

Irreducibility of the Exposed-Point Variety of a Discotope

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27 October 2026

Abstract

A discotope is a finite Minkowski sum of linear images of Euclidean unit balls. We prove that, when every summand has dimension at least two, the complex Zariski closure of its exposed points is irreducible, without any genericity assumption. The proof uses the real-analytic support-point map on the complement of finitely many subspaces of codimension at least two. Under an explicit generic condition on the summand spans, we also prove that every boundary point expressible as a sum of summand-boundary points is a limit of exposed points. Consequently the purely nonlinear part of a generic discotope equals its exposed-point variety and is irreducible. This proves Conjecture 8.2 of Gesmundo and Meroni and the exposed-point formulation in Oberwolfach Report 15/2023. The irreducibility and degree of the entire complex critical locus of the addition map are separate questions.

1 The two formulations

Let $N \geq 1$, let $2 \leq m_i \leq d$, and let $A_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^d$ be injective linear maps. Write

$$B_{m_i} = \{z \in \mathbb{R}^{m_i} : \|z\| \leq 1\}, \quad D_i = A_i B_{m_i}, \quad L_i = \operatorname{im} A_i, \quad \mathcal{D} = D_1 + \cdots + D_N.$$

Thus the discs may be ellipsoids in proper linear subspaces. The symbol ∂D_i means the boundary relative to L_i . We work in the linear span of $\mathcal{D}$, so that $\mathcal{D}$ is full dimensional. This is also the convention of [1, p. 147].

For a compact convex set K and a nonzero vector u , let

$$h_K(u) = \max_{x \in K} \langle u, x \rangle, \quad K^u = \{x \in K : \langle u, x \rangle = h_K(u)\}.$$

An exposed point is a point x for which $K^u = \{x\}$ for some $u \neq 0$. Let $\operatorname{Exp}(\mathcal{D})$ be the set of exposed points, and put

$$\mathcal{E} = \overline{\operatorname{Exp}(\mathcal{D})}^{\operatorname{Zar}} \subset \mathbb{C}^d, \quad X = (\partial D_1 + \cdots + \partial D_N) \cap \partial \mathcal{D}, \quad \mathcal{S} = \overline{X}^{\operatorname{Zar}} \subset \mathbb{C}^d.$$

All Zariski closures in this paper are over $\mathbb{C}$.

The variety $\mathcal{S}$ is the purely nonlinear part of [1, Definition 3.3]. Gesmundo and Meroni conjectured its irreducibility for generic discs of dimension at least two [1, Conjecture 8.2]. Meroni's Oberwolfach formulation asks for the irreducibility of $\mathcal{E}$ under the same dimension and genericity assumptions [2, Conjecture 1, p. 830]. The distinction between $\mathcal{E}$ and $\mathcal{S}$ matters: equality is not automatic.

Theorem 1. *If every $m_i \geq 2$, then $\mathcal{E}$ is irreducible, for every choice of the injective real maps A_i .*

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For the second formulation we use the following span condition:

$$\dim \left(\sum_{i \in J} L_i \right) = \min \left(d, \sum_{i \in J} m_i \right) \quad \text{for every } J \subseteq \{1, \dots, N\}. \quad (\text{GP})$$

It holds on a nonempty Zariski-open set of choices of the matrices A_i .

Theorem 2. *Suppose every $m_i \geq 2$ and (GP) holds. Then*

$$\text{Exp}(\mathcal{D}) \subseteq X \subseteq \overline{\text{Exp}(\mathcal{D})}^{\text{Eucl}}.$$

In particular, $\mathcal{S} = \mathcal{E}$ is irreducible.

This proves the two conjectures just described. The known face description in [1, Section 3] is the starting point. Instead of analyzing the whole critical locus of the complex addition map, we use the connected domain of real support normals and then approximate the exceptional faces. No assertion about the degree or birational geometry of that critical locus is needed.

2 A connected analytic parametrization

We first record two elementary facts, with their proofs.

Lemma 3. *If $K_1, \dots, K_N$ are linear subspaces of $\mathbb{R}^d$ of codimension at least two, then $\mathbb{R}^d \setminus \bigcup_i K_i$ is nonempty, open, and path connected.*

Proof. A finite union of proper linear subspaces does not cover $\mathbb{R}^d$, so the complement is nonempty. It is open because the subspaces are closed. For points a, b in the complement, choose

$$c \notin \bigcup_{i=1}^N ((K_i + \mathbb{R}a) \cup (K_i + \mathbb{R}b)).$$

Every subspace on the right has dimension at most $d-1$, so such a c exists. If $(1-t)a + tc \in K_i$ for some $0 < t \leq 1$, then $c \in K_i + \mathbb{R}a$, a contradiction. The endpoint a also avoids K_i . Thus $[a, c]$ avoids every K_i , as does $[c, b]$. These two segments give the required path. $\square$

Lemma 4. *Let $V \subset \mathbb{R}^d$ be a nonempty connected open set and $f : V \rightarrow \mathbb{R}^n$ a real-analytic map. Then $\overline{f(V)}^{\text{Zar}} \subset \mathbb{C}^n$ is irreducible.*

Proof. Let I be the ideal of complex polynomials vanishing on $f(V)$. If $PQ \in I$ and $P \notin I$, then the real-analytic function $P \circ f$ is nonzero on some nonempty open subset of V . Therefore $Q \circ f$ vanishes on that subset. The identity theorem for real-analytic functions, applied to its real and imaginary parts, gives $Q \circ f = 0$ on all of V . Hence $Q \in I$, so I is prime. It is proper because V is nonempty. The complex algebraic set defined by I is precisely $\overline{f(V)}^{\text{Zar}}$; the prime-ideal criterion proves its irreducibility. $\square$

Set

$$U = \mathbb{R}^d \setminus \bigcup_{i=1}^N \ker A_i^T. \quad (1)$$

For $u \in U$, define

$$p_i(u) = \frac{A_i A_i^T u}{\|A_i^T u\|}, \quad F(u) = \sum_{i=1}^N p_i(u). \quad (2)$$

Lemma 5. *The equality $F(U) = \text{Exp}(\mathcal{D})$ holds.*

Proof. Maximizing $\langle u, A_i z \rangle$ over $\|z\| \leq 1$ is equivalent to maximizing $\langle A_i^T u, z \rangle$. If $A_i^T u \neq 0$, Cauchy–Schwarz gives the unique maximizer $z = A_i^T u / \|A_i^T u\|$, and hence $D_i^u = \{p_i(u)\}$. If $A_i^T u = 0$, then $D_i^u = D_i$.

The support function and the supporting face of a Minkowski sum satisfy

$$h_{\mathcal{D}}(u) = \sum_i h_{D_i}(u), \quad \mathcal{D}^u = \sum_i D_i^u. \quad (3)$$

For completeness, if $x = \sum_i x_i \in \mathcal{D}^u$ with $x_i \in D_i$, the nonnegative numbers $h_{D_i}(u) - \langle u, x_i \rangle$ sum to zero, so each is zero. This proves the second equality; the reverse inclusion follows by addition.

If $u \in U$, (3) gives $\mathcal{D}^u = \{F(u)\}$. If $u \notin U$ and $u \neq 0$, at least one D_i^u is the entire positive-dimensional disc D_i . Fixing points in all other summand faces shows that $\mathcal{D}^u$ contains a translate of that disc, so it is not a singleton. This proves both inclusions. $\square$

Proof of Theorem 1. Because A_i has rank m_i , the subspace $\ker A_i^T$ has codimension $m_i \geq 2$. Lemma 3 shows that U is nonempty and connected. For each i , the polynomial

$$q_i(u) = u^T A_i A_i^T u = \|A_i^T u\|^2$$

is strictly positive on U . Its positive square root, and the reciprocal of that square root, are real analytic there. Thus F is real analytic on U . Lemmas 4 and 5 prove the result. $\square$

No injectivity or immersion property of F is used. Nor is U required to be simply connected: the square roots in (2) are the globally defined positive real ones, not branches chosen on a complex parameter space.

3 Approximating the exceptional faces

Lemma 6. *For fixed dimensions $m_i \leq d$, condition (GP) holds on a nonempty Zariski-open subset of the space of real matrices $(A_1, \dots, A_N)$.*

Proof. For a fixed J , the sum of the images has dimension equal to the rank of the matrix formed by concatenating the blocks A_i , $i \in J$. The condition that its rank be $\min(d, \sum_{i \in J} m_i)$ is Zariski open, as its failure is the simultaneous vanishing of all minors of that size. There are finitely many subsets J .

To verify nonemptiness of their intersection, take all columns of all blocks to be vectors $(1, t, \dots, t^{d-1})^T$, with distinct real parameters t . Any $r \leq d$ such columns are independent, since the minor using their first r coordinates is a nonzero Vandermonde determinant. Every block concatenation therefore has the required rank. $\square$

Proof of Theorem 2. Lemma 5 shows that an exposed point is a sum of points of ∂D_i , and it is on $\partial \mathcal{D}$. Thus $\text{Exp}(\mathcal{D}) \subseteq X$.

Take $x \in X$ and choose a representation

$$x = \sum_{i=1}^N A_i z_i, \quad \|z_i\| = 1.$$

Since $\mathcal{D}$ is full dimensional and $x \in \partial \mathcal{D}$, there is a nonzero supporting normal u at x . Equation (3) implies that $A_i z_i \in D_i^u$ for every i . Let

$$J = \{i : A_i^T u = 0\}.$$

For $i \notin J$ we have $A_i z_i = p_i(u)$. If J is empty, then $x = F(u)$ and there is nothing more to prove.

Otherwise $\sum_{i \in J} L_i \subseteq u^\perp$, so this sum has dimension at most $d - 1$. Condition (GP) consequently implies

$$\sum_{i \in J} m_i \leq d - 1, \quad \dim \left(\sum_{i \in J} L_i \right) = \sum_{i \in J} m_i. \quad (4)$$

The matrix obtained by concatenating A_i for $i \in J$ has full column rank. Its transpose is therefore surjective. We can choose $v \in \mathbb{R}^d$ satisfying

$$A_i^T v = z_i \quad \text{for every } i \in J. \quad (5)$$

For $\varepsilon > 0$ and $i \in J$ this gives

$$A_i^T(u + \varepsilon v) = \varepsilon z_i, \quad p_i(u + \varepsilon v) = A_i z_i.$$

For each $i \notin J$, the vector $A_i^T(u + \varepsilon v)$ remains nonzero for all sufficiently small ε , and $p_i(u + \varepsilon v) \rightarrow p_i(u) = A_i z_i$ as $\varepsilon \rightarrow 0^+$. There are finitely many indices, so for every sufficiently small positive ε the vector $u + \varepsilon v$ belongs to U and

$$F(u + \varepsilon v) \rightarrow x.$$

Lemma 5 now gives $X \subseteq \overline{\text{Exp}(\mathcal{D})}^{\text{Eucl}}$. A complex polynomial vanishing on $\text{Exp}(\mathcal{D})$ vanishes on its Euclidean closure and hence on X . Combining this with $\text{Exp}(\mathcal{D}) \subseteq X$ proves $\mathcal{S} = \mathcal{E}$. Theorem 1 completes the proof. $\square$

4 Why the hypotheses and the distinction matter

Remark 7 (Segment summands). The dimension assumption in Theorem 1 cannot be dropped. In $\mathbb{R}^2$, the sum of the unit disc and the segment $[-e_1, e_1]$ is a stadium. Its exposed points are the two outer open semicircles of the unit circles centered at e_1 and $-e_1$. Each semicircle is Zariski dense in its circle, and the circles are distinct. Thus the exposed-point variety is reducible. Here the exceptional normal space has codimension one, and its complement is disconnected.

Remark 8 (A nongeneric failure of $\mathcal{S} = \mathcal{E}$). Let $D_1 = D_2$ be the unit disc in the xy -plane of $\mathbb{R}^3$, and let D_3 be the unit disc in the xz -plane. The point $w = (0, 0, 1)$ is in X : use $(1, 0, 0) \in \partial D_1$, $(-1, 0, 0) \in \partial D_2$, and $(0, 0, 1) \in \partial D_3$. Its height is maximal, so it is on $\partial \mathcal{D}$.

On the other hand, every exposed point is a sum $(x, y, z) = (a, y, 0) + (b, 0, z)$ with $a^2 + y^2 = 4$, $b^2 + z^2 = 1$, and $a + b = x$. Indeed, the support points of the two identical discs coincide. Eliminating a, b gives

$$G(x, y, z) = (x^2 + y^2 - z^2 - 3)^2 - 4x^2(1 - z^2) = 0.$$

This polynomial vanishes on $\mathcal{E}$, whereas $G(w) = 16$. Thus $w \notin \mathcal{E}$ and $\mathcal{S} \neq \mathcal{E}$. Theorem 1 still applies, but (GP) fails for the first two summands.

The genericity assumption in Theorem 2 is used only to make the simultaneous prescriptions (5) possible. The proof also works under the weaker condition that every subcollection whose span is proper has linearly independent summand spans.

The results concern the exposed-point variety and the purely nonlinear part. They do not establish irreducibility of the entire complex critical locus, birationality of its addition map, a general degree formula, or a characterization of all zonoids.

The support-face description is due to Gesmundo and Meroni; the analytic identity principle and the linear-algebra facts used above are standard. The argument here combines them with the generic perturbation of exceptional normals. A literature search through 27 October 2026 found no prior proof of the two conjectures in the sources checked; this is not a claim of exhaustive priority verification. This manuscript was developed with AI assistance for research, drafting, and checking. The written proof has been self-audited; the accompanying exact algebraic fixtures are supplemental checks, not a formal verification or independent review. This is an unrefereed preprint.

References

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