

The Amoeba Dimension of an Arbitrary Loopless Matroid

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Abstract

Let M be a loopless matroid on a finite set E with rank function r , and let $\Sigma(M) \subseteq \mathbb{R}^E$ be the support of its matroid fan. Draisma, Eggleston, Pendavingh, Rau and Yuen defined $\text{adim}(M)$ as the minimum of $2 \dim(\Sigma(M) + R) - \dim R$ over rational subspaces $R \subseteq \mathbb{R}^E$; for the matroid of a complex linear space this is the dimension of the amoeba of the space. They proved that $\text{adim}(M) = \min \sum_i (2r(P_i) - 1)$, the minimum over all partitions $\{P_1, \dots, P_k\}$ of E , when M is realizable over $\mathbb{C}$, and asked whether this holds for every loopless matroid. The question appears in an Oberwolfach report, in a BIRS problem session and as Conjecture 1.4.1 of their paper. We show that the answer is yes, and that both numbers equal $\dim(\Sigma(M) + \Sigma(M))$; the minimum is attained by a rational subspace in the braid arrangement. The lower bound $\dim(\Sigma(M) + \Sigma(M)) \geq \min \sum_i (2r(P_i) - 1)$ is a theorem of Bernstein, valid for all loopless matroids. The only new ingredient is the inequality $2 \dim(\Phi + R) - \dim R \geq \dim(\Phi + \Phi)$ for every finite union of cones Φ and every subspace R , which follows from the Grassmann formula. Consequently $S \mapsto \text{adim}(M|S)$ is a matroid rank function, which answers one of the sub-questions of the BIRS problem. The others are answered by earlier results: the partition minimum for $M|S$ is a matroid rank function of S (Draisma et al.), and by Bernstein's theorem it equals $\dim(\Sigma(M|S) + \Sigma(M|S))$. The inequality can be strict for other fans. Exact computations on connected non-realizable matroids are consistent with the theorem. This is an unrefereed note.

1 Introduction

For an irreducible subvariety $X \subseteq (\mathbb{C}^*)^n$, the *amoeba* $\mathcal{A}(X)$ is the image of X under $\text{Log}(z) = (\log |z_1|, \dots, \log |z_n|)$. Draisma, Rau and Yuen [DRY20] showed that its dimension is determined by the tropicalization $\text{Trop}(X)$:

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \min \{2 \dim(\text{Trop}(X) + R) - \dim R : R \subseteq \mathbb{R}^n \text{ a rational subspace}\}, \quad (1)$$

where a rational subspace is a subspace spanned by rational vectors and $+$ is the Minkowski sum [DRY20, Corollary 7], [Dra23, Corollary 4]. If $X = V \cap (\mathbb{C}^*)^n$ for a linear subspace $V \subseteq \mathbb{C}^n$ that is not contained in a coordinate hyperplane, then $\text{Trop}(X)$ is the support $\Sigma(M_V)$ of the matroid fan of the matroid M_V of V [DEP⁺24, §1.4]. For an arbitrary loopless matroid M on a finite set E this suggests the invariant [DEP⁺24, (3)]

$$\text{adim}(M) = \min \{2 \dim(\Sigma(M) + R) - \dim R : R \subseteq \mathbb{R}^E \text{ a rational subspace}\}. \quad (2)$$

For a loopless matroid M with rank function r and for $S \subseteq E$ put

$$r'(S) = \min \left\{ \sum_{i=1}^k (2r(P_i) - 1) : \{P_1, \dots, P_k\} \text{ a partition of } S \right\}, \quad (3)$$

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where the parts of a partition are nonempty and $r'(\emptyset) = 0$. Draisma, Eggleston, Pendavingh, Rau and Yuen proved a formula conjectured by Rau: $\dim_{\mathbb{R}} \mathcal{A}(V \cap (\mathbb{C}^*)^n) = r'(E)$ for the matroid of V [DEP⁺24, Theorem 1.3.1]. They also proved that for every loopless matroid M the function r' is the rank function of a matroid M' on E [DEP⁺24, Theorem 1.3.2]. For a partition $P = \{P_1, \dots, P_k\}$ of E let

$$R_P = \text{span}\{\mathbf{1}_{P_1}, \dots, \mathbf{1}_{P_k}\} = \{x \in \mathbb{R}^E : x_i = x_j \text{ whenever } i \text{ and } j \text{ lie in the same part of } P\},$$

where $\mathbf{1}_S$ is the indicator vector of S . This is a rational subspace in the braid arrangement, that is, an intersection of hyperplanes $x_i = x_j$, and every such intersection is R_P for some partition P . Taking $R = R_P$ in (2) gives $\text{adim}(M) \leq r'(E)$ [Dra23, DEP⁺24]; see Lemma 3.3. By (1) and [DEP⁺24, Theorem 1.3.1], equality holds when M is realizable over $\mathbb{C}$. This leaves the following question, which has two forms.

Problem 1.1 ([Dra23, Open question 2], [DEP⁺24, Conjecture 1.4.1], [BHRS23, §3.7, problem 1]). *Let M be a loopless matroid on E .*

(a) *Is $\text{adim}(M) = r'(E)$?*

(b) *Is the minimum in (2) attained by a subspace in the braid arrangement?*

Draisma posed form (a) in his Oberwolfach abstract. He noted there that the inequality $\leq$ is easy, and wrote of the converse: “For the converse we have no idea!” [Dra23, p. 857]. Rau posed form (a) at a problem session of the BIRS workshop 23w5149, together with sub-questions, which we discuss in Section 4 [BHRS23, §3.7]. Form (b) is [DEP⁺24, Conjecture 1.4.1], and the authors add that a direct proof that also covers non-representable matroids would be of interest. The problem is record OWR-12697711-015 of the UnsolvedMath dataset [ula26].

Since every R_P is rational, Lemma 3.3 gives

$$\text{adim}(M) \leq \min_P (2 \dim(\Sigma(M) + R_P) - \dim R_P) \leq r'(E),$$

where P ranges over all partitions of E . So a positive answer to (a) gives a positive answer to (b). The converse would follow from the inequality $2 \dim(\Sigma(M) + R_P) - \dim R_P \geq r'(E)$ for every partition P . In [DEP⁺24, §1.4] the two forms are stated to be equivalent; the argument given there uses the identity $2 \dim(\Sigma(M) + R_P) - \dim R_P = \sum_i (2r(P_i) - 1)$ for every partition P , which in general holds only as an inequality (see the remark after Lemma 3.3). We obtain the inequality needed for the converse only as part of Theorem 1.2, from Corollary 3.2, Theorem 3.4, and Lemma 3.7.

We answer both forms of the question affirmatively.

Theorem 1.2. *Let M be a loopless matroid on a finite set E . Then*

$$\text{adim}(M) = \dim(\Sigma(M) + \Sigma(M)) = r'(E).$$

More precisely, $2 \dim(\Sigma(M) + R) - \dim R \geq \dim(\Sigma(M) + \Sigma(M)) \geq r'(E)$ for every real subspace $R \subseteq \mathbb{R}^E$, and $2 \dim(\Sigma(M) + R_P) - \dim R_P = r'(E)$ for every partition P of E that attains the minimum in (3). In particular, Conjecture 1.4.1 of [DEP⁺24] holds, and the minimum in (2) does not change if R ranges over all real subspaces.

The proof has three ingredients.

- (i) Lemma 3.1: for all subspaces A, B, R of a vector space, $\dim(A + B) + \dim R \leq \dim(A + R) + \dim(B + R)$. Applied to the linear spans of the maximal cones of $\Sigma(M)$, it gives $\text{adim}(M) \geq \dim(\Sigma(M) + \Sigma(M))$ (Corollary 3.2).

- (ii) A theorem of Bernstein [Ber22, Lemma 3.4 and Theorem 3.5], which he stated and proved for all loopless matroids, gives $\dim(\Sigma(M) + \Sigma(M)) \geq r'(E)$ (Theorem 3.4 and Lemma 3.7). The number $r'(E)$ enters through [DEP⁺24, Theorem 1.3.2], or equivalently through Edmonds' theorem on matroids induced by submodular functions [Edm70]. We include a proof for completeness.
- (iii) The upper bound $\text{adim}(M) \leq r'(E)$ of [Dra23, DEP⁺24] (Lemma 3.3).

Only (i) is new, and it is elementary; the substance of the lower bound is Bernstein's. Section 4 discusses the BIRS sub-questions, and gives a two-matroid version of Theorem 1.2 and another proof of the formula of [DEP⁺24] for linear spaces. Section 5 shows that the inequality in (i) can be strict for fans that are not matroid fans. This concerns Open question 1 of [Dra23] (record OWR-12697711-014 of [ula26]), which asks for an algorithm computing the right-hand side of (1) for an arbitrary rational fan in place of $\text{Trop}(X)$; this note does not settle that question. Section 6 reports exact computations, which are consistency checks only.

2 Matroid fans and flags

Throughout, M is a loopless matroid on a finite nonempty set E , with rank function r , closure operator cl and rank $d = r(E)$; we follow [Oxl11] for matroid terminology. For $S \subseteq E$ we write $M|S$ for the restriction of M to S . It is loopless, and $\text{cl}_{M|S}(X) = \text{cl}(X) \cap S$ for $X \subseteq S$. All vector spaces are real and finite-dimensional.

A *maximal flag* F of M is a chain of flats $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = E$. Then $r(F_i) = i$ for all i (note $F_0 = \text{cl}(\emptyset) = \emptyset$ because M is loopless), and the *layers* $F_i \setminus F_{i-1}$, $1 \leq i \leq d$, are nonempty and partition E . Put

$$\sigma_F = \left\{ \sum_{i=1}^d \lambda_i \mathbf{1}_{F_i} : \lambda_1, \dots, \lambda_{d-1} \geq 0, \lambda_d \in \mathbb{R} \right\}, \quad L_F = \text{span}\{\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_d}\}.$$

Thus L_F is the d -dimensional space of vectors that are constant on every layer of F , and it is the linear span of the convex cone σ_F , which contains $\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_{d-1}}$ and $\pm \mathbf{1}_E$. The *matroid fan* (Bergman fan) of M has support

$$\Sigma(M) = \bigcup_F \sigma_F \subseteq \mathbb{R}^E,$$

the union over all maximal flags of M . Up to sign, this is the fine subdivision of Ardila and Klivans [AK06]; with our sign, $\Sigma(M)$ is the set of all $w \in \mathbb{R}^E$ such that on every circuit of M the minimum of w is attained at least twice. Some authors use the opposite sign. Since $\dim(-X + R) = \dim(X + R)$ for every subspace R and $\dim((-X) + (-X)) = \dim(X + X)$, none of the quantities in this note depends on the sign; for instance, Bernstein's Bergman fan [Ber22, Definition 2.16] is $-\Sigma(M)$.

The *dimension* of a finite union of convex cones is the largest dimension of the linear span of one of the cones; it equals the topological dimension of the union. Since $\Sigma(M) + R = \bigcup_F (\sigma_F + R)$, where $\sigma_F + R$ is a convex cone with linear span $L_F + R$, and similarly for the sum of two fans, we have the following.

Lemma 2.1. *For every subspace $R \subseteq \mathbb{R}^E$ and all loopless matroids M, N on E ,*

$$\dim(\Sigma(M) + R) = \max_F \dim(L_F + R), \quad \dim(\Sigma(M) + \Sigma(N)) = \max_{F,G} \dim(L_F + L_G),$$

where F and G range over the maximal flags of M and of N , respectively.

For maximal flags F of M and G of N , let $H_{F,G}$ be the bipartite multigraph whose two vertex classes are the set of layers of F and the set of layers of G , with one edge for each $e \in E$, joining the layer of F that contains e to the layer of G that contains e . A set $I \subseteq E$ of edges is a *forest* if it contains no cycle; two parallel edges form a cycle.

Lemma 2.2 (cf. [Ber22, Lemma 3.3]). *The number $\dim(L_F + L_G)$ is the rank of the cycle matroid of $H_{F,G}$. In particular, if $I \subseteq E$ is a forest in $H_{F,G}$, then $\dim(L_F + L_G) \geq |I|$.*

Proof. Let A be the matrix with rows indexed by E and columns indexed by the vertices of $H_{F,G}$, with $A_{e,B} = 1$ if $e \in B$ and $A_{e,B} = 0$ otherwise. Its columns are the indicator vectors of the layers of F and of G , so its column space is $L_F + L_G$. Multiplying the columns of the layers of G by -1 does not change the rank, and it turns the row of e into the signed incidence vector of the edge e . A set of such rows is linearly independent if and only if the corresponding edges form a forest [Oxl11, §5.1]. Hence $\dim(L_F + L_G) = \text{rank } A$ is the rank of the cycle matroid of $H_{F,G}$. $\square$

3 Proof of the main theorem

3.1 The new inequality

Lemma 3.1. *Let A , B and R be subspaces of a vector space. Then $\dim(A + B) + \dim R \leq \dim(A + R) + \dim(B + R)$.*

Proof. By the Grassmann formula,

$$\dim(A + R) + \dim(B + R) = \dim(A + B + R) + \dim((A + R) \cap (B + R)) \geq \dim(A + B) + \dim R,$$

because $A + B \subseteq A + B + R$ and $R \subseteq (A + R) \cap (B + R)$. $\square$

Corollary 3.2. *Let $\Phi \subseteq \mathbb{R}^n$ be a finite union of convex cones. Then*

$$2 \dim(\Phi + R) - \dim R \geq \dim(\Phi + \Phi) \quad \text{for every subspace } R \subseteq \mathbb{R}^n.$$

In particular, $2 \dim(\Sigma(M) + R) - \dim R \geq \dim(\Sigma(M) + \Sigma(M))$ for every real subspace $R \subseteq \mathbb{R}^E$, and $\text{adim}(M) \geq \dim(\Sigma(M) + \Sigma(M))$.

Proof. Write $\Phi = \bigcup_j C_j$ with convex cones C_j , and let L_j be the linear span of C_j . As in Lemma 2.1, $\dim(\Phi + R) = \max_j \dim(L_j + R)$ and $\dim(\Phi + \Phi) = \max_{j,k} \dim(L_j + L_k)$. By Lemma 3.1, $\dim(L_j + L_k) \leq \dim(L_j + R) + \dim(L_k + R) - \dim R \leq 2 \dim(\Phi + R) - \dim R$ for all j and k . $\square$

3.2 The upper bound

Lemma 3.3 ([Dra23], [DEP⁺24, §1.4]). *Let $P = \{P_1, \dots, P_k\}$ be a partition of E . Then $\dim(L_F + R_P) \leq \sum_{i=1}^k r(P_i)$ for every maximal flag F of M . Consequently $2 \dim(\Sigma(M) + R_P) - \dim R_P \leq \sum_{i=1}^k (2r(P_i) - 1)$, and $\text{adim}(M) \leq r'(E)$.*

Proof. Let $\pi_i: \mathbb{R}^E \rightarrow \mathbb{R}^{P_i}$ be the coordinate projection. Every subspace W of $\mathbb{R}^E = \bigoplus_i \mathbb{R}^{P_i}$ satisfies $W \subseteq \bigoplus_i \pi_i(W)$. The space $\pi_i(L_F + R_P)$ is spanned by the indicator vectors of the sets $F_j \cap P_i$, $1 \leq j \leq d$; the vector $\pi_i(\mathbf{1}_{P_i}) = \mathbf{1}_{F_d \cap P_i}$ is among them. These sets form a chain of flats of the loopless matroid $M|P_i$. Distinct flats in a chain have distinct ranks, and nonempty flats of $M|P_i$ have rank between 1 and $r(P_i)$, so at most $r(P_i)$ of these sets are distinct and nonempty. Hence $\dim \pi_i(L_F + R_P) \leq r(P_i)$ and $\dim(L_F + R_P) \leq \sum_i r(P_i)$. Since $\dim R_P = k$, Lemma 2.1 gives the second claim, and taking the minimum over P gives the third. $\square$

The inequality in the second claim of Lemma 3.3 can be strict. Since $\mathbf{1}_E$ lies in L_F and in R_P , we have $\dim(L_F + R_P) \leq d + k - 1$, so

$$\dim(\Sigma(M) + R_P) \leq \min \left\{ \sum_{i=1}^k r(P_i), d + k - 1 \right\}$$

and $2 \dim(\Sigma(M) + R_P) - \dim R_P \leq 2d + k - 2$. This bound is smaller than $\sum_i (2r(P_i) - 1)$ whenever $\sum_i r(P_i) \geq d + k$. For example, for the uniform matroid $U_{2,4}$ on $\{1, 2, 3, 4\}$ and $P = \{\{1, 2\}, \{3, 4\}\}$ we get $2 \dim(\Sigma(M) + R_P) - \dim R_P = 4$, while $\sum_i (2r(P_i) - 1) = 6$; for the Fano matroid F_7 and a partition of its seven points into two points and the other five, the two numbers are 6 and 8. By Theorem 1.2, equality holds whenever P attains the minimum in (3). So the identity $2 \dim(\Sigma(M) + R_P) - \dim R_P = \sum_i (2r(P_i) - 1)$, stated for every partition in [DEP⁺24, §1.4], holds in general only as the inequality of Lemma 3.3. Apart from their remark that the two forms of Problem 1.1 are equivalent, this inequality is all that [DEP⁺24] use.

3.3 Bernstein's lower bound

Theorem 3.4 (Bernstein [Ber22, Lemma 3.4 and Theorem 3.5]). *Let M and N be loopless matroids on E , and let $I \subseteq E$ satisfy*

$$|J| \leq r_M(J) + r_N(J) - 1 \quad \text{for every nonempty } J \subseteq I. \quad (4)$$

Then there are maximal flags F of M and G of N such that I is a forest in $H_{F,G}$. Consequently $\dim(\Sigma(M) + \Sigma(N)) \geq |I|$.

Condition (4) says that I is independent in the matroid induced by $r_M + r_N - 1$ [Edm70]. Bernstein proved that this holds if and only if the projection of $\Sigma(M) + \Sigma(N)$ to $\mathbb{R}^I$ has dimension $|I|$, for all loopless matroids, realizable or not; his application was to Hadamard products of linear spaces [Ber22, Theorem 3.1]. We need only the implication above. For completeness we include a proof, which is Bernstein's argument arranged as a single greedy procedure.

Lemma 3.5. *Let $I \subseteq E$ and $s = r(I)$, and let $\emptyset = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_s = I$ be a maximal flag of $M|I$. Put $F_j = \text{cl}(K_j)$ for $0 \leq j \leq s$, and extend by any maximal chain of flats $F_s \subsetneq F_{s+1} \subsetneq \cdots \subsetneq F_d = E$. Then F is a maximal flag of M , and its j -th layer meets I in $K_j \setminus K_{j-1}$ if $j \leq s$ and is disjoint from I if $j > s$.*

Proof. Such a chain exists, since $\text{cl}(K \cup \{e\})$ has rank $r(K) + 1$ for every flat K and every $e \notin K$. We have $F_0 = \text{cl}(\emptyset) = \emptyset$, $r(F_j) = r(K_j) = j$ for $j \leq s$, and $F_{j-1} \subseteq F_j$, so F is a maximal flag. For $j \leq s$, $F_j \cap I = \text{cl}_{M|I}(K_j) = K_j$ because K_j is a flat of $M|I$; for $j \geq s$, $F_j \supseteq \text{cl}(I) \supseteq I$. $\square$

Lemma 3.6 ([Ber22, Lemma 3.4]). *Let M and N be matroids on a finite set I such that $|J| \leq r_M(J) + r_N(J) - 1$ for every nonempty $J \subseteq I$. Then M and N are loopless, and there are maximal flags F of M and G of N such that $H_{F,G}$ is a forest.*

Proof. For $J = \{e\}$ the hypothesis gives $r_M(\{e\}) + r_N(\{e\}) \geq 2$, so e is a loop of neither matroid. Since $|J| \leq r_M(J) + r_N(J)$ for all $J \subseteq I$, the matroid union theorem [Oxl11, §11.3] shows that $I = A_1 \cup A_2$ with A_1 independent in M and A_2 independent in N . Extend A_1 to a basis I_1 of M and A_2 to a basis I_2 of N . Then $I = I_1 \cup I_2$. Let $p = |I_1|$ and $q = |I_2|$.

Bijections $\pi_1: I_1 \rightarrow \{1, \dots, p\}$ and $\pi_2: I_2 \rightarrow \{1, \dots, q\}$ define maximal flags F of M and G of N by $F_i = \text{cl}_M(\pi_1^{-1}\{1, \dots, i\})$ and $G_j = \text{cl}_N(\pi_2^{-1}\{1, \dots, j\})$. Since I_1 is independent, the i -th layer of F contains exactly one element of I_1 , namely $\pi_1^{-1}(i)$; likewise for G . We call $\pi_1(e)$ and $\pi_2(e)$ the

positions of e and assign them from the top down. Let $U_1 \subseteq I_1$ and $U_2 \subseteq I_2$ be the sets of elements that have not yet received their positions; initially $U_1 = I_1$ and $U_2 = I_2$. The elements of $I_1 \cap I_2$ receive both positions in the same step, so $U_1 \cap I_2 = U_2 \cap I_1$ throughout. While $U_1 \cup U_2 \neq \emptyset$, perform the first applicable step.

- (a) If some $e \in U_2 \setminus I_1$ satisfies $e \notin \text{cl}_M(U_1)$, set $\pi_2(e) = |U_2|$ and remove e from U_2 .
- (b) Otherwise, if some $e \in U_1 \setminus I_2$ satisfies $e \notin \text{cl}_N(U_2)$, set $\pi_1(e) = |U_1|$ and remove e from U_1 .
- (c) Otherwise let $U = U_1 \cup U_2$. Every element of $U_2 \setminus I_1$ lies in $\text{cl}_M(U_1)$ because (a) does not apply, and $U_2 \cap I_1 \subseteq U_1$. Hence $U \subseteq \text{cl}_M(U_1)$ and $r_M(U) = |U_1|$. Symmetrically $r_N(U) = |U_2|$. As $U \neq \emptyset$, the hypothesis gives $|U_1| + |U_2| - |U_1 \cap U_2| = |U| \leq |U_1| + |U_2| - 1$, so there is an element $x \in U_1 \cap U_2$. Set $\pi_1(x) = |U_1|$ and $\pi_2(x) = |U_2|$, and remove x from U_1 and from U_2 .

Each step removes one element from $U_1 \cup U_2$, so the procedure stops after $|I|$ steps, and then π_1 and π_2 are bijections. If an element receives its position on the side of N in some step, and U_2 denotes the value of this set just before that step, then the positions $1, \dots, |U_2|$ go exactly to the elements of U_2 , so $G_{|U_2|} = \text{cl}_N(U_2)$. The same holds on the side of M .

We show that in each step the edge of the element placed in that step has an endpoint that is not incident with any edge placed earlier. Then every step adds an edge with an endpoint of degree zero, so no cycle is ever created, and $H_{F,G}$ is a forest.

Consider a step of type (a) that places e at position $t = |U_2|$, and let $v = G_t \setminus G_{t-1}$ be the layer of G containing e ; then $v \subseteq G_t = \text{cl}_N(U_2)$. Let $f \in v$ with $f \neq e$. Since e is the only element of I_2 in v , we have $f \in I_1 \setminus I_2$, so f can be placed only in a step of type (b). If that step came earlier, then at that time $f \notin \text{cl}_N(U'_2)$, where $U'_2 \supseteq U_2$ was the value of U_2 at that time; but $f \in v \subseteq \text{cl}_N(U_2) \subseteq \text{cl}_N(U'_2)$, a contradiction. So no edge placed earlier is incident with v . Steps of type (b) are symmetric. In a step of type (c) that places x at position $s = |U_1|$ on the side of M , the layer u of F containing x satisfies $u \subseteq F_s = \text{cl}_M(U_1)$. Every $f \in u$ with $f \neq x$ lies in $I_2 \setminus I_1$ and can be placed only in a step of type (a), and the same argument shows that it was not placed earlier. $\square$

Proof of Theorem 3.4. Condition (4) involves only subsets of I , so Lemma 3.6 applies to $M|I$ and $N|I$. It gives maximal flags F' of $M|I$ and G' of $N|I$ such that $H_{F',G'}$ is a forest. Let F and G be the maximal flags of M and N obtained from F' and G' by Lemma 3.5. Identify the j -th layer of F' with the j -th layer of F , and likewise for G' and G . By Lemma 3.5, every $e \in I$ then joins the same two vertices in $H_{F,G}$ as in $H_{F',G'}$. So I is a forest in $H_{F,G}$, and Lemmas 2.1 and 2.2 give $\dim(\Sigma(M) + \Sigma(N)) \geq \dim(L_F + L_G) \geq |I|$. $\square$

3.4 Proof of Theorem 1.2

Lemma 3.7. *There is a set $I \subseteq E$ with $|I| = r'(E)$ and $|J| \leq 2r(J) - 1$ for every nonempty $J \subseteq I$.*

Proof. By [DEP⁺24, Theorem 1.3.2], whose proof is combinatorial and applies to every loopless matroid, r' is the rank function of a matroid M' on E . Let I be a basis of M' . Then $|I| = r'(E)$, and for every nonempty $J \subseteq I$ we have $|J| = r'(J) \leq 2r(J) - 1$, by the trivial partition $\{J\}$ in (3). $\square$

Lemma 3.7 also follows from Edmonds' rank formula for the matroid induced by $2r - 1$ [Edm70], [ADT26, Theorem 2.7], as in the proof of Proposition 4.2 below.

Proof of Theorem 1.2. Let I be as in Lemma 3.7. Theorem 3.4 with $N = M$ gives $\dim(\Sigma(M) + \Sigma(M)) \geq |I| = r'(E)$. By Corollary 3.2 and Lemma 3.3, for every real subspace R and every partition P attaining the minimum in (3),

$$r'(E) \leq \dim(\Sigma(M) + \Sigma(M)) \leq 2\dim(\Sigma(M) + R) - \dim R, \quad 2\dim(\Sigma(M) + R_P) - \dim R_P \leq r'(E).$$

Taking $R = R_P$ shows that all these quantities equal $r'(E)$. Hence the minimum of $2\dim(\Sigma(M) + R) - \dim R$ over all real subspaces R , and also over all rational ones, is $r'(E)$, and it is attained at R_P . $\square$

4 Consequences

Corollary 4.1. *For $S \subseteq E$ put $f_2(S) = \text{adim}(M|S)$ (with $f_2(\emptyset) = 0$) and $f_3(S) = r'(S)$. Then $f_2 = f_3$, this function is the rank function of the matroid M' of [DEP⁺24, Theorem 1.3.2], and $f_3(S) = \dim(\Sigma(M|S) + \Sigma(M|S))$ for every $S \subseteq E$.*

Proof. For nonempty S the matroid $M|S$ is loopless, and its rank function is the restriction of r ; so its invariant (3), evaluated at its ground set S , is $r'(S)$. Apply Theorem 1.2 to $M|S$, and [DEP⁺24, Theorem 1.3.2]. $\square$

The BIRS problem [BHRS23, §3.7, problem 1] considers, for each $S \subseteq E$, the three numbers $f_1(S) = r(S)$, $f_2(S)$ and $f_3(S)$. It asks whether the last two agree, whether f_2 and f_3 are matroid rank functions, and for an interpretation of f_3 . The only new part of Corollary 4.1 is the equality $f_2 = f_3$, which shows that f_2 is a matroid rank function. That f_3 is a rank function is [DEP⁺24, Theorem 1.3.2], and the description of f_3 in Corollary 4.1 is Bernstein's. Indeed, by Edmonds' rank formula [Edm70] (see [ADT26, Theorem 2.7]), f_3 is the rank function of the matroid induced by $2r - 1$, and by [Ber22, Theorem 3.5] a set $I \subseteq E$ is independent in this matroid if and only if $\dim \pi_I(\Sigma(M) + \Sigma(M)) = |I|$, where $\pi_I: \mathbb{R}^E \rightarrow \mathbb{R}^I$ is the coordinate projection. Since $\pi_S(\Sigma(M) + \Sigma(M))$ is a finite union of convex cones, the largest $|I|$ with $I \subseteq S$ and $\dim \pi_I(\Sigma(M) + \Sigma(M)) = |I|$ is $\dim \pi_S(\Sigma(M) + \Sigma(M))$. So $f_3(S) = \dim \pi_S(\Sigma(M) + \Sigma(M))$. Moreover $\pi_S(\Sigma(M)) = \Sigma(M|S)$: for a maximal flag F of M the sets $F_j \cap S$ form a chain of flats of $M|S$, so $\pi_S(\sigma_F)$ lies in a cone of $\Sigma(M|S)$, and every cone of $\Sigma(M|S)$ is $\pi_S(\sigma_F)$ for a maximal flag F as in Lemma 3.5. Hence $\pi_S(\Sigma(M) + \Sigma(M)) = \Sigma(M|S) + \Sigma(M|S)$, and $f_3(S) = \dim(\Sigma(M|S) + \Sigma(M|S))$ follows from [Ber22] and [Edm70]. Another description of f_3 uses the Dilworth truncation of $M \oplus M$ [DEP⁺24, final remark].

Proposition 4.2. *Let M and N be loopless matroids on E . Then*

$$\begin{aligned} \min_R (\dim(\Sigma(M) + R) + \dim(\Sigma(N) + R) - \dim R) &= \dim(\Sigma(M) + \Sigma(N)) \\ &= \min_P \sum_{i=1}^k (r_M(P_i) + r_N(P_i) - 1), \end{aligned}$$

where R ranges over all real subspaces of $\mathbb{R}^E$ (or over all rational ones) and $P = \{P_1, \dots, P_k\}$ over all partitions of E . The first minimum is attained at R_P for every P attaining the second.

Proof. For maximal flags F of M and G of N and a subspace R , Lemmas 2.1 and 3.1 give $\dim(L_F + L_G) \leq \dim(\Sigma(M) + R) + \dim(\Sigma(N) + R) - \dim R$. So the first minimum is at least $\dim(\Sigma(M) + \Sigma(N))$. The function $g = r_M + r_N - 1$ is monotone and submodular, and $g(\{e\}) = 1$ for all e . By Edmonds' theorem in the form of [ADT26, Theorem 2.7], the largest $I \subseteq E$ with $|J| \leq g(J)$ for all nonempty $J \subseteq I$ has size $\min\{|F_0| + \sum_i g(F_i)\}$, over $F_0 \subseteq E$ and partitions $\{F_i\}$ of $E \setminus F_0$. Since $g(\{e\}) = 1$, this equals $\min_P \sum_i g(P_i)$. Theorem 3.4, applied to such an I , gives $\dim(\Sigma(M) + \Sigma(N)) \geq \min_P \sum_i g(P_i)$. Finally, Lemma 3.3 applied to M and to N gives $\dim(\Sigma(M) + R_P) + \dim(\Sigma(N) + R_P) - \dim R_P \leq \sum_i (r_M(P_i) + r_N(P_i)) - k$ for every partition P . $\square$

The equality of the last two quantities in Proposition 4.2 is Bernstein's theorem [Ber22, Theorem 3.5] combined with Edmonds' rank formula; the new part is the first equality. For $N = M$ we recover Theorem 1.2.

Remark 4.3. If $M = M_V$ for a linear subspace $V \subseteq \mathbb{C}^n$ not contained in a coordinate hyperplane, then (1) and Theorem 1.2 give $\dim_{\mathbb{R}} \mathcal{A}(V \cap (\mathbb{C}^*)^n) = \dim(\Sigma(M_V) + \Sigma(M_V)) = r'(E)$. This is another proof of [DEP⁺24, Theorem 1.3.1], based on [DRY20] and [Ber22]. It is a tropical counterpart of the proof by Antolini, Dewar and Tanigawa [ADT26, §6], who show that $\dim_{\mathbb{R}} \mathcal{A}(V)$ equals the complex dimension of the coordinatewise product $V \star \bar{V}$ of V and its complex conjugate, and then apply Bernstein's description of the algebraic matroid of a Hadamard product [Ber22, Theorem 3.1].

5 Fans that are not matroid fans

For a finite union $\Phi \subseteq \mathbb{R}^n$ of convex cones put

$$\alpha(\Phi) = \min \{ 2 \dim(\Phi + R) - \dim R : R \subseteq \mathbb{R}^n \text{ a rational subspace} \}.$$

Open question 1 of [Dra23] asks for an algorithm that computes $\alpha(\Phi)$ when Φ is the support of a d -dimensional rational polyhedral fan. By Corollary 3.2, $\alpha(\Phi) \geq \dim(\Phi + \Phi)$ for every such Φ , and by Theorem 1.2 equality holds for matroid fans. Equality can fail in general.

Proposition 5.1. *Index the coordinates of $\mathbb{R}^6$ by the six edges of the complete graph K_4 , and for each vertex v of K_4 let S_v be the set of the three edges at v . Let $\Phi = \bigcup_v \mathbb{R}^{S_v}$ be the union of the four coordinate 3-planes $\mathbb{R}^{S_v}$. Then Φ is the support of a pure 3-dimensional rational polyhedral fan, $\dim(\Phi + \Phi) = 5$, and $2 \dim(\Phi + R) - \dim R \geq 6$ for every real subspace $R \subseteq \mathbb{R}^6$, with equality for $R = 0$. In particular $\alpha(\Phi) = 6 > \dim(\Phi + \Phi)$.*

Proof. The coordinate orthants of the four planes form a pure 3-dimensional rational polyhedral fan with support Φ ; two of these orthants meet in a common face, since $\mathbb{R}^{S_v} \cap \mathbb{R}^{S_w} = \mathbb{R}^{S_v \cap S_w}$. For $v \neq w$ the sets S_v and S_w share exactly the edge vw , so $\dim(\mathbb{R}^{S_v} + \mathbb{R}^{S_w}) = 5$; hence $\dim(\Phi + \Phi) = 5$. Let R be a subspace, $u = \dim R$, $a_v = \dim(R \cap \mathbb{R}^{S_v})$ and $a = \min_v a_v$. Then $\dim(R + \mathbb{R}^{S_v}) = u + 3 - a_v$, so $2 \dim(\Phi + R) - \dim R = u + 6 - 2a$. For $v \neq w$ the intersection $(R \cap \mathbb{R}^{S_v}) \cap (R \cap \mathbb{R}^{S_w})$ lies in the line $\mathbb{R}^{\{vw\}}$, so $a_v + a_w \leq u + 1$. Suppose $2a > u$. Then $u + 1 \leq 2a \leq a_v + a_w \leq u + 1$ for all $v \neq w$, so R contains the coordinate line $\mathbb{R}^{\{vw\}}$ for every edge vw . Hence $R = \mathbb{R}^6$, and then $a = 3$ and $2a = u$, a contradiction. Therefore $2a \leq u$ and $2 \dim(\Phi + R) - \dim R \geq 6$, with equality for $R = 0$. $\square$

The fan of Proposition 5.1 is balanced, with all weights equal to 1, but it is not connected in codimension one. Draisma, Rau and Yuen [DRY20, Question 14] ask for an algorithm that computes $\alpha(\Phi)$ when Φ is a balanced, pure-dimensional rational polyhedral complex that is connected in codimension one. The fan of Proposition 5.1 is not of this kind, and we do not know whether $\alpha(\Phi) > \dim(\Phi + \Phi)$ can happen for such complexes. By Theorem 1.2 and Remark 5.2, equality holds for matroid fans and for tropicalizations of irreducible varieties.

Remark 5.2. Let $X \subseteq (\mathbb{C}^*)^n$ be irreducible and $\Phi = \text{Trop}(X)$. By (1) and Corollary 3.2, $\dim_{\mathbb{R}} \mathcal{A}(X) = \alpha(\Phi) \geq \dim(\Phi + \Phi)$. In fact equality holds, so the strict inequality of Proposition 5.1 does not occur for tropicalizations of irreducible varieties. Indeed, let $\bar{X}$ be the complex conjugate of X , and let $Y \subseteq (\mathbb{C}^*)^n$ be the Zariski closure of $\{x \star y : x \in X, y \in \bar{X}\}$, where $\star$ is the coordinatewise product; Y is irreducible. Since the smooth points are dense in X , $\dim_{\mathbb{R}} \mathcal{A}(X)$ is the largest real rank of the differential of Log at a smooth point z of X (compare [DEP⁺24, §2.2]). Since $\text{Log}(z) = \frac{1}{2} \log(z \star \bar{z})$ coordinatewise, this is the real rank of the differential $v \mapsto v \star \bar{z} + z \star \bar{v}$ of $z \mapsto z \star \bar{z}$ on $T_z X$. This map is the composition of $v \mapsto (v, \bar{v})$ with the complex differential δ of the multiplication map $X \times \bar{X} \rightarrow Y$ at $(z, \bar{z})$. The complex span of $\{(v, \bar{v}) : v \in T_z X\}$ is $T_z X \times T_{\bar{z}} \bar{X}$, and the image of our map is a real subspace of $\mathbb{R}^n$, whose real dimension is the complex dimension of its complex span, namely of the image of δ . So the real rank equals the complex rank of δ , which is at most $\dim_{\mathbb{C}} Y$. (The smooth points (x, y) of $X \times \bar{X}$ at which the differential of the multiplication map has

rank greater than $\dim_{\mathbb{C}} Y$ form a Zariski open subset of the irreducible variety $X \times \overline{X}$. If it were nonempty, it would meet the nonempty open set of points mapped to smooth points of Y , where the rank is at most $\dim_{\mathbb{C}} Y$. And $(z, \bar{z})$ is a smooth point of $X \times \overline{X}$.) Finally, $\dim_{\mathbb{C}} Y = \dim \text{Trop}(Y)$ [BG84]; $\text{Trop}(X \times \overline{X}) = \text{Trop}(X) \times \text{Trop}(\overline{X})$, so [ST08, Theorem 1.1], applied to the monomial map $(x, y) \mapsto x \star y$, gives $\text{Trop}(Y) = \text{Trop}(X) + \text{Trop}(\overline{X})$ (see also [Ber22, Lemma 2.15]); and $\text{Trop}(\overline{X}) = \text{Trop}(X)$ because $\mathcal{A}(\overline{X}) = \mathcal{A}(X)$. Hence $\dim_{\mathbb{R}} \mathcal{A}(X) \leq \dim(\Phi + \Phi)$. This extends the identity $\dim_{\mathbb{R}} \mathcal{A}(V) = \dim_{\mathbb{C}}(V \star \overline{V})$ of [ADT26, §6] from linear spaces to irreducible varieties. We have not searched the literature for this identity, and it is not used elsewhere. It does not settle Open question 1 of [Dra23], which concerns arbitrary rational fans.

6 Consistency checks

The computations in this section are consistency checks; no proof depends on them. They use exact arithmetic. For a matroid given by its rank function, the programs compute $r'(E)$ and an optimal partition by dynamic programming over subsets, enumerate all maximal flags, compute $\dim(\Sigma(M) + \Sigma(M))$ as the maximum of $\dim(L_F + L_G)$ over all pairs of layer partitions (Lemmas 2.1 and 2.2), and compute $2 \dim(\Sigma(M) + R_P) - \dim R_P$ for an optimal partition P . They also run the procedure of Lemma 3.6 with every assertion of its proof checked, verify the resulting forest and the rank of $L_F + L_G$ by exact Gaussian elimination, and test Corollary 3.2 on random rational subspaces.

Write $n = |E|$. In the test sets of item 1 of the verification paragraph below, 34 of the 35 single matroids and all 9396 matroids of the exhaustive and random runs satisfy $r'(E) = \min(n, 2d - 1)$, so that the trivial partition or the partition into singletons is optimal; the exception, $F_7 \oplus U_{2,3}$, is disconnected. Table 1 lists 22 connected matroids that are not realizable over $\mathbb{C}$ and satisfy $r'(E) < \min(n, 2d - 1)$, so that their optimal partitions are nontrivial. Here N is one of F_7 , $AG(3, 2)$, the Vámos matroid V_8 , the non-Pappus matroid NP and the non-Desargues matroid ND, none of which is realizable over $\mathbb{C}$ [Oxl11]; $U_{k,k}$ consists of k coloops and $U_{1,m}$ of m parallel elements; $\text{FE}(N \oplus K)$ is the free extension of $N \oplus K$ by one element p , with rank function $r(S \cup \{p\}) = \min(r(S) + 1, r(E))$; and $\text{FE}^2(N \oplus K)$ is the free extension of $\text{FE}(N \oplus K)$. Each matroid contains N as a restriction and so is not realizable over $\mathbb{C}$. Connectivity was checked directly, and the 22 matroids have pairwise distinct numbers of subsets of each size and rank, so they are pairwise non-isomorphic. In every case the optimal partition found consists of $E(N)$, the parallel classes of K and singletons. Both $\dim(\Sigma(M) + \Sigma(M))$, computed over all pairs of layer partitions of maximal flags (at most 1828 of them), and $2 \dim(\Sigma(M) + R_P) - \dim R_P$ equal $r'(E)$.

	$N = F_7$	$AG(3, 2)$	V_8	NP	ND
$\text{FE}(N \oplus U_{2,2})$	(10, 5) $8 < 9$	(11, 6) $10 < 11$	(11, 6) $10 < 11$	(12, 5) $8 < 9$	(13, 5) $8 < 9$
$\text{FE}(N \oplus U_{3,3})$	(11, 6) $9 < 11$	(12, 7) $11 < 12$	(12, 7) $11 < 12$	(13, 6) $9 < 11$	–
$\text{FE}(N \oplus U_{1,2} \oplus U_{1,1})$	(11, 5) $8 < 9$	(12, 6) $10 < 11$	(12, 6) $10 < 11$	(13, 5) $8 < 9$	–
$\text{FE}(N \oplus U_{1,3} \oplus U_{1,1})$	(12, 5) $8 < 9$	(13, 6) $10 < 11$	(13, 6) $10 < 11$	–	–
$\text{FE}(N \oplus U_{2,2} \oplus U_{1,2})$	(12, 6) $9 < 11$	(13, 7) $11 < 13$	(13, 7) $11 < 13$	–	–
$\text{FE}^2(N \oplus U_{3,3})$	(12, 6) $10 < 11$	(13, 7) $12 < 13$	(13, 7) $12 < 13$	–	–

Table 1: The 22 connected matroids of Section 6, which are not realizable over $\mathbb{C}$ and have a nontrivial optimal partition. An entry $(n, d) \ a < b$ means $|E| = n$, rank d , $r'(E) = a$ and $\min(n, 2d - 1) = b$; a dash means $n > 13$ (not computed). In every case $\dim(\Sigma(M) + \Sigma(M))$ and $2 \dim(\Sigma(M) + R_P) - \dim R_P$ equal $r'(E)$.

Verification. The proofs above are complete; the computations only check them.

1. The author's programs `verify_all.py`, `exhaustive_check.py` and `certificates.py` (Python, standard library only) check Theorem 1.2 on 35 matroids, among them F_7 , F_7^* , $AG(3, 2)$, V_8 , the non-Pappus matroid and its dual and the non-Desargues matroid, and Proposition 4.2 on 10 pairs. They also check all 5596 labelled sparse paving matroids of rank 4 on 7 elements, 3800 random sparse paving matroids of rank 4 on 8 and of rank 5 on 9 elements, and 1500 random pairs on 8 elements, and they print explicit flags F , G with $\dim(L_F + L_G) = r'(E)$ for F_7^* , V_8 , the dual of the non-Pappus matroid, $AG(3, 2)$ and F_7 . No check failed.
2. An independent verification run, which was AI-assisted, checked every step of the proofs, re-ran these programs with identical results, and tested the theorem with separately written code on 195 matroids and 54 pairs. These included 22 connected matroids that are not realizable over $\mathbb{C}$ and have a nontrivial optimal partition. No check failed. That run's code is not part of the release.
3. A second independent verification run, also AI-assisted, wrote the program `indep_verify.py` from the text of this note, without the library of item 1, and produced Table 1. The program also checks 16 further connected free extensions of F_7 , F_7^* , $AG(3, 2)$, V_8 and NP, and 11 named, realizable or disconnected matroids, and it checks Proposition 5.1 on all 64 coordinate subspaces and on 3000 random rational subspaces. All checks pass.
4. A further independent verification run, also AI-assisted, checked the sources and every step of the proofs, and re-ran the programs of items 1 and 3 with identical results apart from timings. It wrote its own programs from the definitions and the text of this note, before reading the programs of items 1 and 3. They check Theorem 1.2 on 10 named matroids, on the 22 matroids of Table 1 and 11 further matroids that are not realizable over $\mathbb{C}$, on 115 random sparse paving matroids and on 3 matroids realizable over $\mathbb{Q}$; they evaluate $2\dim(\Sigma(M) + R_P) - \dim R_P$ for all partitions P of E for 60 of these matroids with at most 8 elements and for six matroids on 9 elements; and they check Proposition 4.2 on 18 pairs, the procedure of Lemma 3.6 on 300 random instances and Proposition 5.1 on 4267 subspaces. For all 27 labelled loopless matroids on 4 elements, the minimum of $2\dim(\Sigma(M) + R) - \dim R$ over all 1084 subspaces R of $\mathbb{Q}^4$ spanned by vectors with entries in $\{-1, 0, 1\}$ is $r'(E)$. No check of the results of this note failed. The examples after Lemma 3.3 were found in this run. Its programs and outputs are part of the release.

Scope and priority. Theorem 1.2 answers Open question 2 of [Dra23], Conjecture 1.4.1 of [DEP⁺24] and the question of [BHRS23, §3.7, problem 1] affirmatively for all loopless matroids, with the minimum taken over rational or over real subspaces. Corollary 4.1 answers the sub-question of [BHRS23] whether $S \mapsto \text{adim}(M|S)$ is a matroid rank function; the other sub-questions are answered by earlier results (Section 4). The published literature that we found lists the case of matroids that are not realizable over $\mathbb{C}$ as open: [Dra23], [BHRS23] and [DEP⁺24] pose it, and [ADT26, §6] treats complex linear spaces only. The affirmative answer in the non-realizable case therefore rests on this note, which has not been refereed. The lower bound $\dim(\Sigma(M) + \Sigma(M)) \geq r'(E)$ is Bernstein's theorem [Ber22, Lemma 3.4 and Theorem 3.5], which he stated and proved for all loopless matroids; the upper bound is from [Dra23, DEP⁺24]; the matroid M' is from [DEP⁺24]; and the description $r'(S) = \dim(\Sigma(M|S) + \Sigma(M|S))$ follows from [Ber22] and [Edm70]. The only new ingredient is Lemma 3.1, an elementary consequence of the Grassmann formula. Because the argument is short and uses only published results, it may be known informally. For instance, Antolini, Dewar and Tanigawa combined Bernstein's theorem with amoebas of complex linear spaces, and in their paper Antolini thanks Eggleston, one of the authors of [DEP⁺24], for initiating the discussion on amoebas of linear spaces [ADT26]. We claim only that we found no written account. We searched arXiv

(by topic and for the five authors of [DEP⁺24]), Crossref, OpenAlex, OpenCitations and zbMATH, including the works citing [DEP⁺24] and [Ber22], and ran web searches; we read the full texts of [Dra23], [BHRS23], [DRY20], [DEP⁺24], [Ber22] and [ADT26]. This negative search is not a proof of priority. We did not search the literature for the identity in Remark 5.2.

AI-assisted tools were used to search the problem corpus and the literature, to find the examples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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