

On Reznick's Question about the Cone of Forms with a Sum-of-Squares Odd Power: Explicit Non-Convexity for Sextics in Five and Quartics in Seven Variables

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Abstract

For an even degree m and an odd exponent $2k+1$, let $\Sigma_{n,m}(2k+1)$ be the set of real forms f of degree m in n variables such that f^{2k+1} is a sum of squares of forms. B. Reznick asked (Oberwolfach Report 14/2023, p. 779) whether this set is a closed convex cone. It is a closed cone; whether it is convex is still stated as open by Blekherman, Kozhasov and Reznick (Forum Math. Sigma 14 (2026), e65). A recent preprint of A. Kriebel (Zenodo, 6 October 2026) answers the question in the negative: it proves that $\Sigma_{n,6}(3)$ is not convex for $n = 3 \cdot 10^{62}$, and that for every odd exponent ≥ 3 the set of sextics is not convex in some dimension, by averaging many copies of one seed form written in separate variables. The first negative answer is therefore due to that preprint, and not to this note. The examples of the present note were found after that preprint had been read, and they use its principle with two copies instead of 10^{62} . We give explicit forms p, q such that p^3 and q^3 are sums of squares and $(p+q)^3$ is not: the sextics $p = M_c(x, y, z)$, $q = M_c(u, v, z)$ in five variables, with $M_c = x^4y^2 + x^2y^4 + z^6 - cx^2y^2z^2$ and $c = \frac{41}{16}$, and the quartics $p = Q_c(x, y, z, w)$, $q = Q_c(x', y', z', w)$ in seven variables, with $Q_c = w^4 + x^2y^2 + y^2z^2 + z^2x^2 - cxyzw$ and $c = \frac{13}{4}$. Hence $\Sigma_{n,m}(3)$ is not convex for $n \geq 5$ and even $m \geq 6$, and for $n \geq 7$ and even $m \geq 4$; and, by the many-copies argument of Kriebel's preprint, for every even $m \geq 4$ and every odd exponent ≥ 3 the set is not convex in all sufficiently large dimensions. The two main theorems are computer-assisted: the proofs rest on exact rational certificates (Gram matrices, and a functional that is non-negative on the relevant squares and negative on $(p+q)^3$), found by numerical semidefinite programming and verified in exact arithmetic by a short program. As a by-product, M_c^3 is a sum of squares for all $c \leq \frac{41}{16} = 2.5625$ (proved before: $c \leq (15/13)^{1/3} \approx 1.0489$; reported from experiments: up to $c \approx 2.56548$). The cases $n = 3, 4$ with even $m \geq 6$, the quartics with $n = 4, 5, 6$, and explicit small dimensions for exponents at least 5 remain open; numerical searches found no example there, which is a test and not a proof. This is an unrefereed note.

1 Introduction

1.1 The question

Let n and m be positive integers, m even. A real form (homogeneous polynomial) f of degree m in n variables is *non-negative*, or *psd*, if $f(a) \geq 0$ for all $a \in \mathbb{R}^n$; it is a *sum of squares*, or *sos*, if $f = h_1^2 + \dots + h_r^2$ with real forms h_k , necessarily of degree $m/2$. Let $P_{n,m}$ and $\Sigma_{n,m}$ be the sets of psd and of sos forms of degree m in n variables; both are closed convex cones (the notation is that of Choi and Lam [CL77]). By a theorem of Hilbert [Hil88], $P_{n,m} = \Sigma_{n,m}$ exactly when $n \leq 2$, or $m = 2$, or $(n, m) = (3, 4)$; see [Rez23, p. 778].

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In his Oberwolfach abstract [Rez23, p. 779], B. Reznick defines, for an integer $k \geq 0$,

$$\Sigma_{n,m}(2k+1) = \{f \in P_{n,m} : f^{2k+1} \in \Sigma_{n,(2k+1)m}\},$$

so that $\Sigma_{n,m}(1) = \Sigma_{n,m}$, and writes: “A natural question is whether $\Sigma_{n,m}(2k+1)$ is a closed convex cone.” He adds that closedness is easy and that convexity seems difficult, that he would guess that the answer is negative, and that a positive answer would mean that $(p+q)^{2k+1}$ is sos whenever p^{2k+1} and q^{2k+1} are. The question is record OWR-12697710-006 of the UnsolvedMath collection [ula26], where it is listed as open.

Three remarks fix its meaning. First, the condition $f \in P_{n,m}$ is automatic: if f^{2k+1} is sos, then $f(a)^{2k+1} \geq 0$ and hence $f(a) \geq 0$ for all $a \in \mathbb{R}^n$. Second, the set is a cone, because $(tf)^{2k+1} = t^{2k+1}f^{2k+1}$, and it is closed, being the preimage of the closed cone $\Sigma_{n,(2k+1)m}$ under the continuous map $f \mapsto f^{2k+1}$ [BKR26, Introduction]. A cone is convex if and only if it is closed under addition. So a negative answer for one triple (n, m, k) consists of two forms p, q such that p^{2k+1} and q^{2k+1} are sos and $(p+q)^{2k+1}$ is not; then $\frac{1}{2}(p+q) \notin \Sigma_{n,m}(2k+1)$. Third, in Hilbert’s cases $\Sigma_{n,m}(2k+1) = P_{n,m}$ is convex, so the question concerns $k \geq 1$ and $n \geq 3, m \geq 6$ or $n \geq 4, m \geq 4$.

What was known. Stengle [Ste79] proved that no odd power of a certain psd ternary sextic is sos; for the Motzkin form this was stated in [CDLR82] and is proved in [Rez23, p. 779] and in [BKR26]. In the other direction, all sufficiently large powers of a positive definite form are sos, by a theorem of Scheiderer [Sch12] which generalises a theorem of Reznick [Rez95]. We take this history from [Rez23, pp. 778–779] and [BKR26, Introduction].

Reznick’s abstract states, as a theorem with a hint of the proof, that $(p+q)^{2k+1}$ is sos if p^{2k+1} and q are sos [Rez23, p. 780]; see Lemma 2.4 for the exponent 3. Blekherman, Kozhasov and Reznick [BKR26] prove this (Theorem 5.1; Theorem 42 in the arXiv version) and prove that $p+q \in \Sigma_{n,m}(\ell+\ell'-1)$ if $p \in \Sigma_{n,m}(\ell)$ and $q \in \Sigma_{n,m}(\ell')$ with ℓ, ℓ' odd (Theorem 5.3; Theorem 44 in the arXiv version); Reznick’s abstract has this for $\ell = \ell' = 3$, without proof, and as a conjecture in general [Rez23, p. 780]. Since $\Sigma_{n,m}(2k+1) \subseteq \Sigma_{n,m}(2k+3)$, the union of these sets over all k is therefore a convex cone; it is not closed when $P_{n,m} \neq \Sigma_{n,m}$ [BKR26, Introduction]. Baldi et al. [BBK⁺26, Theorem 2.2] extend Theorem 5.3 to totally real projective varieties. For a fixed exponent, [BKR26] says in its introduction that it is unclear whether $\Sigma_{n,m}(2k+1)$ is convex for $k > 0$, and in its Section 6 that the authors do not know this and expect that the answer does not depend on the parameters. Both passages are in the arXiv version of 31 July 2024 (pp. 4 and 22) and, unchanged, in the journal version published on 27 April 2026 (pp. 4 and 23).

1.2 The prior answer

The question has a negative answer in the preprint [Kri26] of A. Kriebel (Zenodo, version 1.0, dated 6 October 2026). Its Theorem 1 states the following. Let $N = 10^{62}$ and let $p_i = x_i^4 y_i^2 + x_i^2 y_i^4 + z_i^6 - x_i^2 y_i^2 z_i^2$ ($1 \leq i \leq N$) be copies of one form in N disjoint triples of variables. Then every p_i lies in $\Sigma_{3N,6}(3)$ and the average $N^{-1} \sum_i p_i$ does not, so $\Sigma_{n,6}(3)$ is not convex for $n = 3 \cdot 10^{62}$; moreover, for every odd $\ell \geq 3$ there is a finite $n(\ell)$ such that $\Sigma_{n(\ell),6}(\ell)$ is not convex. The proof separates a form which is not sos from the squares by a linear functional (Lemma 2 there), and shows that the product of N copies of the functional, one for each block of variables, is non-negative on squares and takes on the ℓ -th power of the sum of the N copies of the form a value that is a polynomial in N with negative leading coefficient (Lemma 1 there). Both lemmas are stated for forms of any even degree, and the preprint notes that the conclusion holds for every non-negative form which is not sos and has an sos ℓ -th power. For $\ell = 3$ the preprint gives an explicit rational functional; the cube of its seed, which it attributes to Berg, Christensen and Jensen [BCJ79], is sos by an identity of [BKR26, Section 6]. The preprint says that it draws no conclusion for three variables or for the smallest dimension. It remarks

that the examples persist when variables are added, and that a pair of forms violating convexity exists in the same dimension (a partial sum and the next summand), without making the pair explicit.

The first negative answer to Reznick's question is therefore due to Kriebel's preprint, and not to the present note. The examples below were found after that preprint had been read, and they use its principle: copies of one seed form written in separate variables. What changes is the number of copies. Two copies, which share one variable, suffice instead of 10^{62} , if the seed is taken close to the boundary of the cone and if the functional is not required to be a product.

1.3 Results

For a real number c let

$$M_c(x, y, z) = x^4y^2 + x^2y^4 + z^6 - cx^2y^2z^2, \quad Q_c(x, y, z, w) = w^4 + x^2y^2 + y^2z^2 + z^2x^2 - cxyzw.$$

M_3 is the Motzkin form and Q_4 is the quartic of Choi and Lam; the family M_c is considered in [Rez23, p. 779] and [BKR26, Section 6].

Theorem 1.1 (five variables, sextics). *Let $c = \frac{41}{16}$ and $p = M_c(x, y, z)$, $q = M_c(u, v, z)$, two sextic forms in the five variables (x, y, u, v, z) . Then p^3 and q^3 are sums of squares of forms of degree 9, and $(p + q)^3$ is not a sum of squares of real polynomials. Hence $p, q \in \Sigma_{5,6}(3)$ and $\frac{1}{2}(p + q) \notin \Sigma_{5,6}(3)$.*

Theorem 1.2 (seven variables, quartics). *Let $c = \frac{13}{4}$ and $p = Q_c(x, y, z, w)$, $q = Q_c(x', y', z', w)$, two quartic forms in the seven variables (x, y, z, x', y', z', w) . Then p^3 and q^3 are sums of squares of forms of degree 6, and $(p + q)^3$ is not a sum of squares of real polynomials. Hence $p, q \in \Sigma_{7,4}(3)$ and $\frac{1}{2}(p + q) \notin \Sigma_{7,4}(3)$.*

The proofs of Theorems 1.1 and 1.2 are computer-assisted. For each theorem a data file contains positive definite rational Gram matrices of the cube of the seed form, and a table of rational numbers which defines a linear functional Λ whose moment matrix on the monomials of half the Newton polytope of $(p + q)^3$ is positive definite, while $\Lambda((p + q)^3) < 0$. That these finitely many rational identities and inequalities hold is verified in exact arithmetic by a short Python program, in less than a second; that they imply the theorems is proved in Sections 2 and 3. Numerical semidefinite programming was used only to find the certificates.

Corollary 1.3. $\Sigma_{n,m}(3)$ is not convex if $n \geq 5$ and $m \geq 6$ is even, and if $n \geq 7$ and $m \geq 4$ is even.

Theorem 1.4 (every odd exponent, large dimension; after [Kri26]). *For every even $m \geq 4$ and every odd $\ell \geq 3$ there is a number $n_0(m, \ell)$ such that $\Sigma_{n,m}(\ell)$ is not convex for all $n \geq n_0(m, \ell)$.*

Theorem 1.4 is proved by the argument of [Kri26, Lemmas 1 and 2], written out in Section 4. For $m = 6$ it is the second assertion of [Kri26, Theorem 1] with the remark on additional variables made there, and even $m \geq 8$ follows by Lemma 2.3(b). What is added is $m = 4$: the argument needs a form that is not sos and has an sos cube, which Theorem 1.2 supplies for quartics. The number $n_0(m, \ell)$ is not explicit.

Proposition 1.5 (the two families). (a) M_c^3 is a sum of squares for every $c \leq \frac{41}{16}$, and Q_c^3 is a sum of squares for every c with $|c| \leq \frac{13}{4}$.

(b) For real t let p_t, q_t be the forms of Theorem 1.1 (case I), respectively of Theorem 1.2 (case II), with t in place of c , and let Λ be the functional of that theorem. Then $\varphi(t) = \Lambda((p_t + q_t)^3)$ is the cubic polynomial

$$\begin{aligned} 10^{13}\varphi_{\text{I}}(t) &= -713422739406t^3 + 6692802744270t^2 - 21578549424822t + 23337150265011, \\ 10^{13}\varphi_{\text{II}}(t) &= -570065651754t^3 + 6422042875791t^2 - 28987219462803t + 45237034859253 \end{aligned}$$

in case I and in case II. Each is strictly decreasing on $\mathbb{R}$ and has exactly one real root: $t_I \in (2.5516680, 2.5516681)$ and $t_{II} \in (3.1196392, 3.1196393)$. For every t with $t_I < t \leq \frac{41}{16}$, respectively $t_{II} < t \leq \frac{13}{4}$, the forms p_t^3 and q_t^3 are sums of squares and $(p_t + q_t)^3$ is not.

Part (a) says $c_1 \geq \frac{41}{16} = 2.5625$ in the notation of [BKR26, Section 6], where $(-\infty, c_1]$ is the set of all c such that M_c^3 is sos (Theorem 6.3 there; Theorem 47 in the arXiv version). That paper proves $(15/13)^{1/3} \approx 1.04886 \leq c_1$ by an explicit identity and reports the experimental value $c_1 \approx 2.56548$, attributed to a private communication of P. Parrilo (p. 23 of the arXiv version, p. 24 of the journal version). The Oberwolfach abstract has $c^3 \leq \frac{15}{13}$ and mentions, without proof, an improvement to about 1.1336 [Rez23, p. 779]. Part (a) does not prove the experimental value, with which it is consistent.

Remark 1.6. In Theorems 1.1 and 1.2 neither p nor q is sos; this is forced, since otherwise $(p + q)^3$ would be sos by Lemma 2.4. The form $(p + q)^5$ is a sum of squares, by [BKR26, Theorem 5.3] or directly: $(p + q)^5 = p^3[(p + \frac{5}{2}q)^2 + \frac{15}{4}q^2] + q^3[(q + \frac{5}{2}p)^2 + \frac{15}{4}p^2]$. So the examples say nothing against the convexity of the union over k . Moreover, $P = p + q$ is an explicit psd form such that P and P^3 are not sos and P^5 is sos. Reznick asks whether such forms exist and guesses that they do, without an example [Rez23, p. 780]. By [BKR26, Theorem 5.3], the sum of every pair of forms which violates the convexity of $\Sigma_{n,m}(3)$ is of this kind; so the existence of such forms follows already from [Kri26], where no form is made explicit. We found no explicit example in the sources that we read, and make no claim that none was known.

1.4 What is new, what is not, and what stays open

The negative answer is not new, and neither is the principle of copies in separate variables with a functional that is negative on a power of their sum: both are in [Kri26], and Theorem 1.4 is that principle with a quartic seed. Added here, as far as we know (see “Scope and priority” in Section 6), are the explicit pairs in five and in seven variables with certificates that can be checked exactly; Corollary 1.3, where [Kri26] has $m = 6$ and $n \geq 3 \cdot 10^{62}$; the case $m = 4$ of Theorem 1.4; and Proposition 1.5.

The set $\Sigma_{n,m}(3)$ is convex in Hilbert’s cases, and it is not convex for $n \geq 5$, $m \geq 6$ and for $n \geq 7$, $m = 4$; the cases $n = 3, 4$ with $m \geq 6$ and $n = 4, 5, 6$ with $m = 4$ are open. For an odd exponent $\ell \geq 5$ it is not convex for $n \geq n_0(m, \ell)$, and no explicit dimension is known. Numerical searches in the open cases with exponent 3 found no example (Section 6.1). This is a test, not a proof, and its reach is small: in three and four variables the quantities to be decided are far below the accuracy of double precision arithmetic.

2 Tools

We write $X^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ for $\gamma \in \mathbb{N}^n$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, and $\text{supp}(P)$ for the set of all γ such that the monomial X^γ occurs in the polynomial P with a non-zero coefficient. The Newton polytope $\text{New}(P)$ is the convex hull of $\text{supp}(P)$. “Sum of squares” always means a sum of squares of polynomials with real coefficients.

Lemma 2.1 (Newton polytope lemma). *Let $P = h_1^2 + \cdots + h_r^2$ with real polynomials h_k in n variables, let $\omega \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}$. If $\omega \cdot \gamma \geq \kappa$ for all $\gamma \in \text{supp}(P)$, then $\omega \cdot \alpha \geq \kappa/2$ for all $\alpha \in \text{supp}(h_k)$, $k = 1, \dots, r$.*

Proof. We may assume that not all h_k are zero. Let μ be the least value of $\omega \cdot \alpha$ over all α in $\text{supp}(h_1) \cup \cdots \cup \text{supp}(h_r)$, and write $h_k = h_k^* + h_k^+$, where h_k^* collects the terms of h_k with $\omega \cdot \alpha = \mu$.

In $h_k^2 = (h_k^*)^2 + 2h_k^*h_k^+ + (h_k^+)^2$ every monomial X^γ of the first summand has $\omega \cdot \gamma = 2\mu$, and every monomial of the other two has $\omega \cdot \gamma > 2\mu$. Hence the sum of the terms of P with $\omega \cdot \gamma = 2\mu$ is $S = \sum_k (h_k^*)^2$. Some h_k^* is a non-zero polynomial, so $h_k^*(a) \neq 0$ for some $a \in \mathbb{R}^n$, and then $S(a) > 0$. So $S \neq 0$: there is $\gamma \in \text{supp}(P)$ with $\omega \cdot \gamma = 2\mu$, and $2\mu \geq \kappa$. $\square$

Real coefficients are essential. With $\omega = \pm(1, \dots, 1)$ the lemma shows that if a form of degree $2D$ is a sum of squares of polynomials h_k , then every h_k is a form of degree D . Used for all ω , it says $\text{New}(h_k) \subseteq \frac{1}{2} \text{New}(P)$, which is the fact quoted in [Rez23, p. 778] from [Rez78].

Lemma 2.2 (duality). *Let $B \subset \mathbb{N}^n$ be finite and let Λ be a real-valued function on $B + B$, extended to a linear functional on the span of the monomials X^γ , $\gamma \in B + B$. Suppose that the symmetric matrix $M_\Lambda = (\Lambda(X^{\alpha+\beta}))_{\alpha, \beta \in B}$ is positive semidefinite. If $P = h_1^2 + \dots + h_r^2$ with $\text{supp}(h_k) \subseteq B$ for all k , then $\Lambda(P) \geq 0$.*

Proof. Write $h_k = \sum_{\alpha \in B} c_{k,\alpha} X^\alpha$. Then $P = \sum_k \sum_{\alpha, \beta} c_{k,\alpha} c_{k,\beta} X^{\alpha+\beta}$, and therefore $\Lambda(P) = \sum_k c_k^\top M_\Lambda c_k \geq 0$. $\square$

Any table of real numbers indexed by $B + B$ defines a functional Λ ; if its matrix is positive semidefinite and $\Lambda(P) < 0$, then P is not a sum of squares of polynomials supported on B .

Lemma 2.3. (a) *Let $n \leq n'$ and $P \in \mathbb{R}[x_1, \dots, x_n]$. If P is a sum of squares in $\mathbb{R}[x_1, \dots, x_{n'}]$, then it is a sum of squares in $\mathbb{R}[x_1, \dots, x_n]$.*

(b) *If $x_1^2 P$ is a sum of squares, then P is a sum of squares.*

Proof. (a) Put $x_{n+1} = \dots = x_{n'} = 0$ in a representation of P . (b) Let $x_1^2 P = \sum_k h_k^2$. For real a with $a_1 = 0$ we get $\sum_k h_k(a)^2 = 0$. So the polynomial $h_k(0, x_2, \dots, x_n)$ vanishes on $\mathbb{R}^{n-1}$ and is zero, that is, x_1 divides h_k , and $P = \sum_k (h_k/x_1)^2$. $\square$

Lemma 2.4 (Reznick [Rez23, p. 780]; [BKR26, Theorem 5.1]). *If P_1^3 and P_2 are sums of squares, then so is $(P_1 + P_2)^3$.*

Proof. $(P_1 + P_2)^3 = P_1^3 + P_2[\frac{3}{4}(2P_1 + P_2)^2 + \frac{1}{4}P_2^2]$, and a product of sums of squares is a sum of squares. $\square$

3 The two examples

3.1 The data

We refer to Theorem 1.1 as case I and to Theorem 1.2 as case II. The seed form is $f = M_{41/16}$ in (x, y, z) , respectively $f = Q_{13/4}$ in (x, y, z, w) ; its degree is $m = 6$, respectively $m = 4$, and p and q are f written in two sets of variables which share the last variable. Exponent vectors are written (a, b, c, d, e) for $x^a y^b u^c v^d z^e$ in case I and (a, b, c, a', b', c', d) for $x^a y^b z^c x'^{a'} y'^{b'} z'^{c'} w^d$ in case II. Let

$$B_I = \{(a, b, c, d, e) \in \mathbb{N}^5 : a + b + c + d + e = 9, 2a \geq b, 2b \geq a, 2c \geq d, 2d \geq c\},$$

$$B_{II} = \{(a, b, c, a', b', c', d) \in \mathbb{N}^7 : a + b + c + a' + b' + c' + d = 6, (a, b, c) \text{ and } (a', b', c') \text{ triangular}\},$$

where a triple is called *triangular* if none of its entries exceeds the sum of the other two. The inequalities are $\omega \cdot \alpha \geq 0$ for the weight vectors $\omega = (2, -1, 0, 0, 0), (-1, 2, 0, 0, 0), (0, 0, 2, -1, 0), (0, 0, -1, 2, 0)$ in case I, and $\omega = (1, 1, -1, 0, 0, 0, 0), (1, -1, 1, 0, 0, 0, 0), (-1, 1, 1, 0, 0, 0, 0)$ and the same three vectors for the primed coordinates in case II. Let $A_I \subset \mathbb{N}^3$ and $A_{II} \subset \mathbb{N}^4$ be the corresponding sets for the seed alone: $A_I = \{(a, b, e) : a + b + e = 9, 2a \geq b, 2b \geq a\}$ and $A_{II} = \{(a, b, c, d) : a + b + c + d = 6, (a, b, c) \text{ triangular}\}$. A direct count, which the program repeats, gives $|A_I| = 19$, $|B_I| = 91$, $|A_{II}| = 24$ and $|B_{II}| = 99$.

Lemma 3.1 (half the Newton polytope). *In both cases $\text{New}(p+q)$ is a simplex Δ , and B is the set of lattice points of $\frac{3}{2}\Delta = \frac{1}{2}\text{New}((p+q)^3)$. Every $\gamma \in \text{supp}((p+q)^3)$ satisfies $|\gamma| = 3m$ and $\omega \cdot \gamma \geq 0$ for the weight vectors ω listed above. Consequently, if $(p+q)^3 = h_1^2 + \dots + h_r^2$ with real polynomials h_k , then $\text{supp}(h_k) \subseteq B$ for all k .*

Proof. Case I. The monomials of $p+q$ are x^4y^2 , x^2y^4 , u^4v^2 , u^2v^4 , z^6 and $x^2y^2z^2$, $u^2v^2z^2$. The exponent vectors $v_1, \dots, v_5$ of the first five are linearly independent, and those of the last two are $\frac{1}{3}(v_1 + v_2 + v_5)$ and $\frac{1}{3}(v_3 + v_4 + v_5)$. So $\text{New}(p+q)$ is the simplex Δ with the vertices $v_1, \dots, v_5$. A vector $\alpha = (a, b, c, d, e) \in \mathbb{R}^5$ equals $\frac{3}{2} \sum_i t_i v_i$ if and only if

$$t_1 = \frac{2a-b}{9}, \quad t_2 = \frac{2b-a}{9}, \quad t_3 = \frac{2c-d}{9}, \quad t_4 = \frac{2d-c}{9}, \quad t_5 = \frac{e}{9},$$

and then $\sum_i t_i = \frac{1}{9}(a+b+c+d+e)$. Hence $\alpha \in \frac{3}{2}\Delta$ if and only if $|\alpha| = 9$, $e \geq 0$ and $\omega \cdot \alpha \geq 0$ for the four weight vectors; the lattice points with this property are the elements of B_I (the inequalities imply $a, b, c, d \geq 0$).

Case II. The monomials of $p+q$ are w^4 , the six monomials x^2y^2 , y^2z^2 , z^2x^2 , $x'^2y'^2$, $y'^2z'^2$, $z'^2x'^2$, and $xyzw$, $x'y'z'w$. The exponent vectors of the first seven are linearly independent, and those of $xyzw$ and $x'y'z'w$ are the mean of the vectors of w^4 , x^2y^2 , y^2z^2 , z^2x^2 , respectively of w^4 , $x'^2y'^2$, $y'^2z'^2$, $z'^2x'^2$. So $\text{New}(p+q)$ is the simplex Δ on these seven vertices. For $(a, b, c) = t_1(3, 3, 0) + t_2(0, 3, 3) + t_3(3, 0, 3)$ one finds $t_1 = \frac{a+b-c}{6}$, $t_2 = \frac{b+c-a}{6}$, $t_3 = \frac{a+c-b}{6}$; in the same way for the primed coordinates, and the coefficient of the vertex of w^4 is $\frac{d}{6}$. The sum of the seven coefficients is $\frac{1}{6}|\alpha|$. Hence $\alpha \in \frac{3}{2}\Delta$ if and only if $|\alpha| = 6$, $d \geq 0$ and both triples are triangular, and the lattice points with this property are the elements of B_{II} .

Both cases. Every monomial of $(p+q)^3$ is a product of three monomials of $p+q$, so $\text{supp}((p+q)^3) \subseteq 3\Delta$; and $\text{New}((p+q)^3) = 3\Delta$, because X^{3v} occurs in $(p+q)^3$ for every vertex v of Δ . Each weight vector ω satisfies $\omega \cdot v \geq 0$ at the vertices of Δ , hence on 3Δ . Now let $(p+q)^3 = \sum_k h_k^2$. By Lemma 2.1 with $\omega = \pm(1, \dots, 1)$, every h_k is a form of degree $\frac{3}{2}m$, and by Lemma 2.1 with the weight vectors listed above and $\kappa = 0$, every $\alpha \in \text{supp}(h_k)$ satisfies the inequalities that define B . $\square$

The parity vector of $\gamma \in \mathbb{N}^n$ is its image $\bar{\gamma}$ in $\mathbb{F}_2^n$. Let

$$V_I = \{0\} \subset \mathbb{F}_2^5, \quad V_{II} = \{0, (1, 1, 1, 0, 0, 1), (0, 0, 0, 1, 1, 1), (1, 1, 1, 1, 1, 0)\} \subset \mathbb{F}_2^7,$$

and $T = \{\gamma \in B + B : \bar{\gamma} \in V\}$. In case I, $T = \{2\alpha : \alpha \in B_I\}$ has 91 elements (if $\alpha + \beta$ is even, then $\frac{1}{2}(\alpha + \beta)$ is a lattice point of the convex set $\frac{3}{2}\Delta$); in case II, T has 168 elements. The space V is spanned by the parity vectors of the monomials of $p+q$, and therefore contains those of the monomials of $(p+q)^3$.

3.2 The certificates and what the exact program checks

For each case the package (Section 6) contains a file with the following rational numbers, written as quotients of integers; the files are `certificate_n5_m6.json` and `certificate_n7_m4.json`.

Gram blocks. A partition of A into four blocks $A_1, \dots, A_4$ and four symmetric matrices $G_1, \dots, G_4$, where G_i is indexed by A_i . In case I the blocks are the four classes of A_I modulo 2, of sizes 5, 5, 5, 4; in case II they are the four classes of A_{II} modulo 2 and modulo the vector $(1, 1, 1, 1)$, each of size 6. The entries have denominators dividing $63 \cdot 10^{13}$ in case I and $126 \cdot 10^{13}$ in case II.

The table. A number λ_γ for every $\gamma \in T$, with denominator dividing 10^{13} : 91 numbers in case I and 168 in case II. It defines the functional Λ on the span of the monomials X^γ , $\gamma \in B + B$, by

$$\Lambda(X^\gamma) = \lambda_\gamma \text{ if } \gamma \in T, \quad \Lambda(X^\gamma) = 0 \text{ if } \gamma \in (B + B) \setminus T.$$

By this definition the entry $\Lambda(X^{\alpha+\beta})$ of M_Λ vanishes unless $\bar{\alpha} - \bar{\beta} \in V$, so that M_Λ is, after a permutation, block diagonal with one block for each class of B modulo 2 and V . There are 16 classes in both cases: of sizes 3, 3, 3, 3, 3, 4, 4, 4, 4, 7, 7, 9, 9, 9, 10 in case I, and nine of size 3, six of size 10 and one of size 12 in case II.

Proposition 3.2 (exact verification). *For the data in the two files the following holds in each case.*

- (A) $f^3 = \sum_{i=1}^4 \sum_{\alpha, \beta \in A_i} (G_i)_{\alpha\beta} X^{\alpha+\beta}$ in $\mathbb{Q}[X]$, and every G_i is positive definite.
- (B) The matrix $M_\Lambda = (\Lambda(X^{\alpha+\beta}))_{\alpha, \beta \in B}$, of size 91 in case I and 99 in case II, is positive definite.
- (C) The value of Λ on $(p+q)^3$ is

$$\begin{aligned} \Lambda((p+q)^3) &= -\frac{29836645080831}{20480000000000000} \approx -1.4569 \cdot 10^{-3} && \text{in case I,} \\ \Lambda((p+q)^3) &= -\frac{22652335069827}{320000000000000} \approx -7.0789 \cdot 10^{-2} && \text{in case II.} \end{aligned}$$

Proof. This is a finite computation with rational numbers. It is carried out by the program `verify_note.py` of the package, which uses only integers and exact fractions of the Python standard library; its recorded output is in the package. In the program, p, q , the vertices of Δ , the weight vectors and generators of V are typed in; from a file it reads only the matrices G_i with their index sets and the table. It checks the following.

(A) The index sets A_i are disjoint and their union is A ; every G_i is symmetric; the identity in (A) holds, as an equality of two polynomials with rational coefficients; every G_i is positive definite. Positive definiteness is tested twice: all pivots of the elimination $G = LDL^\top$ without pivoting are positive, and all leading principal minors, computed by fraction-free elimination on an integer multiple of the matrix, are positive. The smallest pivot is about $2.20 \cdot 10^{-4}$ in case I and $8.35 \cdot 10^{-3}$ in case II.

(B1) The statements of Lemma 3.1, which are proved above, are tested again: the vertices are monomials of $p+q$ and affinely independent; all monomials of $p+q$ have non-negative barycentric coordinates; B , defined by the inequalities, equals the set of lattice points of $\frac{3}{2}\Delta$, computed from barycentric coordinates by exact linear algebra; every monomial of $(p+q)^3$ has degree $3m$ and non-negative weights; and every product of three monomials of $p+q$ is X^γ with $\gamma \in B+B$, hence with $\gamma \in T$, so that Λ is defined on $(p+q)^3$ and on the polynomials of Proposition 1.5(b).

(B2) The parity vectors of the monomials of $(p+q)^3$ span V ; the table has exactly one entry for every $\gamma \in T$ and no other entry; the full matrix M_Λ , built from the table without any use of the block structure, is positive definite (all 91, respectively 99, pivots of the exact elimination are positive); and every block is positive definite by both tests. In the lexicographic order of B the smallest pivot is about $3.75 \cdot 10^{-7}$ in case I and $1.25 \cdot 10^{-5}$ in case II.

(C) The value $\Lambda((p+q)^3)$ is computed from the expanded polynomial and compared with the fraction in (C). $\square$

A symmetric matrix is positive definite if and only if elimination without pivoting meets only positive pivots; the k -th leading principal minor is the product of the first k pivots. A positive definite rational matrix $G = LDL^\top$, with L unit lower triangular and D diagonal, gives $\sum_{\alpha, \beta} G_{\alpha\beta} X^{\alpha+\beta} = \sum_j D_{jj} (\sum_\alpha L_{\alpha j} X^\alpha)^2$ with $D_{jj} > 0$.

Proof of Theorems 1.1 and 1.2. By Proposition 3.2(A) and the last remark, f^3 is a sum of 19, respectively 24, squares of forms of degree $\frac{3}{2}m$ with rational coefficients, multiplied by positive rational numbers. Renaming the variables gives the same for p^3 and q^3 . Suppose that $(p+q)^3 = h_1^2 + \dots + h_r^2$ with real polynomials h_k . By Lemma 3.1, $\text{supp}(h_k) \subseteq B$ for all k . By Proposition 3.2(B) and

Lemma 2.2, $\Lambda((p+q)^3) \geq 0$. This contradicts Proposition 3.2(C). Finally $(\frac{1}{2}(p+q))^3 = \frac{1}{8}(p+q)^3$ is not sos, so $\frac{1}{2}(p+q) \notin \Sigma_{n,m}(3)$. $\square$

Remark 3.3. The space V is not needed for the proof; any table with a positive semidefinite matrix and a negative value would do. It reflects a symmetry: averaging a functional over the group of those changes of sign of the variables which fix p and q does not change its value on $(p+q)^3$ and makes it vanish on the monomials with parity outside V .

4 More variables, higher degree, other exponents

Proof of Corollary 1.3. Let (p, q) be the pair of Theorem 1.1, with $(n_0, m_0) = (5, 6)$ and the variable $\xi = z$, or the pair of Theorem 1.2, with $(n_0, m_0) = (7, 4)$ and $\xi = w$. Let $n \geq n_0$ and $m = m_0 + 2j$ with $j \geq 0$, and regard $p' = \xi^{2j}p$ and $q' = \xi^{2j}q$ as forms of degree m in n variables. Then $p'^3 = (\xi^{3j})^2 p^3$ and q'^3 are sums of squares. If $(p' + q')^3 = \xi^{6j}(p+q)^3$ were a sum of squares of polynomials in n variables, it would be one in the n_0 variables by Lemma 2.3(a), and then $(p+q)^3$ would be a sum of squares by Lemma 2.3(b), applied $3j$ times. This contradicts Theorems 1.1 and 1.2. So $p', q' \in \Sigma_{n,m}(3)$ and $\frac{1}{2}(p' + q') \notin \Sigma_{n,m}(3)$. $\square$

The next proposition is [Kri26, Lemmas 1 and 2] in one statement. The preprint cites the literature for its ingredients; we did not read those sources and give complete proofs.

Proposition 4.1 (many copies). *Let f be a real form of even degree m_0 in s variables which is not a sum of squares, and let $\ell \geq 3$ be odd. For $N \geq 1$ let $F_N = f(X^{(1)}) + \dots + f(X^{(N)})$, where $X^{(1)}, \dots, X^{(N)}$ are N disjoint sets of s variables. Then there is N_0 such that F_N^ℓ is not a sum of squares for all $N \geq N_0$.*

Proof. Put $d = \ell m_0/2$, and let $\mathbb{R}[X]_{\leq e}$ denote the space of real polynomials of degree at most e in the s variables X .

Step 1: the cone C of sums of squares of polynomials of degree at most d is closed in $\mathbb{R}[X]_{\leq 2d}$. Let v be the vector of all monomials of degree at most d . Then $C = \{v^\top Q v : Q \text{ positive semidefinite}\}$. Let γ be the integral against the standard Gaussian density on $\mathbb{R}^s$. The matrix $\Gamma = \gamma(vv^\top)$ is positive definite, since $c^\top \Gamma c = \gamma((c \cdot v)^2) > 0$ for $c \neq 0$, and $\gamma(v^\top Q v) = \text{tr}(Q\Gamma) \geq \lambda_{\min}(\Gamma) \text{tr}(Q)$ for positive semidefinite Q . If $v^\top Q_j v$ converges to P , then $\text{tr}(Q_j)$ is bounded, a subsequence of the Q_j converges to a positive semidefinite Q , and $P = v^\top Q v \in C$.

Step 2: a functional. The form f lies in $\mathbb{R}[X]_{\leq 2d}$ and not in C . Since C is a closed convex cone in a space of finite dimension, there is a linear functional λ_0 on $\mathbb{R}[X]_{\leq 2d}$ with $\lambda_0 \geq 0$ on C and $\lambda_0(f) < 0$. Let $\lambda = \lambda_0 + \varepsilon \gamma$ with $\varepsilon > 0$ so small that $\lambda(f) < 0$. Then $\lambda(h^2) \geq 0$ for all h of degree at most d , and $\lambda(1) \geq \varepsilon > 0$. After scaling, $\lambda(1) = 1$. Put $a_j = \lambda(f^j)$ for $0 \leq j \leq \ell$; then $a_0 = 1$ and $a_1 < 0$. The matrix $M_\lambda = (\lambda(X^{\alpha+\beta}))_{|\alpha|, |\beta| \leq d}$ is positive semidefinite.

Step 3: the product. On the polynomials of total degree at most $2d$ in the sN variables define Λ_N on monomials by $\Lambda_N(\mu_1(X^{(1)}) \dots \mu_N(X^{(N)})) = \lambda(\mu_1) \dots \lambda(\mu_N)$. Then $\Lambda_N(\prod_i H_i(X^{(i)})) = \prod_i \lambda(H_i)$ whenever $\sum_i \deg H_i \leq 2d$. For a polynomial H of total degree at most d , $\Lambda_N(H^2)$ is a quadratic form in the coefficients of H , and its matrix has the entries $\prod_i M_\lambda[\alpha_i, \beta_i]$: it is a principal submatrix of the N -fold Kronecker power of M_λ , hence positive semidefinite. So $\Lambda_N(H^2) \geq 0$.

Step 4: the count. Expand F_N^ℓ into the N^ℓ products $f(X^{(i_1)}) \dots f(X^{(i_\ell)})$ and group the ℓ -tuples $(i_1, \dots, i_\ell)$ by the partition π of $\{1, \dots, \ell\}$ into the classes of equal indices. A given π with $|\pi|$ blocks arises from $N(N-1) \dots (N-|\pi|+1)$ tuples, and each of them contributes $\prod_{\beta \in \pi} a_{|\beta|}$. Hence

$$\Lambda_N(F_N^\ell) = \sum_{\pi} N(N-1) \dots (N-|\pi|+1) \prod_{\beta \in \pi} a_{|\beta|},$$

a polynomial in N of degree ℓ with leading coefficient $a_1^\ell < 0$, because only the partition into singletons has ℓ blocks and ℓ is odd. So $\Lambda_N(F_N^\ell) < 0$ for all $N \geq N_0$.

Step 5. If F_N^ℓ , a form of degree $2d$, were a sum of squares of polynomials, these would have degree d by Lemma 2.1, and Step 3 would give $\Lambda_N(F_N^\ell) \geq 0$. $\square$

Proof of Theorem 1.4. Let $f = Q_{13/4}$, a form of degree $m_0 = 4$ in $s = 4$ variables. It is not sos: by Lemma 2.1 the squares could only contain the monomials w^2, xy, yz, zx , and $xyzw$ is not a product of two of them. Its cube is sos by Proposition 3.2(A), hence so is $f^\ell = f^3(f^{(\ell-3)/2})^2$ for every odd $\ell \geq 3$. Fix ℓ , let N_0 be as in Proposition 4.1, and let $n \geq 4N_0$, $m = 4 + 2j$ with $j \geq 0$. Let ξ be one of the variables. The forms $\xi^{2j}f(X^{(i)})$, $1 \leq i \leq N_0$, are forms of degree m in n variables whose ℓ -th powers are sos, so they lie in $\Sigma_{n,m}(\ell)$. Their sum is $\xi^{2j}F_{N_0}$, and its ℓ -th power $\xi^{2j\ell}F_{N_0}^\ell$ is not sos, by Lemma 2.3 and Proposition 4.1. A convex cone contains the sum of finitely many of its elements. So $\Sigma_{n,m}(\ell)$ is not convex, and $n_0(m, \ell) = 4N_0$ will do. $\square$

With the seed M_1 and the identity of [BKR26, Section 6] for its cube, this is the proof of [Kri26] for sextics.

5 The two families

Proof of Proposition 1.5. (a) Let $c \leq \frac{41}{16}$ and $\delta = \frac{41}{16} - c \geq 0$. Then $M_c = P_1 + P_2$ with $P_1 = M_{41/16}$ and $P_2 = \delta(xyz)^2$. Here P_1^3 is sos by Proposition 3.2(A) and P_2 is sos, so M_c^3 is sos by Lemma 2.4. This is the monotonicity argument of [BKR26, Theorem 6.3]. For the quartics let $0 < c \leq \frac{13}{4}$ and $\varepsilon = (13/(4c))^4 - 1 \geq 0$. By Proposition 3.2(A) and Lemma 2.4, the cube of $Q_{13/4} + \varepsilon w^4$ is sos. The substitution $w \mapsto (1 + \varepsilon)^{-1/4}w = \frac{4c}{13}w$ turns $Q_{13/4} + \varepsilon w^4$ into Q_c , and a linear substitution maps sums of squares to sums of squares. So Q_c^3 is sos. The form Q_0 is itself sos, and $Q_{-c}(x, y, z, w) = Q_c(-x, y, z, w)$.

(b) We have $p_t + q_t = S - tR$, where S is the sum of the terms of $p + q$ with positive coefficients and $R = (x^2y^2 + u^2v^2)z^2$ in case I, $R = (xyz + x'y'z')w$ in case II. The monomials of S^3, S^2R, SR^2 and R^3 have their exponents in T (proof of Proposition 3.2, (B1)), so that Λ is defined on these polynomials. Hence

$$\varphi(t) = \Lambda(S^3) - 3t\Lambda(S^2R) + 3t^2\Lambda(SR^2) - t^3\Lambda(R^3).$$

The four values of Λ are computed exactly by `verify_note.py` from the tables of Section 3.2, with the result stated in the proposition; the program also compares the cubic with a direct evaluation of $\Lambda((p_t + q_t)^3)$ at five rational values of t . Write $\varphi = a_3t^3 + a_2t^2 + a_1t + a_0$. In both cases $a_3 < 0$ and $a_2^2 - 3a_1a_3 < 0$ (the values are about $-1.39 \cdot 10^{-2}$ and $-8.33 \cdot 10^{-2}$), so the quadratic polynomial φ' is negative on $\mathbb{R}$, and φ is strictly decreasing with exactly one real root. The root is enclosed by the signs of φ at the two rational endpoints of the stated intervals, which the program evaluates exactly.

Now let t be larger than this root, so that $\varphi(t) < 0$. The monomials of $p_t + q_t$ are among those of $p + q$, so the conclusion of Lemma 3.1 holds for $(p_t + q_t)^3$ with the same set B : a representation as a sum of squares would be supported on B . By Proposition 3.2(B) and Lemma 2.2 it would give $\varphi(t) \geq 0$. So $(p_t + q_t)^3$ is not sos. If moreover $t \leq \frac{41}{16}$, respectively $t \leq \frac{13}{4}$, then p_t^3 and q_t^3 are sos by (a). $\square$

So the examples are not isolated: the same two functionals work on intervals of parameters of length about 0.011 and 0.130. By the argument for (a), the set of all c such that Q_c^3 is sos is an interval $[-c_Q, c_Q]$; here $\frac{13}{4} \leq c_Q < 4$, since no odd power of Q_4 is sos [BKR26, Introduction]. Numerically $c_Q \approx 3.30151$, and $c_1 \approx 2.56548$ as reported in [BKR26].

6 Computations and reproducibility

How the certificates were found. Numerically. For a polynomial P and a list of blocks of monomials, the *margin* is the largest τ such that $P - \tau R$ has a positive semidefinite Gram matrix on the blocks, R being the sum of the squares of the monomials; P is sos on the blocks if and only if the margin is non-negative. The dual problem minimises $\Lambda(P)$ over the functionals with $\Lambda(R) = 1$ and positive semidefinite moment blocks. Both were solved in floating point with the solver Clarabel. A Gram matrix of f^3 with large smallest eigenvalue was rounded to denominator 10^{13} and projected exactly onto the affine space of the Gram matrices of f^3 ; a functional minimising $\Lambda((p+q)^3)$ with moment blocks at least η times the identity ($\eta = 2 \cdot 10^{-7}$, respectively 10^{-5}) was rounded to denominator 10^{13} . The rounded numbers are verified as they stand; nothing of the floating-point stage enters the proofs. The seeds lie near the largest parameters with an sos cube; in floating-point tests $(p_t + q_t)^3$ is sos for $t = 2.55$ and not for $t = 2.5525$ in case I.

How they are verified. Exactly, by `verify_note.py` (Python 3, standard library, about 600 lines, 0.3 seconds), as described in Proposition 3.2. Six other programs have verified the same files exactly: two of the first results (standard library, block by block; SymPy and python-flint, exact determinants), two of verification run A (standard library, full matrices and blocks, elimination and leading principal minors; SymPy), one of run B (full matrices; integer minors, and the characteristic polynomial computed with python-flint), and one of verification run 2, made on the final text (integers only; leading principal minors of the full matrices for two orders of B). Run 2 also proved positive definiteness in a second way, without exact elimination: for a floating-point Cholesky factor L of $M - \delta I$, taken as a matrix of rational numbers, the exact residual $M - LL^T$ is strictly diagonally dominant with positive diagonal. The programs of runs A and B were written from the statements, independently of each other and before the first programs were opened; `verify_note.py` was written with knowledge of these; the program of run 2 was written from the statements of this note, after `verify_note.py` had been read. All seven obtain the fractions of Proposition 3.2(C).

The package contains the certificate files with checksums, all programs named above with their recorded outputs, the generator of the certificates, the numerical explorations, and a `README.md` with the command lines. To check Theorems 1.1 and 1.2 it suffices to read Sections 2 and 3 and to run `python3 verify_note.py`.

6.1 Numerical searches in the open cases

With one exception, which is marked below, everything in this subsection comes from floating-point or multiprecision semidefinite programming without exact certificates: tests, not proofs.

Two searches for pairs with exponent 3 in the open cases $(n, m) = (3, 6), (4, 6), (4, 4), (5, 4), (6, 4)$ found none. The first, made with the first results, tested sums of two copies of M_c sharing two or three variables, and the local convexity of $\Sigma_{3,6}(3)$ at boundary points in a 10-dimensional space of ternary sextics. The second, made by verification run B, minimised the margin of the cube over sums of two to four forms obtained from M_c , Q_c and $x^4y^2 + y^4z^2 + z^4x^2 - cx^2y^2z^2$, with c just below the thresholds, by linear substitutions and products with sums of squares (several thousand semidefinite programs); as a control, the same search finds the pair of Theorem 1.1 in five variables within seconds.

The reach of these searches is small. Only these special families were examined; the 28-dimensional space of ternary sextics was not searched. A double precision solver cannot decide the sign of a margin below about 10^{-6} . Five points with margins near $-2 \cdot 10^{-6}$, one for $(4, 4)$ and four for $(4, 6)$, came from solves that ended with reduced accuracy. The first is not an example, and this is proved: it is a sum $H_1 + H_2$, where H_1 arises from $Q_{c'}$ with $c' \approx 0.99$ by rescaling and renaming the variables and H_2 is a sum of three squares of monomials with positive coefficients; so its cube is sos by Proposition 1.5(a) and Lemma 2.4. (The multiprecision solver SDPA-GMP gives this point the margin $+3.3 \cdot 10^{-12}$;

this is the coefficient of a vertex monomial of the normalised cube, an upper bound for the margin.) Recomputed with the same solver, three of the other four points have the margins $+6.6 \cdot 10^{-13}$ once and zero within the output resolution of 10^{-17} twice; the fourth, close to one of these, was not recomputed. And the true margins are tiny: multiprecision spot checks of sums of two boundary elements in three and four variables gave non-negative values, most of them between 10^{-14} and 10^{-6} ; and for the Motzkin form M_3 and $\rho = x^2 + y^2 + z^2$ the margin of $(M_3 + \beta\rho^3)^3$ was positive at the tested values $\beta \geq 10^{-4}$ and negative at those $\leq 5 \cdot 10^{-5}$, so that near M_3 the boundaries of $\Sigma_{3,6}(3)$ and of $P_{3,6}$ are separated only by a layer of about this thickness (the number β_3 of [BKR26, Introduction] appears to lie between the two values). A double precision search cannot resolve this thin layer, in which a ternary example would have to be found. For exponents $\ell \geq 5$ the numerical runs were unreliable.

Verification. The results were first obtained with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. Runs A and B examined the first written version of the results, and neither saw the other; the third, called run 2 in the package, examined the final text of this note.

Run A compared the statement with [Rez23] and both versions of [BKR26]; proved Lemmas 2.1 to 2.4 and 3.1 again; verified both certificates with its two programs; checked that its program rejects five kinds of altered certificates and returns a positive value for Gaussian moments; searched numerically for a sum-of-squares representation of $(p + q)^3$ and found none; re-derived the proofs of Corollary 1.3 and Theorem 1.4 and of Proposition 1.5(a) for M_c ; and observed the cubics of Proposition 1.5(b). Run B rebuilt both examples numerically from the statements, with and without the block structure (margins $-1.5 \cdot 10^{-3}$ and $-7.7 \cdot 10^{-2}$ for $(p + q)^3$); verified both certificates exactly with its own program; repeated all 18 recorded runs of the first programs, reproducing the outputs (up to running times and, in one floating-point exploration, the values of solves with reduced accuracy) and regenerating both certificates byte for byte; made the second search of Section 6.1; and searched the literature. Both runs confirmed the theorems; neither found a mathematical error. Their corrections concern the description of the prior preprint and of the literature, the digits quoted for numerical thresholds, the documentation of the package, and one check missing in the SymPy program of the first results (that the monomials of $p + q$ lie in the simplex it takes as input). All were made; the verification report of the package lists them.

When this note was written, `verify_note.py` was written and run; the cubics of Proposition 1.5(b) were derived again, with their monotonicity; all programs of the first results and the exact programs of runs A and B were run again, with the same results as before; and the sources were read again. Added at that stage, after runs A and B: Proposition 1.5(a) for the quartics Q_c with $|c| < \frac{13}{4}$, which the earlier stages had only tested numerically, and with it the interval for case II in Proposition 1.5(b); Proposition 4.1 in the form with disjoint copies; the comparison with [Rez23, p. 780] in Remark 1.6; and the multiprecision value for the point with $(n, m) = (4, 4)$ in Section 6.1.

Run 2 examined these additions first and then the whole text. It read [Rez23], the passages of [BKR26] that are used here and [Kri26] again, and compared every sentence of this note about [Kri26] with the preprint; checked all proofs line by line; read `verify_note.py` line by line (a passing run proves Proposition 3.2); verified both certificates with its own program and proved the positive definiteness of all matrices a second time (Section 6); recomputed the cubics of Proposition 1.5(b) with an indeterminate t and counted their real roots by Sturm's theorem; checked the identities of Proposition 1.5(a) and of Remark 1.6 as identities in the parameter; tested the count in the proof of Proposition 4.1 by enumeration for $\ell \leq 7$; recomputed the data printed in [Kri26]; ran the exact programs of the package again from the archive, and the multiprecision program for the point with $(n, m) = (4, 4)$, reproducing the recorded outputs; and repeated the literature search. It found no

mathematical error and no further source. It required corrections of the text and of the package, which were made; the verification report of the package lists them. Three concern the mathematics: the proofs of Proposition 3.2 and of Proposition 1.5(b) now say, and `verify_note.py` checks explicitly, that all products of three monomials of $p + q$ have their exponents in T , so that Λ is defined on $(p_t + q_t)^3$ for every t ; Proposition 4.1 is stated for even degree; and the smallest pivots of M_Λ are those of the lexicographic order of B . The others concern the abstract (the open cases include the higher even degrees in three and four variables), the description of [Rez23] and of [Kri26], the older references and the package. Two statements were added by this run and have not been examined by another one: the proof that the point with $(n, m) = (4, 4)$ lies in the cone (Section 6.1), and the sentence of Remark 1.6 on what follows from [Kri26]. The numerical searches were not repeated by run 2.

Scope and priority. The question is B. Reznick’s [Rez23, p. 779]; it is still stated as open in [BKR26], in the introduction and in Section 6 of both versions. The first negative answer is due to A. Kriebel [Kri26]: non-convexity for $(n, m, 2k + 1) = (3 \cdot 10^{62}, 6, 3)$ and, for every odd exponent ≥ 3 , in some dimension, by averaging many copies of one seed form in separate variables. The examples of this note were found after that preprint had been read, and they use its principle with two copies; Theorem 1.4 and Proposition 4.1 are its argument. The explicit data printed in [Kri26] (the identity for the cube of its seed, its moment matrix of size 220, three moments, the sign at $N = 10^{62}$) were recomputed exactly by all three runs and with the first results, with agreement. Added here (Section 1.4): the explicit pairs of Theorems 1.1 and 1.2; Corollary 1.3; the quartic case of Theorem 1.4; and Proposition 1.5, whose bound $\frac{41}{16}$ is to be compared with the proved 1.04886 and the experimental 2.56548 of [BKR26, Section 6] and with the “about 1.1336” of [Rez23, p. 779]. Open: $n = 3$ and $n = 4$ for sextics and for higher even degrees, quartics in 4 to 6 variables, and explicit small dimensions for exponents $\ell \geq 5$. The numerical searches of Section 6.1 found no example there, which is a test and not a proof, and double precision arithmetic cannot resolve the thin layer described there. Theorems 1.1 and 1.2 are computer-assisted in the sense explained after Theorem 1.2.

What was read. The abstract [Rez23] completely. Of [BKR26]: in the arXiv version (v1, the only one) the introduction and Sections 5 and 6; in the journal version the introduction, the statements of Theorems 5.1 and 5.3, and Section 6; its Sections 2 to 4 were not read. The preprint [Kri26] completely, in the file deposited on Zenodo (version 1.0, the only one; accessed on 11 October 2026, when the record showed a modification date of 10 October 2026); the verification archive attached to it was not read. Of [BBK⁺26]: the abstract and the statement of Theorem 2.2; a search of its text found no statement on a fixed exponent. The older papers [Hil88, CL77, Rez78, Ste79, BCJ79, CDLR82, Rez95] were not read, and of [Sch12] only the abstract; what is said about them here is taken from [Rez23, BKR26, Kri26], and nothing in this note depends on them.

Searches. On 11 October 2026: queries to the arXiv API, Crossref, OpenAlex, Semantic Scholar, zbMATH, Zenodo and DataCite, and nine web searches, made with the first results, by the three verification runs and at the writing. The works citing [BKR26] that are listed in OpenAlex and in Semantic Scholar are [Kri26], [BBK⁺26] and one paper on a different cone (arXiv:2305.14848); OpenAlex lists no work that cites [Kri26]. No source was found with an example in small dimension, for three variables or for quartics, and none with a proved bound for c_1 beyond that of [BKR26]. A search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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