{2/(3\sqrt{3})}$; a regular hexagon of area a has circumradius $\kappa\sqrt{a}$.

Proposition 4.1. *For every $N \geq 1$ there is a partition of S into N sets with $\mathcal{F} \leq (1 - \delta)c_H + 3\delta \leq c_H + 3\delta_N$, where $\delta = 1 - Na$ with $\sqrt{a} = 1/(\sqrt{N} + 4\kappa)$, and $\delta = \delta_N \leq 8\kappa N^{-1/2} + 16\kappa^2 N^{-1}$.*

Proof. Take the honeycomb tiling of the plane by regular hexagons of area a , with circumradius $R = \kappa\sqrt{a}$. A hexagon whose centre lies in $[R, 1 - R]^2$ is contained in the disc of radius R about its centre and hence in S . Every point of $[2R, 1 - 2R]^2$ lies in a hexagon whose centre is within distance R , hence in $[R, 1 - R]^2$; so these hexagons cover a square of area $(1 - 4R)^2 = Na$ ($4R = 4\kappa\sqrt{a}$ and $1 - 4R = \sqrt{N}\sqrt{a}$), and there are at least $Na/a = N$ of them. Choose N of them, $H_1, \dots, H_N$ (closed, overlapping only in null sets) and let $\delta = 1 - Na$ be the measure of the remainder $R_0 = S \setminus \bigcup_i H_i$. Put $\Omega_1 = H_1 \cup R_0$ and $\Omega_i = H_i$ for $i \geq 2$. Then $m = a$ and $\sum_i (|\Omega_i| - m) = \delta$. For $i \geq 2$, $|\Omega_i| \mathcal{A}(\Omega_i) = c_H a$. For Ω_1 use the disc B' concentric with H_1 and of area $a + \delta$; with B the concentric disc of area a , $B' \setminus B$ is an annulus of area δ and $\Omega_1 \triangle B' \subset (H_1 \triangle B) \cup (R_0 \triangle (B' \setminus B))$, so $|\Omega_1| \mathcal{A}(\Omega_1) \leq c_H a + 2\delta$. Therefore $\mathcal{F} \leq Na c_H + 2\delta + \delta = (1 - \delta)c_H + 3\delta$. Finally $\delta = 1 - N(\sqrt{N} + 4\kappa)^{-2} = (8\kappa\sqrt{N} + 16\kappa^2)(\sqrt{N} + 4\kappa)^{-2} \leq 8\kappa N^{-1/2} + 16\kappa^2 N^{-1}$. $\square$

Proof of Theorem 1.1 and Corollary 1.2. (i) was proved in Section 2, (ii) is Proposition 4.1. For Corollary 1.2: since $m \leq 1/N$ always, and $\mathcal{F} \geq \sum_i (|\Omega_i| - m) = 1 - Nm$, either $m \geq 1/(2N)$ and $\mathcal{F} \geq c_H + 0.03/N$ by (i), or $m < 1/(2N)$ and $\mathcal{F} > 1/2 > c_H + 0.03/N$. The remaining statements follow from (i), (ii) and $\delta_N \rightarrow 0$. The optimal constant for S is $\liminf_N I_N = c_H$, and any universal constant valid for all domains is at most this value. $\square$

Proposition 4.2 (Combinatorial hexagonality). *Let $\Omega_1, \dots, \Omega_N$ be a partition of S with $m > 0$ and let $\eta > 0$. There are discs B'_j with $|B'_j| = |\Omega_j|$ and pairwise distinct centres whose power cells Q'_j in S satisfy*

$$\#\{j : Q'_j \text{ is not a hexagon}\} \leq \frac{\mathcal{F} + \eta - c_H - 0.06m}{0.0055m} \leq N \frac{\mathcal{F} + \eta - c_H}{0.0055(1 - \mathcal{F})}.$$

Hence along any sequence of partitions with $\mathcal{F} \rightarrow c_H$ the proportion of non-hexagonal cells tends to 0.

Proof. Let B_j be optimal discs; by Lemma 2.1, $\Phi(B) \leq \mathcal{F}$. Move the centres slightly to make them distinct so that Φ increases by at most η (continuity), and apply (8). The last inequality uses $Nm \geq 1 - \mathcal{F}$ (from $\mathcal{F} \geq 1 - Nm$) and $\mathcal{F} < 1$ (otherwise there is nothing to prove). $\square$

This statement concerns the power cells of near-optimal discs of the pieces; it does not say that the pieces Ω_j themselves are close to hexagons. A geometric stability statement would require a quantitative stability version of the moment theorem, of the kind studied in [Fej01], and is not attempted.

5 The weighted functional

Recall $\mathcal{F}_\alpha = \alpha \sum_i |\Omega_i| \mathcal{A}(\Omega_i) + \sum_i (|\Omega_i| - m)$. Put $\beta = 1/\alpha$.

Proof of Theorem 1.3. Lower bound for $\alpha \leq 1/c_H$. As in Lemma 2.1, $\mathcal{F}_\alpha \geq \alpha \Phi_\beta$ with $\Phi_\beta = 2|S \setminus \bigcup_j B_j| + \beta \sum_j (s_j - m)$ and $\beta \geq c_H$. In the proof of Theorem 2.2 replace the term $\sigma_j - 1$ by $\beta(\sigma_j - 1)$. The same computation gives $\sigma_j h_{k_j}(t_j) + \beta(\sigma_j - 1) = (\sigma_j - 1)(G_j + \beta) + G_j + c_H \sigma_j + \mu(y_j - \sigma_j)$, and $G_j + \beta \geq \beta - c_H \geq 0$ by Proposition 3.5(b); empty cells contribute $\beta(\sigma_j - 1) \geq 0$. Hence $\Phi_\beta \geq c_H + 0.06m$ and $\mathcal{F}_\alpha \geq \alpha c_H + 0.06\alpha m > \alpha c_H$. For $\alpha \geq 1/c_H$, since $\mathcal{F}_\alpha$ is non-decreasing in α , $\mathcal{F}_\alpha \geq \mathcal{F}_{1/c_H} \geq 1 + 0.06m/c_H \geq 1$ (if $m = 0$, $\mathcal{F}_\alpha \geq 1$ directly). Thus $I_N(\alpha) \geq \min\{\alpha c_H, 1\}$ for all N .

Upper bounds. The partition of Proposition 4.1 gives $\mathcal{F}_\alpha \leq \alpha(1 - \delta)c_H + (2\alpha + 1)\delta \leq \alpha c_H + (2\alpha + 1)\delta_N$. For the bound 1 we use a covering lemma.

Claim. For every $\varepsilon > 0$ there are finitely many pairwise disjoint closed discs in S whose union covers all but at most ε of $|S|$. Indeed, let U_0 be the interior of S . Given an open set U_i , choose finitely many closed dyadic squares in U_i with disjoint interiors and total area at least $|U_i|/2$, and in each square a closed disc of radius 0.45 times its side, which covers a fraction ≥ 0.63 of it; let U_{i+1} be U_i minus these discs. Then U_{i+1} is open, disjoint from the discs chosen so far, and $|U_{i+1}| \leq 0.7|U_i|$. After finitely many steps $|U_i| < \varepsilon$.

Given ε , let $D_1, \dots, D_n$ be the discs of the claim and, for $N > n$, split the remainder $S \setminus \bigcup D_i$ (of measure $\leq \varepsilon$, and positive because S is connected and is not a disc) into $N - n$ measurable sets of positive measure. Discs have asymmetry 0 and every set has $|\Omega| \mathcal{A}(\Omega) \leq 2|\Omega|$, so $\mathcal{F}_\alpha \leq 2\alpha\varepsilon + \sum_i |\Omega_i| = 1 + 2\alpha\varepsilon$ (using $m \geq 0$). Hence $\limsup_N I_N(\alpha) \leq 1$.

Combining the bounds, $\lim_N I_N(\alpha) = \alpha c_H$ for $\alpha \leq 1/c_H$ and $= 1$ for $\alpha \geq 1/c_H$.

Honeycomb partitions. The partitions P_N of Proposition 4.1 satisfy $\limsup_N \mathcal{F}_\alpha(P_N) \leq \alpha c_H$ by the upper bound above. The $N - 1$ regular hexagons alone contribute

$$\mathcal{F}_\alpha(P_N) \geq \alpha(N - 1)a c_H = \alpha(1 - \delta - a) c_H.$$

Hence $\mathcal{F}_\alpha(P_N) \rightarrow \alpha c_H$ and

$$\mathcal{F}_\alpha(P_N) - I_N(\alpha) \rightarrow \alpha c_H - \min\{\alpha c_H, 1\},$$

which vanishes if and only if $\alpha \leq 1/c_H$. At $\alpha = 1/c_H$ the partitions into discs and a remainder have $\mathcal{F}_\alpha \rightarrow 1 = \alpha c_H$ as well. $\square$

So the honeycomb partitions stop being asymptotically minimal exactly when αc_H exceeds 1, that is at $\alpha_0 = 1/c_H = 13.42899\dots$ (consistent with the expectation $\alpha_0 \geq 1$ in [Ste14]), and at α_0 itself two very different families of partitions are asymptotically extremal. This is a statement about the limit $N \rightarrow \infty$; we do not determine the minimisers of $\mathcal{F}_\alpha$ for a given N .

6 Remarks and open questions

Remark 6.1 (Convex polygons). Lemma 3.3 is stated for a convex p -gon K , and the proof of Theorem 2.2 uses nothing else about the square. It gives, for the unnormalised functional on K , $\Phi \geq c_H|K| + \nu(6-p)m$. For $p \geq 7$ this is weaker than the square case, and we do not discuss the upper bound for polygons.

Problem 6.2. *Determine the true order of $I_N - c_H$; we proved $c/N \leq I_N - c_H \leq CN^{-1/2}$. Is the infimum I_N attained for each N , and what do minimisers look like for small N ?*

Problem 6.3. *Is $c_2 = c_H$ for Steinerberger’s universal constant, i.e. does $\mathcal{F} \geq c_H - o(1)$ hold for every bounded open domain?*

Verification. The claims were checked as follows.

1. Every derivative and closed-form identity in Lemma 3.4 and Appendix A was checked symbolically with `sympy`, and the formulas for $K'(\delta)$ were also compared with numerical differentiation of the defining integral in 400 random cases covering all four regimes (`lemmaM_sympy.py`).
2. The constants λ, μ, c_H and the table were computed with 60 digits (`mpmath`) and re-computed in rigorous interval arithmetic with exact rational endpoints; c_H was also obtained by direct quadrature of the hexagon-minus-disc area.
3. Lemma 3.3 and the edge-to-edge statements of Appendix B were tested in exact rational arithmetic on 20,051 clipped power diagrams, including lattice configurations with many ties, and the bound was attained in 7294 of them.
4. Lemma 3.2 was tested on 180,000 random cases (random convex polygons with up to 9 vertices, each with three admissible values of k , the point inside or outside) and by adversarial minimisation for up to 12 sides; the chain of inequalities of Section 3, including the refined inequality (8), was tested on random disc systems in the square, and Theorem 1.1(i) on 364 actual partitions of the square into convex pieces, computing the true functional with the optimal disc of every piece.
5. A first set of independent verification runs, also AI-assisted, examined the argument before the two appendices and the covering lemma were written out. With separately written code they rederived the constants at 90 digits, tested the moment lemma on 60,000 random polygons, the Euler count on 4,000 random systems and on lattice configurations (no violation, and the bound is sharp), and the tangent inequality (with a direct geometric evaluation of h_k for $k = 3, \dots, 40$ and for selected k up to 3000), minimised Φ over disc systems for $N \leq 8$ (minima 0.18 to 0.45, always above $c_H + 0.06m$ by at least 0.046), and minimised the same functional on a flat torus, where the minimum equals c_H times the area for $N = 2, 6, 8$ and is larger for $N = 1, 3, 4$.
6. A second independent verification run, also AI-assisted, examined the final text, including Appendices A and B, the covering lemma and Theorem 1.3, and compared the definitions and the question with the printed Report 49/2013 and with [Ste14] (arXiv version). With newly written code it re-derived every derivative and identity of Appendix A symbolically (31 checks, the four-case formula for $K'(\delta)$ also against quadrature of K in 1200 random cases), tested Lemma 3.2 on 360,000 random combinations of a convex polygon, a point (inside, on the boundary, outside) and a radius, checked Lemma 3.3 and the edge-to-edge statements of Appendix B in exact rational arithmetic on 4,000 clipped power diagrams (the square and convex p -gons, $p = 3, 5, 6, 7$), recomputed the constants and Table 1 at 100 digits and by direct geometry for

$k \leq 200$, tested the chain of Section 3 and the weighted inequality on 6,000 random disc systems, Theorem 1.1(i) on 198 further partitions of the square, the upper-bound construction (hexagon counts for 900 pairs of N and lattice position, a rasterised asymmetry bound for the glued piece), and the covering lemma by an exact-geometry simulation. It also reran all programs of the first supplement from an extracted copy of the source archive, with identical outputs. It found no mathematical error; the points of wording it raised are corrected in this version.

All programs, outputs and the interval certificate are in the supplement. These computations are evidence; the proof is the argument above.

Scope and priority. Theorem 1.1 answers the question of [Ste13] for the square $[0, 1]^2$. It does not prove the universal constant for arbitrary domains, nor a geometric stability or uniqueness statement, nor the true rate of $I_N - c_H$. We read Steinerberger’s paper [Ste14] (arXiv version) and the extended abstract [Ste13], and of [BC21] the statements of Lemmas 2.3 and 2.6. The other cited works ([Fej72, Gru99, Hal01, BPT14, Fej01]) were checked for bibliographic data in Crossref and zbMATH, and are cited for the attribution of the method; we did not rely on their statements, and the moment lemma and the Euler count are proved here. We searched arXiv, OpenAlex, Semantic Scholar, zbMATH and the web for work citing [Ste14] or treating the sharp constant, and found no paper that proves or restates it; we could not search MathSciNet, Google Scholar or journal sites. The toolkit is classical, and this negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the argument, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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A Proof of the moment lemma

Fix $\rho > 0$ and write $B = B(0, \rho)$. We prove Lemma 3.2. All identities below were verified symbolically (supplement).

Reduction. If z is not in Q let z' be its nearest point in Q ; then $|x - z| \geq |x - z'|$ for $x \in Q$, so $|Q \setminus B(z, \rho)| \geq |Q \setminus B(z', \rho)|$. A boundary point z' is a limit of interior points and $z \mapsto |Q \setminus B(z, \rho)|$ is continuous. So we may assume that $z = 0$ is an interior point of Q . Insert extra vertices on the sides so that Q has exactly k vertices $v_1, \dots, v_k$ (several consecutive vertices may be collinear). The triangles $T_i = \text{conv}\{0, v_i, v_{i+1}\}$ have disjoint interiors, areas $\tau_i > 0$, apex angles $2\gamma_i \in (0, \pi)$, $\sum_i \gamma_i = \pi$, $\sum_i \tau_i = |Q|$.

Wedges. For $0 < \gamma < \pi/2$, $|\delta| < \pi/2 - \gamma$ and $d > 0$ let $W(\gamma, \delta, d)$ be the triangle $\{(r, \theta) : |\theta| \leq \gamma, 0 \leq r \leq d/\cos(\theta - \delta)\}$ in polar coordinates: apex 0, bisector $\theta = 0$, base on the line at distance d whose foot has polar angle δ . Every triangle with apex 0 and apex angle $2\gamma < \pi$ is, after a rotation, such a wedge: its base lies on a line at some distance $d > 0$ from 0 whose foot has polar angle δ , and every ray $|\theta| \leq \gamma$ meets this line, so $|\theta - \delta| < \pi/2$ for $|\theta| \leq \gamma$, that is $|\delta| < \pi/2 - \gamma$. With $R(\theta) = d/\cos(\theta - \delta)$ and $x = \theta - \delta$, its area is

$$\tau = \frac{1}{2} \int_{-\gamma}^{\gamma} R^2 d\theta = \frac{1}{2} d^2 (t_a + t_b), \quad t_a = \tan(\gamma + \delta), \quad t_b = \tan(\gamma - \delta),$$

and the area outside B is $E = \frac{1}{2} \int_{-\gamma}^{\gamma} (R^2 - \rho^2)_+ d\theta$.

Lemma A.1 (Isosceles wedges are best). *Fix γ and τ , and let $d = d(\delta)$ be determined by $\tau = \frac{1}{2} d^2 (t_a + t_b)$. Then $K(\delta) := E(W(\gamma, \delta, d(\delta))) \geq K(0)$ for all $|\delta| < \pi/2 - \gamma$.*

Proof. By symmetry take $\delta \geq 0$. Put $w = t_a - t_b \geq 0$. Since $t_a + t_b = \sin 2\gamma / (\cos(\gamma + \delta) \cos(\gamma - \delta)) > 0$, and $\frac{d}{d\delta}(t_a + t_b) = \sec^2(\gamma + \delta) - \sec^2(\gamma - \delta) = (t_a - t_b)(t_a + t_b)$, we get

$$(d^2)' = -w d^2. \tag{9}$$

For fixed θ , $\partial_\delta R^2 = (d^2)' \sec^2 x - 2d^2 \sec^2 x \tan x = -d^2 \sec^2 x (w + 2 \tan x)$, and $\frac{d}{dx}(\tan^2 x + w \tan x) = \sec^2 x (w + 2 \tan x)$. Let $p(T) = T^2 + wT$; then $p(t_b) = p(-t_a) = t_a t_b$. The function $(u - \rho^2)_+$ is Lipschitz, R^2 is C^1 in δ , and $R(\theta) = \rho$ for at most two values of θ ; so K is differentiable with

$$K'(\delta) = \frac{1}{2} \int_{\{R > \rho\}} \partial_\delta R^2 d\theta = -\frac{1}{2} d^2 \sum [p(\tan x'') - p(\tan x')],$$

the sum over the maximal intervals $[x', x'']$ of $\{x \in [-\gamma - \delta, \gamma - \delta] : |x| > x_0\}$, where $x_0 = \arccos(d/\rho)$ if $d < \rho$ and $x_0 = 0$ otherwise; put $T_0 = \tan x_0 \geq 0$. Let $\delta > 0$ (so $t_a > |t_b|$). There are four cases.

- $\gamma + \delta \leq x_0$: the set is empty and $K' = 0$.
- $\gamma - \delta > x_0$: two intervals, $[-\gamma - \delta, -x_0]$ and $[x_0, \gamma - \delta]$, and the sum equals $[p(-T_0) - p(-t_a)] + [p(t_b) - p(T_0)] = -2wT_0$; thus $K' = d^2 w T_0 \geq 0$.

- $-x_0 \leq \gamma - \delta \leq x_0 < \gamma + \delta$: one interval $[-\gamma - \delta, -x_0]$, the sum is $T_0^2 - wT_0 - t_a t_b$, and $K' = \frac{1}{2}d^2(t_a - T_0)(t_b + T_0) \geq 0$ because $t_a > T_0$ and $t_b \geq -T_0$.
- $\gamma - \delta < -x_0$: one interval, the whole $[-\gamma - \delta, \gamma - \delta]$, the sum is $p(t_b) - p(-t_a) = 0$ and $K' = 0$.

Thus $K' \geq 0$ on $(0, \pi/2 - \gamma)$ and K is continuous, so $K(\delta) \geq K(0)$. $\square$

The isosceles function. For $\tau > 0$, $0 < \gamma < \pi/2$ let $\varepsilon(\tau, \gamma) = K(0)$, the area outside B of the isosceles wedge $W(\gamma, 0, d)$ with $d^2 = s := \tau \cot \gamma$:

$$\varepsilon(\tau, \gamma) = \begin{cases} \tau - \rho^2 \gamma & s \geq \rho^2, \\ \tau - \rho^2 \gamma + F(s) & \rho^2 \cos^2 \gamma < s < \rho^2, \\ 0 & s \leq \rho^2 \cos^2 \gamma, \end{cases} \quad F(s) = \rho^2 \arccos \frac{\sqrt{s}}{\rho} - \sqrt{s(\rho^2 - s)}. \quad (10)$$

(The integrand $s \sec^2 \theta - \rho^2$ is ≥ 0 on $[-\gamma, \gamma]$ in the first case, ≤ 0 in the third, and positive exactly for $|\theta| > \arccos(\sqrt{s}/\rho)$ in the second.) A regular k -gon of area y about 0 is the union of k isosceles wedges $W(\pi/k, 0, a)$ of area y/k , so

$$M_k(y, \pi \rho^2) = k \varepsilon(y/k, \pi/k). \quad (11)$$

Lemma A.2. ε is convex on $(0, \infty) \times (0, \pi/2)$.

Proof. Let $A(s) = \rho^2/(2s^{3/2}\sqrt{\rho^2 - s})$ and $B(s) = \sqrt{(\rho^2 - s)/s}$. Then $F' = -B$ and $F'' = A > 0$ on $(0, \rho^2)$, and $F(\rho^2) = 0$, $F'(s) \rightarrow 0$ as $s \rightarrow \rho^2$.

C^1 . At the boundary $s = \rho^2$, ($\tau = \rho^2 \tan \gamma$) the second expression has the same value and gradient $(1, -\rho^2)$ as the first. At $s = \rho^2 \cos^2 \gamma$ ($\tau = \rho^2 \sin \gamma \cos \gamma$) one finds $\varepsilon = 0$, $\varepsilon_\tau = 1 - B \cot \gamma = 0$ and $\varepsilon_\gamma = -\rho^2 + B\tau \csc^2 \gamma = 0$. So $\varepsilon \in C^1$.

Hessian in the middle regime. With $s = \tau \cot \gamma$,

$$\varepsilon_{\tau\tau} = A \cot^2 \gamma > 0, \quad \varepsilon_{\tau\gamma} = \csc^2 \gamma (B - A\tau \cot \gamma), \quad \varepsilon_{\gamma\gamma} = A\tau^2 \csc^4 \gamma - 2B\tau \csc^2 \gamma \cot \gamma,$$

and, using $AB = \rho^2/(2s^2)$,

$$\varepsilon_{\tau\tau}\varepsilon_{\gamma\gamma} - \varepsilon_{\tau\gamma}^2 = \csc^2 \gamma B^2 \left(\frac{\rho^2}{\rho^2 - s} - \csc^2 \gamma \right) = \frac{\csc^4 \gamma B^2 (s - \rho^2 \cos^2 \gamma)}{\rho^2 - s} \geq 0$$

precisely in the middle regime $s > \rho^2 \cos^2 \gamma$. The first and third regimes are linear. So the Hessian is positive semi-definite wherever ε is C^2 .

Convexity. The regime boundaries are the graphs $\tau = \rho^2 \tan \gamma$ (strictly convex) and $\tau = \rho^2 \sin \gamma \cos \gamma$ (strictly concave) over $(0, \pi/2)$. A line meets each of them in at most two points, so along any line the function $g(t) = \varepsilon(\text{line})$ is C^1 , and g' is non-decreasing on each of at most five intervals; being continuous, g' is non-decreasing and g is convex. $\square$

Conclusion. By Lemma A.1 each T_i satisfies $|T_i \setminus B| \geq \varepsilon(\tau_i, \gamma_i)$. By Lemma A.2, Jensen's inequality, $\sum \gamma_i = \pi$, $\sum \tau_i = |Q|$ and (11),

$$|Q \setminus B| = \sum_{i=1}^k |T_i \setminus B| \geq \sum_{i=1}^k \varepsilon(\tau_i, \gamma_i) \geq k \varepsilon\left(\frac{|Q|}{k}, \frac{\pi}{k}\right) = M_k(|Q|, \pi \rho^2). \quad \square$$

B Proof of the Euler count

Let $K, c_j, r_j, P_j, Q_j, J', N', k_j$ be as in Lemma 3.3; the c_j are distinct, so ℓ_{ij} is a line for $i \neq j$. Cells with $|Q_j| = 0$ are discarded; all cells below belong to J' .

B.1: edge-to-edge. On ℓ_{ij} we have $\text{pow}_i = \text{pow}_j$, so for $x \in \ell_{ij}$: $x \in P_i \iff x \in P_j$. Hence $P_i \cap P_j = P_i \cap \ell_{ij} = P_j \cap \ell_{ij}$ and

$$Q_i \cap Q_j = Q_i \cap \ell_{ij} = Q_j \cap \ell_{ij}.$$

Q_i lies in a closed half-plane bounded by ℓ_{ij} , and Q_j in the opposite one. So $Q_i \cap Q_j$ is empty, a common vertex, or a common side of Q_i and Q_j . Consequently: if v is a vertex of Q_i and $v \in Q_l$, then v is a vertex of Q_l (the common face of the two polygons contains v as an extreme point, and extreme points of a face of a convex set are extreme points of the set).

B.2: shared sides. Let s be a side of Q_i not contained in ∂K , on the line L . Then L is not a side line of K (otherwise $L \cap K \subset \partial K$), so $L \cap \partial K$ has at most two points and we can choose x in the relative interior of s with $x \in \text{int } K$. Take a small disc $D \ni x$, $D \subset \text{int } K$, with $D \cap Q_i = D \cap L^-$ (L^- the closed half-plane containing Q_i), and meeting no cell P_l with $x \notin P_l$. For each $l \neq i$ with $x \in P_l$, the line ℓ_{il} passes through x and has $P_i \supset D \cap L^-$ on one side, so $\ell_{il} = L$. The open half-disc $D \setminus L^-$ is covered by these finitely many P_l , so one of them, P_l , has $|P_l \cap D \setminus L^-| > 0$, i.e. $l \in J'$, and $Q_i \cap Q_l = Q_i \cap L = s$ by B.1. Another cell $Q_{l'}$ with this property lies on the same side of L as Q_l and has s as a side, so it overlaps Q_l in positive area near x , which is impossible. So *every side of a cell is either contained in ∂K or is a side of exactly one other cell.*

B.3: the plane graph. Let V be the set of vertices of the cells Q_j , $j \in J'$, and E the set of their sides; a side shared by two cells is counted once. By B.1 two distinct elements of E meet at most in a common endpoint, so (V, E) is a plane graph. The union of the cells with positive area is a closed subset of K of full measure and hence equals K . The bounded faces of (V, E) are the interiors of the Q_j , $j \in J'$ (a boundary point of Q_l in $\text{int } Q_j$ would give Q_l and Q_j an overlap of positive area), and the unbounded face is $\mathbb{R}^2 \setminus K$. Thus there are $N' + 1$ faces and Euler's formula for a plane graph with $C \geq 1$ components gives

$$|V| - |E| + N' + 1 = 1 + C \geq 2. \quad (12)$$

Let E_∂ be the number of sides contained in ∂K . Each side of K is covered by the cells, so it contains a side of some cell; hence $E_\partial \geq p$ and, by B.2,

$$\sum_{j \in J'} k_j = 2|E| - E_\partial \leq 2|E| - p. \quad (13)$$

Degrees. Let $v \in V$. By B.1, every cell containing v has v as a vertex, so near v it is a closed angular sector of opening $< \pi$ bounded by two sides issuing from v . These sectors have disjoint interiors and cover a neighbourhood of v in K . If $v \in \text{int } K$ they tile the full angle 2π , so there are at least 3 of them and exactly as many distinct bounding rays; if v lies in the relative interior of a side of K they tile an angle π , so there are $n \geq 2$ sectors and $n + 1 \geq 3$ rays; if v is a corner of K there are $n \geq 1$ sectors and $n + 1 \geq 2$ rays. Each ray is part of an edge ending at v . Every corner of K belongs to V (it lies in some cell and is an extreme point of K). Hence $\deg v \geq 3$ except at the p corners where $\deg v \geq 2$, so

$$2|E| = \sum_{v \in V} \deg v \geq 3|V| - p. \quad (14)$$

From (12), $|V| \geq |E| - N' + 1$; with (14), $2|E| \geq 3|E| - 3N' + 3 - p$, i.e. $|E| \leq 3N' + p - 3$. By (13), $\sum_{j \in J'} k_j \leq 6N' + 2p - 6 - p = 6N' + p - 6$. For the square $p = 4$. $\square$