

The Elements with Property (s) of a Skew Brace Need Not Form an Ideal: On a Question of Colazzo, Ferrara and Trombetti

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Abstract

An element x of a skew brace B has property (s), in the sense of Colazzo, Ferrara and Trombetti, if it has only finitely many conjugates of each of the kinds $x * b$, $b * x$, $b + x - b$ and $b \circ x \circ b^{-1}$. In her abstract in Oberwolfach Report 9/2023, Colazzo asks whether the elements with property (s) of a skew brace form an ideal; by a result of the three authors they do so in two-sided braces. We answer the question in the negative. On $\mathbb{Z} \times \mathbb{Z}/4$, with componentwise addition, the multiplication $(m, a) \circ (n, b) = (m + \varepsilon(a)n, a + b + 2ab)$, where $\varepsilon(a) = 1$ for $a \in \{0, 1\}$ and $\varepsilon(a) = -1$ for $a \in \{2, 3\}$, defines a left brace in which the elements with property (s) are those of $\mathbb{Z} \times \{0, 1\}$; this set is not closed under addition and not invariant under the maps λ_b . A sub-brace with additive group $\mathbb{Z} \times \mathbb{Z}/2$ has the same property. The verification is elementary and is given in full. On the positive side, in every skew brace these elements form a normal subgroup of the multiplicative group, so only the additive conditions can fail; and they form an ideal in two-sided skew braces, in braces with torsion-free additive group, in bi-skew braces, and in skew braces whose multiplicative group is an FC-group. In semidirect products of a trivial brace by a finite skew brace Q the set in question is given by a normal subgroup of the multiplicative group of Q , which can be prescribed; this yields examples for each way in which the additive conditions can fail, among them a skew brace with torsion-free additive group, a centrally nilpotent brace of exponent 4, and the structure skew brace of a non-degenerate solution of the Yang–Baxter equation with six elements. The question, the notion and the positive answer for two-sided braces are due to Colazzo, Ferrara and Trombetti; the brace on $\mathbb{Z}/4$ underlying the example is Example 3.8 of their paper, and the semidirect product of skew braces is a known construction. That the elements with property (s) form a subgroup of the multiplicative group follows at once from a characterisation in a recent preprint of Di Matteo and Ferrara together with a result of Cascella, Properzi and Van Antwerpen; it also follows from a result of Mondal and Yadav on a set which we show to be the same. The statement on two-sided skew braces is a short consequence of the first two results. Our proofs are included for completeness. The example also shows that Lemma 3.12 of the paper of Colazzo, Ferrara and Trombetti needs an additional hypothesis; it holds for skew braces with property (S), where it is used. The literature was searched within the limits stated in the text (one citing paper was seen in abstract only). A search that finds nothing is not a proof of novelty, and no priority is claimed. This is an unrefereed note.

1 Introduction

1.1 The question

A *skew (left) brace* [GV17] is a set B with two group structures $(B, +)$ and $(B, \circ)$ such that

$$a \circ (b + c) = a \circ b - a + a \circ c \quad \text{for all } a, b, c \in B;$$

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the group $(B, +)$ is not assumed to be abelian. If it is abelian, B is a *(left) brace*, or a skew brace of *abelian type*. The two groups have the same neutral element 0; we write $-a$ and a^{-1} for the inverses of a in $(B, +)$ and in $(B, \circ)$. The maps $\lambda_a : b \mapsto -a + a \circ b$ are automorphisms of $(B, +)$, and $a \mapsto \lambda_a$ is a homomorphism $(B, \circ) \rightarrow \text{Aut}(B, +)$. One puts $a * b = \lambda_a(b) - b = -a + a \circ b - b$. A *left ideal* of B is a subgroup I of $(B, +)$ with $\lambda_a(I) \subseteq I$ for all $a \in B$; it is a *strong left ideal* if it is normal in $(B, +)$, and an *ideal* if it is normal in $(B, +)$ and in $(B, \circ)$.

Colazzo, Ferrara and Trombetti [CFT25, Section 3] attach to an element $x \in B$ the sets

$$\text{Fix}^r(x) = \{b \in B : x * b = 0\}, \quad \text{Fix}^l(x) = \{b \in B : b * x = 0\},$$

which are subgroups of $(B, +)$ and of $(B, \circ)$ respectively, and the centralisers C_x^+ and C_x° of x in $(B, +)$ and in $(B, \circ)$.

Definition 1.1 ([CFT25, Definition 3.7]). An element x of a skew brace B has *property (s)*, or is an *(s)-element*, if the two indices

$$|(B, +) : \text{Fix}^r(x) \cap C_x^+| \quad \text{and} \quad |(B, \circ) : \text{Fix}^l(x) \cap C_x^\circ|$$

are finite. The skew brace B has *property (S)* if all its elements have property (s). We write $S(B)$ for the set of all (s)-elements of B .

Equivalently, x has only finitely many “conjugates” of each of the four kinds $x * b$, $b * x$, $b + x - b$ and $b \circ x \circ b^{-1}$, $b \in B$ ([CFT25, Notation 3.9]; see Lemma 2.1). In a finite skew brace every element has property (s), and for the trivial skew brace on a group G (where $a \circ b = a + b$) the set $S(B)$ is the FC-centre of G .

In her abstract [Col23] in the report of the Oberwolfach mini-workshop on skew braces and the Yang–Baxter equation, which reports on the joint work [CFT25], Colazzo states three open problems. The first is posed for an arbitrary skew brace B and asks [Col23, Question 1]: “Is the set of all elements with property (s) an ideal?” The abstract adds that the answer is affirmative for two-sided skew braces of abelian type; a skew brace is *two-sided* if also $(a + b) \circ c = a \circ c - c + b \circ c$ for all a, b, c . In [CFT25] the positive case is Proposition 3.10, for two-sided braces, with a sketch of the proof, and the authors say after it that they could not decide the case of skew braces. The question is record OWR-12697693-003 of the UnsolvedMath collection [ula26], where it is listed as open.

For a reader who compares with the sources: [Col23] cites the arXiv preprint of [CFT25] with the numbering of its first version, in which property (s) is Definition 3.5; the second arXiv version has the numbering of the journal version, which we use. In [Col23, p. 543] the display of the definition contains the two subgroups of Definition 1.1; in the line after it the set $\text{Fix}^l(x)$ is printed without its superscript.

1.2 Results

The answer to the question is negative, already for braces.

Theorem 1.2. *For $a \in \mathbb{Z}/4$ let $\varepsilon(a) = 1$ if $a \in \{0, 1\}$ and $\varepsilon(a) = -1$ if $a \in \{2, 3\}$. On the set $B_A = \mathbb{Z} \times \mathbb{Z}/4$ define*

$$(m, a) + (n, b) = (m + n, a + b), \quad (m, a) \circ (n, b) = (m + \varepsilon(a)n, a + b + 2ab).$$

Then $(B_A, +, \circ)$ is a left brace, and

$$S(B_A) = \mathbb{Z} \times \{0, 1\}.$$

The element $x = (0, 1)$ is central in $(B_A, +)$ and in $(B_A, \circ)$, and both indices of Definition 1.1 are equal to 2 for it; but $x + x = (0, 2)$ and $-x = \lambda_x(x) = (0, 3)$ are not (s)-elements. Hence $S(B_A)$ is

not a subgroup of $(B_A, +)$ and is not invariant under the maps λ_b : it is not an ideal, not a left ideal and not a sub-skew brace of B_A . It is a normal subgroup of index 2 of $(B_A, \circ)$.

The subset $C = \{(m, a) \in B_A : a \equiv m \pmod{2}\}$ is a sub-brace of B_A with $(C, +) \cong \mathbb{Z} \times \mathbb{Z}/2$, and $S(C) = \{(m, a) \in C : a \in \{0, 1\}\}$; so $(1, 1) \in S(C)$, while $(1, 1) + (1, 1) = (2, 2)$ and $\lambda_{(1,1)}(1, 1) = (1, 3)$ are not in $S(C)$.

The proof (Section 3) consists of six short verifications in $\mathbb{Z}$ and $\mathbb{Z}/4$ and is given in full. The answer does not depend on the conventions (left or right skew braces, the order of the terms in $a * b$, the definition of an ideal); see Remark 3.2. A brace whose (s)-elements do not form an ideal has an infinite additive group which is not torsion-free, by Theorem 1.4(b) below; among the finitely generated abelian groups of this kind, $\mathbb{Z} \times \mathbb{Z}/2$ has the smallest rank and the smallest torsion subgroup. The brace B_A is finitely generated, right nilpotent and soluble, and has finite multipermutation level (Remark 3.3).

What remains true in general is the multiplicative half of the statement.

Theorem 1.3. *For every skew brace B the set $S(B)$ is a normal subgroup of $(B, \circ)$. Consequently $S(B)$ is a left ideal of B if and only if it is invariant under all maps λ_b , and it is an ideal of B if and only if $b + \lambda_c(x) - b \in S(B)$ for all $x \in S(B)$ and $b, c \in B$.*

So only the additive conditions can fail. They hold in the following classes.

Theorem 1.4. *The set $S(B)$ is an ideal of the skew brace B in each of the following cases.*

- (a) *B is a two-sided skew brace (the additive group need not be abelian).*
- (b) *B is a brace whose additive group is torsion-free. In this case $S(B)$ is the set of all $x \in \text{Soc}(B)$ for which the orbit $\{\lambda_b(x) : b \in B\}$ is finite.*
- (c) *B is a bi-skew brace, that is, $(B, \circ, +)$ is a skew brace as well.*
- (d) *The multiplicative group $(B, \circ)$ is an FC-group.*

Here $\text{Soc}(B) = \text{Ker}(\lambda) \cap Z(B, +)$ is the socle. In (b) the word “brace” cannot be replaced by “skew brace”: in Example 6.5 the group $(B, +)$ is torsion-free and finitely generated, and $S(B)$ is a subgroup of finite index of $(B, +)$ that is neither normal nor λ -invariant. Structure braces of involutive non-degenerate solutions of the Yang–Baxter equation are covered by (b) (Remark 5.3).

The brace B_A is a semidirect product of the trivial brace $\mathbb{Z}$ by the brace $Q_4 = (\mathbb{Z}/4, +, \circ)$ with $a \circ b = a + b + 2ab$. In this construction the (s)-elements can be computed, and they reproduce any given normal subgroup of the multiplicative group of a finite skew brace.

Theorem 1.5. *Let Q be a finite skew brace, M a normal subgroup of $(Q, \circ)$, V a torsion-free abelian group and $\alpha : (Q, \circ) \rightarrow \text{Aut}(V)$ a homomorphism with kernel M (for instance $V = \mathbb{Z}[\Gamma]$ with $\Gamma = (Q, \circ)/M$ acting by left multiplication). On $B = V \times Q$ let*

$$(v, q) + (w, p) = (v + w, q + p), \quad (v, q) \circ (w, p) = (v + \alpha_q(w), q \circ p).$$

Then B is a skew brace and $S(B) = V \times M$. The set $S(B)$ is a subgroup of $(B, +)$, is λ -invariant, is invariant under conjugation in $(B, +)$, or is an ideal of B , if and only if M has the corresponding property in Q .

Theorem 1.2 is the case $Q = Q_4$, $M = \{0, 1\}$, $V = \mathbb{Z}$. Other choices give a brace in which $S(B)$ is a normal subgroup of both groups but is not λ -invariant, and a skew brace in which $S(B)$ is a left ideal that is not normal in $(B, +)$ (Examples 6.3 and 6.4); so each of the additive conditions fails

separately. The underlying computation (Lemma 6.1) does not need V to be torsion-free; it also gives a brace of exponent 4 which is left nilpotent, right nilpotent and centrally nilpotent and in which the (s)-elements are not closed under addition (Example 6.6).

The paper [CFT25] is motivated by the structure skew braces of finite solutions of the Yang–Baxter equation. For these the answer is negative as well.

Theorem 1.6. *Let $X = \{a_0, a_1, a_2, b_0, b_1, b_2\}$, with indices in $\mathbb{Z}/3$, and*

$$r(a_k, a_l) = (a_{-l}, a_{k-l}), \quad r(a_k, b_l) = (b_{k+l}, a_k), \quad r(b_k, a_l) = (a_l, b_{k-l}), \quad r(b_k, b_l) = (b_l, b_k).$$

Then (X, r) is a bijective non-degenerate solution of the Yang–Baxter equation which is not involutive. In its structure skew brace $G = G(X, r)$, with canonical map $\iota: X \rightarrow G$, the element $\iota(a_0)$ has property (s), while $-\iota(a_1) + \iota(a_0) + \iota(a_1) = \iota(a_2)$ has not. Hence $S(G)$ is not normal in $(G, +)$ and is not an ideal of G .

The proof uses three results of [GV17], and nothing else from the literature: the solution attached to a skew brace and its restriction to invariant subsets (Theorem 3.1, Corollary 3.3), and the existence and the universal property of the structure skew brace (Theorem 3.9).

Finally, the brace B_A satisfies the hypothesis of [CFT25, Lemma 3.12] but not its conclusion; in Remark 8.2 we write this out, locate the step of the printed proof that needs an additional hypothesis, and observe that the conclusion holds for skew braces with property (S) (Proposition 8.1), which is the case in which the lemma is used in [CFT25].

1.3 Credits and limits

The question, the notion of an (s)-element and the positive answer for two-sided braces are due to Colazzo, Ferrara and Trombetti [Col23, CFT25]; the brace Q_4 is [CFT25, Example 3.8], where it is quoted from the literature. The semidirect product of a trivial skew brace by a skew brace, of which B_A and the skew braces of Theorem 1.5 are instances, is a known construction [CPV26, Definition 2.11]; it is used with a multiplicative character, for a different question, in [DF26, Section 4.3].

Casella, Properzi and Van Antwerpen [CPV26] show that the set $\theta_f(B)$ of the elements x for which $\{b + x - b\}$ and $\{\lambda_b(x)\}$ are finite is a strong left ideal [CPV26, Proposition 5.8], and an ideal if B is two-sided [CPV26, Proposition 5.9]. Di Matteo and Ferrara prove in the preprint [DF26] that $S(B) = \theta_f(B) \cap \text{FC}(B, \circ)$ [DF26, Corollary 5.4]. From these results it follows at once that $S(B)$ is a subgroup of $(B, \circ)$, which is part of Theorem 1.3; and Theorem 1.4(a) for skew braces of non-abelian type is a short consequence of them and of [CPV26, Remark 2.4] (Remark 5.2). Mondal and Yadav [MY27] consider the set $\text{FCI}(B)$ of the elements x for which the subgroup $\text{Fix}^r(x) \cap \text{Fix}^l(x) \cap C_x^+ \cap C_x^\circ$ of $(B, \circ)$ has finite index. They show that it is a subgroup of $(B, \circ)$, and an ideal if B is a two-sided brace; they state that it is a sub-skew brace if B is two-sided, bi-skew or λ -homomorphic (that is, if λ is also a homomorphism of $(B, +)$); and they remark that it contains $S(B)$. In fact the two sets are equal (Remark 4.4), so that the first of these statements is a second source for the fact that $S(B)$ is a subgroup of $(B, \circ)$. In Theorem 1.4(b), the fact that the set described there is an ideal consisting of (s)-elements is [CPV26, Theorem 5.22 and Corollary 5.23]. Our proofs of Theorem 1.3 and of Theorem 1.4(a), (b) do not use these results; they are included for completeness. For Theorem 1.4(c), (d) we use the criterion of [DF26], with a proof which is a variant of theirs (Proposition 4.3).

We found no source that answers the question or contains the example. Which sources were read and in which versions, and the limits of the search, are stated in the paragraph “Scope and priority” at the end of the note; in particular one citing paper was seen in abstract only, and [GV17, CPV26, MY27] were read in the arXiv versions only. A search that finds nothing is not a proof of novelty, and no priority is claimed.

2 Preliminaries

Throughout, B is a skew brace. From the definition of λ one has

$$a \circ b = a + \lambda_a(b), \quad a + b = a \circ \lambda_a^{-1}(b), \quad -a = \lambda_a(a^{-1}), \quad a^{-1} = \lambda_{a^{-1}}(-a) \quad (1)$$

for all $a, b \in B$. In particular, a λ -invariant subgroup of $(B, +)$ is a subgroup of $(B, \circ)$, and a λ -invariant subgroup of $(B, \circ)$ is a subgroup of $(B, +)$. For $x \in B$ we consider the four families

$$T_x = \{x * b\}, \quad [x]_+ = \{b + x - b\}, \quad [x]_\lambda = \{\lambda_b(x)\}, \quad [x]_\circ = \{b \circ x \circ b^{-1}\} \quad (b \in B).$$

Note that $\{b * x : b \in B\} = [x]_\lambda - x$.

Lemma 2.1. *For every $x \in B$,*

$$|(B, +) : \text{Fix}^r(x)| = |T_x|, \quad |(B, +) : C_x^+| = |[x]_+|, \quad |(B, \circ) : \text{Fix}^l(x)| = |[x]_\lambda|, \quad |(B, \circ) : C_x^\circ| = |[x]_\circ|.$$

Hence $x \in S(B)$ if and only if the four sets T_x , $[x]_+$, $[x]_\lambda$ and $[x]_\circ$ are finite.

Proof. The set $\text{Fix}^r(x)$ is the subgroup of fixed points of the automorphism λ_x of $(B, +)$, and $x * b = x * b'$ if and only if $\lambda_x(-b' + b) = -b' + b$, that is, $b \in b' + \text{Fix}^r(x)$. The set $\text{Fix}^l(x)$ is the stabiliser of x for the action λ of $(B, \circ)$ on the set B . The other two equalities are the orbit-stabiliser theorem for conjugation. Finally, the intersection of two subgroups of a group has finite index if and only if both have. $\square$

Lemma 2.2. *For all $a, b, c, k, u, x \in B$:*

$$(E1) \quad a * (b + c) = a * b + b + a * c - b;$$

$$(E2) \quad (a \circ b) * c = a * (b * c) + b * c + a * c;$$

$$(E3) \quad u + (x * b) - u = -(x * u) + x * (u + b); \text{ hence } [t]_+ \subseteq -T_x + T_x \text{ for every } t \in T_x;$$

$$(E4) \quad \lambda_k(x * b) = (k \circ x \circ k^{-1}) * \lambda_k(b); \text{ hence } T_{k \circ x \circ k^{-1}} = \lambda_k(T_x);$$

$$(E5) \quad \text{if } c = b \circ x \circ b^{-1}, \text{ then } c = b + \lambda_b(x) - b - (c * b).$$

Proof. (E1) $\lambda_a(b+c) - c - b = (a*b+b) + (a*c+c) - c - b$. (E2) $\lambda_a \lambda_b(c) - c = \lambda_a(b*c+c) - c = (a*(b*c) + b*c) + (a*c+c) - c$. (E3) is (E1) for the triple (x, u, b) . (E4) $\lambda_k(\lambda_x(b) - b) = \lambda_k \lambda_x \lambda_k^{-1}(\lambda_k(b)) - \lambda_k(b)$, and $\lambda_k \lambda_x \lambda_k^{-1} = \lambda_{k \circ x \circ k^{-1}}$. (E5) The equation $c \circ b = b \circ x$ reads $c + (c * b) + b = b + \lambda_b(x)$ by (1). $\square$

The identities (E1) and (E2) are recalled in [CFT25, Section 2.1]. As there, let $G = (B, +) \rtimes_\lambda (B, \circ)$ be the set $B \times B$ with the multiplication $(n, k)(n', k') = (n + \lambda_k(n'), k \circ k')$. We write $\text{FC}(\Gamma)$ for the FC-centre of a group Γ , the set of its elements with finitely many conjugates; it is a characteristic subgroup, and Γ is an *FC-group* if $\text{FC}(\Gamma) = \Gamma$.

Lemma 2.3. *In the group $G = (B, +) \rtimes_\lambda (B, \circ)$, with $x_k = k \circ x \circ k^{-1}$:*

$$(G1) \quad (n, k)(x, 0)(n, k)^{-1} = (n + \lambda_k(x) - n, 0);$$

$$(G2) \quad (n, k)(0, x)(n, k)^{-1} = (-(x_k * n), x_k);$$

$$(G3) \quad (0, c)(b, 0)(0, c)^{-1}(b, 0)^{-1} = (c * b, 0).$$

Consequently, $(x, 0) \in \text{FC}(G)$ if and only if $[x]_+$ and $[x]_\lambda$ are finite; $(0, x) \in \text{FC}(G)$ if and only if $[x]_\circ$ and T_x are finite; and

$$S(B) = S_1 \cap S_2, \quad S_1 = \{x : (x, 0) \in \text{FC}(G)\}, \quad S_2 = \{x : (0, x) \in \text{FC}(G)\}.$$

Proof. We have $(n, k)^{-1} = (\lambda_{k^{-1}}(-n), k^{-1})$. So (G3) follows from $(0, c)(b, 0)(0, c^{-1})(-b, 0) = (\lambda_c(b), 0)(-b, 0) = (c * b, 0)$, and for (G1) and (G2) we compute

$$\begin{aligned} (n, k)(x, 0)(n, k)^{-1} &= (n + \lambda_k(x), k)(\lambda_{k^{-1}}(-n), k^{-1}) = (n + \lambda_k(x) - n, 0), \\ (n, k)(0, x)(n, k)^{-1} &= (n, k \circ x)(\lambda_{k^{-1}}(-n), k^{-1}) = (n - \lambda_{x_k}(n), x_k), \end{aligned}$$

where $n - \lambda_{x_k}(n) = -(x_k * n)$.

By (G1) the conjugacy class of $(x, 0)$ is the union of the sets $[y]_+ \times \{0\}$, $y \in [x]_\lambda$, and $[\lambda_k(x)]_+ = \lambda_k([x]_+)$. By (G2) the class of $(0, x)$ is the set of all $(-t, y)$ with $y \in [x]_\circ$ and $t \in T_y$, and $|T_y| = |T_x|$ by (E4). The last statement follows from Lemma 2.1. $\square$

The identity (G3) is equation $(\star)$ of [CFT25], and S_1 is the set $\theta_f(B)$ of [CPV26].

3 The example

Proof of Theorem 1.2. The product $2ab \in \mathbb{Z}/4$ depends only on a and b modulo 2, so $a \circ b = a + b + 2ab$ is well defined on $\mathbb{Z}/4$.

(A1) $(\mathbb{Z}/4, \circ)$ is a Klein four-group. The operation is commutative, and $(a \circ b) \circ c = a + b + c + 2(ab + ac + bc) + 4abc$ is symmetric in a, b, c , so it is associative. The element 0 is neutral, and $a \circ a = 2a(a + 1) = 0$ because $a(a + 1)$ is even. Explicitly, $1 \circ 1 = 2 \circ 2 = 3 \circ 3 = 0$, $1 \circ 2 = 3$, $1 \circ 3 = 2$ and $2 \circ 3 = 1$.

(A2) $\varepsilon(a \circ b) = \varepsilon(a)\varepsilon(b)$. By the table in (A1), $\{0, 1\}$ is a subgroup of $(\mathbb{Z}/4, \circ)$ and $\{2, 3\}$ is its other coset.

(A3) $(B_A, \circ)$ is a group. By (A1) and (A2),

$$((m, a) \circ (n, b)) \circ (p, c) = (m + \varepsilon(a)n + \varepsilon(a)\varepsilon(b)p, a \circ b \circ c) = (m, a) \circ ((n, b) \circ (p, c));$$

the neutral element is $(0, 0)$, and $(m, a)^{-1} = (-\varepsilon(a)m, a)$.

(A4) *The skew left distributive law.* For $x = (m, a)$, $y = (n, b)$, $z = (p, c)$ the first coordinates of $x \circ (y + z)$ and of $x \circ y - x + x \circ z$ are $m + \varepsilon(a)(n + p) = (m + \varepsilon(a)n) - m + (m + \varepsilon(a)p)$, and the second coordinates are $a + (b + c) + 2a(b + c) = (a + b + 2ab) - a + (a + c + 2ac)$.

So B_A is a brace. From the definitions we obtain, for $x = (m, a)$ and $y = (n, b)$,

$$\begin{aligned} \lambda_x(y) &= (\varepsilon(a)n, b + 2ab), & x * y &= ((\varepsilon(a) - 1)n, 2ab), \\ y \circ x \circ y^{-1} &= ((1 - \varepsilon(a))n + \varepsilon(b)m, a), & y * x &= ((\varepsilon(b) - 1)m, 2ab); \end{aligned} \tag{2}$$

for the third formula note that $y \circ x = (n + \varepsilon(b)m, b \circ a)$, $y^{-1} = (-\varepsilon(b)n, b)$, $\varepsilon(b \circ a)\varepsilon(b) = \varepsilon(a)$ and $b \circ a \circ b = a$.

(A5) If $a \in \{2, 3\}$, then $x = (m, a)$ is not an (s)-element. Indeed $x * y = (-2n, 2ab)$, so $\text{Fix}^r(x) \subseteq \{0\} \times \mathbb{Z}/4$ is finite and has infinite index in $(B_A, +)$.

(A6) If $a \in \{0, 1\}$, then $x = (m, a)$ is an (s)-element. Now $x * y = (0, 2ab)$, so $\text{Fix}^r(x) \supseteq \mathbb{Z} \times \{0, 2\}$, and $C_x^+ = B_A$; the first index is at most 2. For $y = (n, 0)$ we have $y * x = (0, 0)$ and $y \circ x = (n + m, a) = (m + \varepsilon(a)n, a) = x \circ y$. Hence $\text{Fix}^l(x) \cap C_x^\circ \supseteq \mathbb{Z} \times \{0\}$, which is the kernel of the homomorphism $(m, a) \mapsto a$ from $(B_A, \circ)$ onto $(\mathbb{Z}/4, \circ)$ and has index 4.

By (A5) and (A6), $S(B_A) = \mathbb{Z} \times \{0, 1\}$; this is the kernel of the homomorphism $(m, a) \mapsto \varepsilon(a)$ of $(B_A, \circ)$, hence a normal subgroup of index 2. For $x = (0, 1)$ the formulas (2) give $x * y = y * x = (0, 2b)$ and $y \circ x \circ y^{-1} = x$; so $\text{Fix}^r(x) = \text{Fix}^l(x) = \mathbb{Z} \times \{0, 2\}$, the element x is central in both groups, and both indices are 2 (note that $\{0, 2\}$ is a subgroup of $(\mathbb{Z}/4, \circ)$). Finally $x + x = (0, 2)$, $-x = (0, 3)$ and $\lambda_x(x) = (0, 1 + 2) = (0, 3)$ are not in $S(B_A)$.

The sub-brace C . The first coordinate of $(m, a) \circ (n, b)$ is congruent to $m + n$ and the second to $a + b$ modulo 2, and $(m, a)^{-1} = (-\varepsilon(a)m, a)$; so C is closed under $+$, $-$, $\circ$ and inversion in $(B_A, \circ)$, and it is a sub-brace. Every $(m, a) \in C$ is $m(1, 1) + (0, a - m)$ with $a - m \in \{0, 2\}$, so $(C, +) = \mathbb{Z}(1, 1) \oplus (\mathbb{Z}/2)(0, 2) \cong \mathbb{Z} \times \mathbb{Z}/2$. Let $x = (m, a) \in C$. If $a \in \{2, 3\}$, then the elements $y = (2k, 0) \in C$ give the infinitely many values $x * y = (-4k, 0)$, so $x \notin S(C)$ by Lemma 2.1. If $a \in \{0, 1\}$, the four families of x in C are contained in those of x in B_A , which are finite; so $x \in S(C)$. The last statement of the theorem follows, with $\lambda_{(1,1)}(1, 1) = (\varepsilon(1) \cdot 1, 1 + 2) = (1, 3)$. $\square$

Remark 3.1. The element $(0, 1)$ is central of order 2 in $(B_A, \circ)$, and $(0, 2)$ inverts $\mathbb{Z} \times \{0\}$; so $(B_A, \circ) = (\mathbb{Z} \times \{0, 2\}) \times \langle (0, 1) \rangle \cong D_\infty \times C_2$, and $S(B_A)$ is its FC-centre. Similarly $(C, \circ) \cong D_\infty$ is generated by $t = (1, 1)$ and the involution $s = (0, 2)$, with $s \circ t \circ s^{-1} = t^{-1}$, and $S(C) = \langle t \rangle$. The brace B_A is not two-sided, as it must be by Theorem 1.4(a): $((0, 1) + (0, 1)) \circ (1, 0) = (-1, 2)$, but $(0, 1) \circ (1, 0) - (1, 0) + (0, 1) \circ (1, 0) = (1, 2)$.

Remark 3.2 (Conventions). The negative answer does not depend on the conventions.

(i) As $(B_A, +)$ is abelian, the expressions $-a + a \circ b - b$, $a \circ b - a - b$ and $-b - a + a \circ b$ coincide, and B_A is equal to its opposite skew brace $(B_A, +^{\text{op}}, \circ)$.

(ii) With $u \circ' v = v \circ u$, the triple $(B_A, +, \circ')$ is a right brace, and the mirror image of Definition 1.1 gives the same set $\mathbb{Z} \times \{0, 1\}$. If the formulas of Definition 1.1 are applied literally to $(B_A, +, \circ')$, the sets Fix^r and Fix^l exchange their roles. For $x = (0, 1)$ this changes nothing, because $\text{Fix}^r(x) = \text{Fix}^l(x)$ is a subgroup of index 2 of both groups and x is central in both. And $x + x = (0, 2)$ fails every variant of the definition which requires finitely many multiplicative conjugates or finitely many elements $x * b$: by (2) its centraliser in $(B_A, \circ)$ is $\{0\} \times \mathbb{Z}/4$, and $T_{(0,2)} = 2\mathbb{Z} \times \{0\}$.

(iii) By (2) the sets $\{b * x\}$ and $[x]_+$ are finite for every $x \in B_A$, and T_x is finite exactly when $[x]_\circ$ is, that is, for $x \in \mathbb{Z} \times \{0, 1\}$. So the variants of the definition that keep only the elements $x * b$, $b * x$, or only the conjugates in the two groups, define the same set.

(iv) In [GV17, Definition 2.1] an ideal is a normal subgroup I of $(B, \circ)$ with $\lambda_a(I) \subseteq I$ and $a + I = I + a$ for all a ; by (1) this agrees with the definition of Section 1. As $S(B_A)$ is neither closed under addition nor λ -invariant, it is not an ideal, a left ideal, a strong left ideal or a sub-skew brace under any of the usual definitions.

Remark 3.3 (Further properties of B_A). By (2), $\lambda_{(m,a)} = \text{id}$ if and only if $a = 0$; so $\text{Soc}(B_A) = \text{Ker}(\lambda) = \mathbb{Z} \times \{0\}$ has index 4, and the sets $[x]_\lambda$ and $[x]_+$ have at most 4 elements. Thus B_A is a θ_f -skew brace in the sense of [CPV26], and for these [CPV26, Proposition 5.17] states that the (s)-elements are the FC-elements of the multiplicative group. With that proposition the equality $S(B_A) = \text{FC}(B_A, \circ)$ is immediate; our proof does not use it.

For a skew brace B let $B^{(2)}$ be the subgroup of $(B, +)$ generated by all elements $a * b$, and $B^{(n+1)}$ the subgroup generated by the elements $y * b$ with $y \in B^{(n)}$, $b \in B$. By (2), $B_A^{(2)} = 2\mathbb{Z} \times \{0, 2\}$, $B_A^{(3)} = 2\mathbb{Z} \times \{0\}$ and $B_A^{(4)} = 0$: the brace B_A is right nilpotent. It is not left nilpotent: the series $B^1 = B$, $B^{n+1} = B * B^n$ does not reach 0, since $B_A * (2^k \mathbb{Z} \times \{0\}) = 2^{k+1} \mathbb{Z} \times \{0\}$. The subsets $0 \subset \mathbb{Z} \times \{0\} \subset \mathbb{Z} \times \{0, 2\} \subset B_A$ form the socle series of B_A (the kernel of the homomorphism $(m, a) \mapsto a$ onto Q_4 , and the preimage of $\text{Soc}(Q_4) = \{0, 2\}$); they are ideals, and the factors are trivial braces. So B_A is soluble and has multipermutation level 3, in the usual terminology; and $(B_A, +)$ is generated by $(1, 0)$ and $(0, 1)$. None of these properties, therefore, implies a positive answer.

4 The multiplicative group

Proof of Theorem 1.3. We use Lemma 2.3 and its notation. The map $k \mapsto (0, k)$ is an injective homomorphism $(B, \circ) \rightarrow G$, so S_2 , the preimage of the normal subgroup $\text{FC}(G)$, is a normal subgroup of $(B, \circ)$. The map $n \mapsto (n, 0)$ is an isomorphism of $(B, +)$ onto the normal subgroup $N = B \times \{0\}$ of G , and S_1 is the preimage of $N \cap \text{FC}(G)$, which is normal in G . Hence S_1 is a normal subgroup of $(B, +)$, and by (G1) with $n = 0$ it is λ -invariant. So S_1 is a strong left ideal, and in particular a subgroup of $(B, \circ)$ by (1). Therefore $S(B) = S_1 \cap S_2$ is a subgroup of $(B, \circ)$.

Let $x \in S(B)$, $b \in B$ and $c = b \circ x \circ b^{-1}$. Then $c \in S_2$. By (E5),

$$c = (b + \lambda_b(x) - b) - (c * b).$$

The first term lies in S_1 , because S_1 is λ -invariant and normal in $(B, +)$. By (G3), $(c * b, 0) = f \cdot (gf^{-1}g^{-1})$ with $f = (0, c) \in \text{FC}(G)$ and $g = (b, 0)$; as $\text{FC}(G)$ is a normal subgroup of G , this gives $(c * b, 0) \in \text{FC}(G)$, that is, $c * b \in S_1$. Hence $c \in S_1 \cap S_2 = S(B)$.

Now the consequences. If $S(B)$ is λ -invariant and $a, b \in S(B)$, then $a + b = a \circ \lambda_a^{-1}(b)$ and $-a = \lambda_a(a^{-1})$ lie in $S(B)$ by (1), because $S(B)$ is a subgroup of $(B, \circ)$; so $S(B)$ is a left ideal. The condition $b + \lambda_c(x) - b \in S(B)$ for all $x \in S(B)$ and $b, c \in B$ says that $S(B)$ is λ -invariant ($b = 0$) and invariant under conjugation in $(B, +)$ ($c = 0$). Then $S(B)$ is a strong left ideal, and it is normal in $(B, \circ)$ by the first part. $\square$

Remark 4.1. That $S_1 = \theta_f(B)$ is a strong left ideal is [CPV26, Proposition 5.8]. It is in general not normal in $(B, \circ)$ [CPV26, Example 5.10]; the second paragraph of the proof shows that a multiplicative conjugate of an element of $S_1 \cap S_2$ nevertheless stays in S_1 .

For Theorem 1.4(c) and (d) we need a criterion of Di Matteo and Ferrara, which answers [CPV26, Question 5.21]. We include a proof; it is a variant of the proof in [DF26, Proposition 5.1 and Theorem 5.2]. Both rest on the following lemma of B. H. Neumann [Neu54], whose short proof we recall.

Lemma 4.2 (Neumann). *If a group Γ is the union of finitely many left cosets $g_1H_1, \dots, g_nH_n$ of subgroups $H_1, \dots, H_n$, then at least one of the subgroups H_i has finite index in Γ .*

Proof. Induction on the number d of distinct subgroups among $H_1, \dots, H_n$. If $d = 1$, then $|\Gamma : H_1| \leq n$. Let $d > 1$ and $H = H_n$. If the cosets g_iH_i with $H_i = H$ cover Γ , then $|\Gamma : H|$ is finite. Otherwise some coset gH is not among them; it is disjoint from them and therefore contained in the union of the cosets g_iH_i with $H_i \neq H$. Then every coset $g_jH = (g_jg^{-1})gH$ with $H_j = H$ is contained in the union of the cosets $g_jg^{-1}g_iH_i$ with $H_i \neq H$. So Γ is the union of finitely many cosets of the $d - 1$ subgroups different from H , and the induction hypothesis applies. $\square$

Proposition 4.3 ([DF26, Corollary 5.4]). *An element x of a skew brace B is an (s)-element if and only if the three sets $[x]_+$, $[x]_\lambda$ and $[x]_\circ$ are finite. In other words, $S(B) = S_1 \cap \text{FC}(B, \circ)$.*

Proof. By Lemma 2.1 we have to show that T_x is finite if the three sets are finite. The subgroup $C = C_x^\circ$ has finite index in $(B, \circ)$. For $b \in C$ the equation $x \circ b = b \circ x$ reads $x + \lambda_x(b) = b + \lambda_b(x)$, so that

$$x * b = \lambda_x(b) - b = -x + b + \lambda_b(x) - b \in -x + \bigcup_{y \in [x]_\lambda} [y]_+,$$

which is a finite set, since $[\lambda_k(x)]_+ = \lambda_k([x]_+)$. As $x * b = x * b'$ if and only if $b \in b' + \text{Fix}^r(x)$, the set C is contained in finitely many cosets $b_1 + \text{Fix}^r(x), \dots, b_m + \text{Fix}^r(x)$. Let $c_1, \dots, c_r$ be representatives of the left cosets of C in $(B, \circ)$. Since $c \circ b = c + \lambda_c(b)$,

$$B = \bigcup_j c_j \circ C \subseteq \bigcup_{i,j} (c_j + \lambda_{c_j}(b_i) + \lambda_{c_j}(\text{Fix}^r(x))).$$

So the group $(B, +)$ is the union of finitely many cosets of the subgroups $\lambda_{c_j}(\text{Fix}^r(x))$. By Lemma 4.2 one of these has finite index, and then so has $\text{Fix}^r(x)$, because λ_{c_j} is an automorphism of $(B, +)$. Hence T_x is finite. $\square$

Remark 4.4 (The set $\text{FCI}(B)$ of Mondal and Yadav). For $x \in B$ let $\text{Ann}_B(x) = \text{Fix}^r(x) \cap \text{Fix}^l(x) \cap C_x^+ \cap C_x^\circ$; this is a subgroup of $(B, \circ)$ [CFT25, Theorem 3.3]. In [MY27, Section 2] it is called the brace centraliser of x , and $\text{FCI}(B)$ is the set of all $x \in B$ for which it has finite index in $(B, \circ)$. It is shown there that $\text{FCI}(B)$ is a subgroup of $(B, \circ)$, and that it is an ideal if B is a two-sided brace; and it is stated that it is a sub-skew brace if B is two-sided, bi-skew or λ -homomorphic (Propositions 2.3 and 2.4 of the arXiv version of [MY27], where bi-skew braces are called symmetric). The inclusion $S(B) \subseteq \text{FCI}(B)$ is remarked there. In fact

$$\text{FCI}(B) = S(B) \quad \text{for every skew brace } B.$$

So the first of these statements says that $S(B)$ is a subgroup of $(B, \circ)$, which is part of Theorem 1.3, and the second is [CFT25, Proposition 3.10].

For the proof let first $x \in S(B)$. The subgroups $\text{Fix}^r(y) = \lambda_k(\text{Fix}^r(x))$ of $(B, +)$ with $y = k \circ x \circ k^{-1} \in [x]_\circ$, and $C_y^+ = \lambda_k(C_x^+)$ with $y = \lambda_k(x) \in [x]_\lambda$, are finite in number and of finite index, and the maps λ_b permute each of the two families. So their intersection L is a left ideal of finite index in $(B, +)$, and $L \subseteq \text{Fix}^r(x) \cap C_x^+$. As $b \circ L = b + \lambda_b(L) = b + L$ for all b , the subgroup L has finite index in $(B, \circ)$ as well. Hence $\text{Ann}_B(x) \supseteq L \cap \text{Fix}^l(x) \cap C_x^\circ$ has finite index in $(B, \circ)$. Conversely, let $H = \text{Ann}_B(x)$ have finite index in $(B, \circ)$, and let $c_1, \dots, c_r$ be representatives of its left cosets. Then $\text{Fix}^l(x) \cap C_x^\circ \supseteq H$ has finite index in $(B, \circ)$. The subgroup $R = \text{Fix}^r(x) \cap C_x^+$ of $(B, +)$ contains H , so that $B = \bigcup_j c_j \circ H \subseteq \bigcup_j (c_j + \lambda_{c_j}(R))$. By Lemma 4.2 one of the subgroups $\lambda_{c_j}(R)$, and therefore R , has finite index in $(B, +)$. So $x \in S(B)$.

5 Classes with a positive answer

5.1 Two-sided skew braces

Lemma 5.1. *Let B be a two-sided skew brace.*

(T1) *For every $c \in B$ the map $\iota_c: b \mapsto c \circ b \circ c^{-1}$ is an automorphism of the skew brace B . In particular it maps the four families of x onto those of $\iota_c(x)$, and $\iota_c(S(B)) = S(B)$.*

(T2) *$(a + b) * c = -b + a * c + b + b * c$ and $(-a) * c = a - a * c - a$ for all $a, b, c \in B$.*

Proof. (T1) Two applications of the right distributive law give $(u + v + w) \circ d = u \circ d - d + v \circ d - d + w \circ d$, and applied to $(c + (-c)) \circ c^{-1} = c^{-1}$ the law gives $(-c) \circ c^{-1} = c^{-1} + c^{-1}$. Hence

$$\iota_c(a + b) = (c \circ a - c + c \circ b) \circ c^{-1} = \iota_c(a) - c^{-1} + (c^{-1} + c^{-1}) - c^{-1} + \iota_c(b) = \iota_c(a) + \iota_c(b).$$

(T2) $-(a + b) + (a + b) \circ c - c = -b - a + a \circ c - c + b \circ c - c = -b + (a * c) + b + (b * c)$; the second formula is the case $b = -a$. $\square$

The statement (T1) is [CPV26, Remark 2.4].

Proof of Theorem 1.4(a). Let $S = S(B)$. We use Lemma 2.1 throughout.

Step 1: S is a subgroup of $(B, +)$. Clearly $0 \in S$. Let $x, y \in S$. Then $[x - y]_+ \subseteq [x]_+ - [y]_+$ and $[x - y]_\lambda \subseteq [x]_\lambda - [y]_\lambda$, and $[x - y]_\circ \subseteq [x]_\circ - [y]_\circ$ by (T1). By (T2),

$$(x - y) * c = y + x * c - y + (y - y * c - y) = y + x * c - y * c - y \in y + T_x - T_y - y.$$

So the four families of $x - y$ are finite.

Step 2: S is normal in $(B, +)$. Let $x \in S$, $g \in B$ and $z = g + x - g$. Then $[z]_+ = [x]_+$. Since $\lambda_k(z) \in [\lambda_k(x)]_+ = \lambda_k([x]_+)$, the set $[z]_\lambda$ lies in the union of the finitely many finite sets $[y]_+$, $y \in [x]_\lambda$. In the same way, by (T1), $[z]_\circ$ lies in the union of the sets $[y]_+ = \iota_k([x]_+)$, $y = \iota_k(x) \in [x]_\circ$. For T_z put $w = g * c$ and $t = x * c$. By (T2), applied to $(g + x) + (-g)$ and to $g + x$,

$$\begin{aligned} z * c &= g + (g + x) * c - g + (g - w - g) = g + (-x + w + x + t) - w - g \\ &= g + (-x + (w + x - w)) + (w + t - w) - g. \end{aligned}$$

Here $w + x - w \in [x]_+$, and $w + t - w \in [t]_+ \subseteq -T_x + T_x$ by (E3). So T_z is finite.

Step 3: S is normal in $(B, \circ)$. This follows from (T1) (or from Theorem 1.3).

*Step 4: if $y \in S$ and $b \in B$, then $t = y * b \in S$.* First, $[t]_+ \subseteq -T_y + T_y$ by (E3). By (E4), $\lambda_k(t) = \iota_k(y) * \lambda_k(b)$, and by (T1), $\iota_k(t) = \iota_k(y) * \iota_k(b)$; both lie in $T_{\iota_k(y)}$. The sets $T_{y'}, y' \in [y]_\circ$, are finite in number, and each has $|T_y|$ elements by (E4). So $[t]_\lambda$ and $[t]_\circ$ are finite. For T_t write $t = (-y) + (y \circ b) + (-b)$ and put $p = y * c$, $q = y * (b * c)$, $r = b * c$. By (T2), applied twice, and by (E2),

$$\begin{aligned} t * c &= b + ((-y) + (y \circ b)) * c - b + (b - r - b) \\ &= b + [-(y \circ b) + (y - p - y) + (y \circ b) + (q + r + p)] - r - b \\ &= b + [-(y \circ b) + y - p - y + (y \circ b) + q + (r + p - r)] - b. \end{aligned}$$

Here $p, q \in T_y$, and $r + p - r \in [p]_+ \subseteq -T_y + T_y$ by (E3). So T_t is finite.

Step 5: S is λ -invariant. Let $x \in S$, $b \in B$ and $y = \iota_b(x)$; then $y \in S$ by Step 3. From $y \circ b = b \circ x$ and (1) we get

$$\lambda_b(x) = -b + y \circ b = -b + (y + y * b) + b,$$

which lies in S by Steps 4, 1 and 2. □

Remark 5.2. With Proposition 4.3 there is a shorter proof. By that proposition $S(B) = S_1 \cap \text{FC}(B, \circ)$, and S_1 is a strong left ideal. Let B be two-sided, $x, y \in S(B)$ and $g, b \in B$. By (T1), $[x - y]_\circ \subseteq [x]_\circ - [y]_\circ$, and the sets $[g + x - g]_\circ$ and $[\lambda_b(x)]_\circ$ lie in the union of the sets $\iota_k([x]_+)$, respectively $\iota_k([x]_\lambda)$, taken over the finitely many elements $\iota_k(x)$ of $[x]_\circ$. Hence $x - y$, $g + x - g$ and $\lambda_b(x)$ lie in $S_1 \cap \text{FC}(B, \circ)$; so $S(B)$ is a strong left ideal, and an ideal by Theorem 1.3. This is the derivation from [DF26, Corollary 5.4] and [CPV26, Proposition 5.9, Remark 2.4] mentioned in the introduction. The proof given above uses neither Proposition 4.3 nor Lemma 4.2.

5.2 Braces with torsion-free additive group

Proof of Theorem 1.4(b). Let B be a brace with torsion-free additive group. As $(B, +)$ is abelian, $\text{Soc}(B) = \text{Ker}(\lambda)$, and $\lambda_x - \text{id}$ is an endomorphism of $(B, +)$ with image T_x . If $x \in S(B)$, then T_x is a finite subgroup of a torsion-free group; so $T_x = 0$, that is, $x \in \text{Ker}(\lambda)$, and $[x]_\lambda$ is finite.

Conversely, let $x \in \text{Ker}(\lambda)$ with $[x]_\lambda$ finite. Then $T_x = \{0\}$ and $[x]_+ = \{x\}$. For $b \in B$ we have $x \circ b^{-1} = x + b^{-1}$ by (1), and the left distributive law gives

$$b \circ x \circ b^{-1} = b \circ (x + b^{-1}) = b \circ x - b + b \circ b^{-1} = \lambda_b(x). \quad (3)$$

So $[x]_\circ = [x]_\lambda$ is finite, and $x \in S(B)$ by Lemma 2.1.

Thus $S(B)$ is the set described in the theorem. On the subgroup $\text{Ker}(\lambda)$ of $(B, \circ)$ the operations $\circ$ and $+$ coincide by (1), so $\text{Ker}(\lambda)$ is a subgroup of $(B, +)$; and $[x - y]_\lambda \subseteq [x]_\lambda - [y]_\lambda$. Hence $S(B)$ is a subgroup of $(B, +)$. It is λ -invariant, because $\lambda_b(x) \in \text{Ker}(\lambda)$ by (3) and $[\lambda_b(x)]_\lambda = [x]_\lambda$. So $S(B)$ is a left ideal of a brace, and it is normal in $(B, \circ)$ by (3) (or by Theorem 1.3): it is an ideal. □

The same proof shows that $S(B)$ is an ideal in every brace with $S(B) \subseteq \text{Soc}(B)$. By Theorem 1.2 torsion-freeness cannot be omitted, and by Example 6.5 commutativity of the addition cannot.

Remark 5.3 (Involutive solutions). Let (X, r) be a non-degenerate solution of the Yang–Baxter equation, $r(x, y) = (\sigma_x(y), \tau_y(x))$. By the presentation recalled in [CFT25, Section 2.1], the additive group of its structure skew brace is generated by X with the relations $x + u = u + \sigma_u(\tau_y(x))$ for $x, y \in X$ and $u = \sigma_x(y)$. If r is involutive, then $\sigma_u(\tau_y(x)) = x$, the relations say that the generators commute, and the additive group is free abelian. So by Theorem 1.4(b) the (s)-elements of the structure brace of an involutive non-degenerate solution form an ideal; by Theorem 1.6 this does not extend to all non-involutive solutions.

5.3 Bi-skew braces and skew braces with an FC multiplicative group

Proof of Theorem 1.4(d). If $(B, \circ)$ is an FC-group, then $S(B) = S_1$ by Proposition 4.3. This is a strong left ideal by the proof of Theorem 1.3, and it is normal in $(B, \circ)$ by Theorem 1.3. $\square$

The mirror hypothesis does not suffice: in B_A the additive group is abelian and all orbits $[x]_\lambda$ are finite, so that $S_1 = B_A$.

Bi-skew braces were introduced by Childs [Chi19]. The condition that $(B, \circ, +)$ is a skew brace reads $a + (b \circ c) = (a + b) \circ a^{-1} \circ (a + c)$ for all a, b, c . By (1) the left-hand side is $a \circ \lambda_a^{-1}(b \circ c)$ and the right-hand side is $a \circ \lambda_a^{-1}(b) \circ \lambda_a^{-1}(c)$. So B is a bi-skew brace if and only if every λ_a is an automorphism of $(B, \circ)$ as well.

Proof of Theorem 1.4(c). Let B be a bi-skew brace and $x \in S(B)$.

(i) For $a \in B$ we have $[\lambda_a(x)]_+ = \lambda_a([x]_+)$, $[\lambda_a(x)]_\circ = \lambda_a([x]_\circ)$, because λ_a is an automorphism of both groups, and $[\lambda_a(x)]_\lambda = [x]_\lambda$. So $\lambda_a(x) \in S(B)$ by Proposition 4.3.

(ii) Let $B' = (B, \circ, +)$. Its λ -map is $\lambda'_a(b) = a^{-1} \circ (a + b) = \lambda_a^{-1}(b)$ by (1). So the additive class, the multiplicative class and the λ' -orbit of x in B' are $[x]_\circ$, $[x]_+$ and $[x]_\lambda$, and $S(B') = S(B)$ by Proposition 4.3. By Theorem 1.3, applied to B' , the set $S(B)$ is a normal subgroup of the multiplicative group of B' , which is $(B, +)$.

(iii) By (i), (ii) and Theorem 1.3 for B , the set $S(B)$ is a λ -invariant normal subgroup of $(B, +)$ and a normal subgroup of $(B, \circ)$. $\square$

6 Semidirect products

Let V be an abelian group, Q a skew brace and $\alpha: (Q, \circ) \rightarrow \text{Aut}(V)$, $q \mapsto \alpha_q$, a homomorphism. On $B = V \times Q$ let

$$(v, q) + (w, p) = (v + w, q + p), \quad (v, q) \circ (w, p) = (v + \alpha_q(w), q \circ p). \quad (4)$$

This is the semidirect product $V \rtimes_\alpha Q$ of the trivial brace V by the skew brace Q [CPV26, Definition 2.11].

Lemma 6.1. *The set $B = V \rtimes_\alpha Q$ with the operations (4) is a skew brace. For $x = (v, q)$ and $y = (w, p)$ in B ,*

$$\begin{aligned} \lambda_x(y) &= (\alpha_q(w), \lambda_q(p)), & x * y &= ((\alpha_q - 1)w, q * p), \\ y + x - y &= (v, p + q - p), & y \circ x \circ y^{-1} &= (\alpha_p(v) - (\alpha_{q'} - 1)w, q'), \end{aligned} \quad (5)$$

where $q' = p \circ q \circ p^{-1}$. An element (v, q) of B is an (s)-element if and only if $q \in S(Q)$, the subgroup $(\alpha_q - 1)V$ of V is finite, and the orbit $\alpha(Q)v = \{\alpha_p(v) : p \in Q\}$ is finite.

Proof. The group $(B, +)$ is the direct product of V and $(Q, +)$, and $(B, \circ)$ is the semidirect product of V by $(Q, \circ)$. For $z = (w', p')$ both $x \circ (y + z)$ and $x \circ y - x + x \circ z$ are equal to $(v + \alpha_q(w) + \alpha_q(w'), q \circ p - q + q \circ p')$, because V is abelian. The first three formulas of (5) follow from the definitions, and the fourth from $y^{-1} = (-\alpha_{p^{-1}}(w), p^{-1})$ and $y \circ x = (w + \alpha_p(v), p \circ q)$. Hence

$$T_x = (\alpha_q - 1)V \times T_q, \quad [x]_+ = \{v\} \times [q]_+, \quad [x]_\lambda = \{(\alpha_p(v), \lambda_p(q)) : p \in Q\}.$$

If $x \in S(B)$, then the projections of the four families of x show that $q \in S(Q)$ (for $[q]_\circ$ take $w = 0$), that $(\alpha_q - 1)V$ is finite, and that $\alpha(Q)v$ is finite. Conversely, assume the three conditions. Then T_x and $[x]_+$ are finite, $[x]_\lambda \subseteq \alpha(Q)v \times [q]_\lambda$ is finite, and $[x]_\circ$ is contained in the set of all $(u - d, q')$ with $u \in \alpha(Q)v$, $q' \in [q]_\circ$ and $d \in (\alpha_{q'} - 1)V$; this set is finite, because $(\alpha_{q'} - 1)V = \alpha_p((\alpha_q - 1)V)$ for $q' = p \circ q \circ p^{-1}$. So $x \in S(B)$ by Lemma 2.1. $\square$

Proof of Theorem 1.5. In Lemma 6.1 we have $S(Q) = Q$, and all orbits $\alpha(Q)v$ are finite, because Q is finite. The group $(\alpha_q - 1)V$ is a subgroup of the torsion-free group V , so it is finite if and only if $\alpha_q = 1$, that is, $q \in M$. Hence $S(B) = V \times M$. By (4) and (5), $V \times M$ is a subgroup of $(B, +)$, is λ -invariant, or is invariant under conjugation in $(B, +)$, if and only if M has the same property in Q . As $V \times M$ is a normal subgroup of $(B, \circ)$ and M a normal subgroup of $(Q, \circ)$, the statement on ideals follows. For $V = \mathbb{Z}[\Gamma]$, $\Gamma = (Q, \circ)/M$, the action of q by left multiplication with its coset is a homomorphism $(Q, \circ) \rightarrow \text{Aut}(V)$ with kernel M , and V is free abelian. $\square$

Example 6.2 (The brace B_A and its variants). For $Q = Q_4$, $V = \mathbb{Z}$ and $\alpha = \varepsilon$ we get B_A , and Theorem 1.5 gives $S(B_A) = \mathbb{Z} \times \{0, 1\}$ again: the kernel $M = \{0, 1\}$ of ε is not a subgroup of $(\mathbb{Z}/4, +)$, and it is not λ -invariant, as $\lambda_1(1) = 3$. If $\mathbb{Z}$ is replaced by an abelian group V such that $2V$ is infinite, on which $\varepsilon(a)$ acts as ± 1 , then Lemma 6.1 gives $S = V \times \{0, 1\}$; for an infinite elementary abelian 3-group V both groups of this brace are locally finite of finite exponent. On the other hand $a \mapsto (-1)^a$ is a homomorphism of $(\mathbb{Z}/4, \circ)$ as well, whose kernel $\{0, 2\} = \text{Soc}(Q_4)$ is an ideal of Q_4 ; for the corresponding semidirect product the set $S = \mathbb{Z} \times \{0, 2\}$ is an ideal.

Example 6.3 (A normal subgroup of both groups that is not λ -invariant). Let $R = \mathbb{F}_2x \oplus \mathbb{F}_2y \oplus \mathbb{F}_2xy$ be the commutative ring without identity in which $x^2 = y^2 = 0$, and let Q be the set R with the addition of R and $u \circ u' = u + u' + uu'$. The left distributive law follows from the distributivity in R ; the operation $\circ$ is associative and commutative with neutral element 0, and $u \circ u = 2u + u^2 = 0$. So Q is a two-sided brace of order 8 with $(Q, \circ) \cong C_2^3$. For $u = ax + by + cxy$ let α_u be the diagonal matrix with entries $(-1)^b$ and $(-1)^{c+ab}$, acting on $V = \mathbb{Z}^2$. Since $u \circ u' = (a+a')x + (b+b')y + (c+c'+ab'+a'b)xy$ and $(c+c'+ab'+a'b) + (a+a')(b+b') \equiv (c+ab) + (c'+a'b') \pmod{2}$, the map α is a homomorphism of $(Q, \circ)$; its kernel is $M = \{0, x\}$. Let $B_B = \mathbb{Z}^2 \rtimes_\alpha Q$, a brace with additive group $\mathbb{Z}^2 \times (\mathbb{Z}/2)^3$. By Theorem 1.5, $S(B_B) = \mathbb{Z}^2 \times \{0, x\}$. This is a subgroup of the abelian group $(B_B, +)$ and a normal subgroup of $(B_B, \circ)$, but it is not λ -invariant: $\lambda_y(x) = x + xy \notin M$, so $\lambda_{(0,y)}(0, x) = (0, x + xy) \notin S(B_B)$.

Example 6.4 (A left ideal that is not normal in the additive group). Let $S_3 = \langle r, s \rangle$ with $r^3 = s^2 = 1$ and $srs = r^{-1}$, and let Q_6 be the set S_3 with the addition $p + q = pq$ and the multiplication $(r^i s^j) \circ q = r^i q s^j$. Then $(r^i s^j) \circ (r^k s^l) = r^{i+k} s^{j+l}$, so $(Q_6, \circ)$ is cyclic of order 6; and $u \circ (pq) = (u \circ p) u^{-1} (u \circ q)$, so Q_6 is a skew brace, with $\lambda_{r^i s^j}(q) = s^j q s^j$; it is the skew brace of the exact factorization $S_3 = \langle r \rangle \langle s \rangle$ in the sense of [GV17, Example 1.6]. Let ω be a primitive third root of unity, $V = \mathbb{Z}[\omega]$, and let $\alpha_{r^i s^j}$ be the multiplication by ω^i ; this is a homomorphism of $(Q_6, \circ)$ with kernel $M = \{1, s\}$. Let $B_C = \mathbb{Z}[\omega] \rtimes_\alpha Q_6$; its additive group is $\mathbb{Z}^2 \times S_3$. By Theorem 1.5, $S(B_C) = \mathbb{Z}[\omega] \times \{1, s\}$. The subgroup M of $(Q_6, +) = S_3$ is λ -invariant, since $\lambda_q(s) = s$ for all q , and it is not normal. So $S(B_C)$ is a left ideal of B_C which is not normal in $(B_C, +)$: $(0, r) + (0, s) - (0, r) = (0, rsr^{-1}) \notin S(B_C)$.

skew brace B	$(B, +)$	the set $S(B)$ is		
		a subgroup of $(B, +)$	normal in $(B, +)$	λ -invariant
B_A, C, B_U	abelian	no	–	no
B_B	abelian	yes	yes	no
B_C	not abelian	yes	no	yes
B_T	not abelian	yes	no	no

Table 1: The set $S(B)$ in the examples of Theorem 1.2 and of Examples 6.3 to 6.6. In all of them it is a normal subgroup of $(B, \circ)$.

Example 6.5 (Torsion-free additive group, non-abelian type). Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, a matrix of order 3, and let Q_T be the set $\mathbb{Z}^2 \times \mathbb{Z}$ with

$$(a, k) + (a', k') = (a + A^k a', k + k'), \quad (a, k) \circ (a', k') = (a + a', k + k').$$

The group $(Q_T, +) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is torsion-free and not abelian, and $(Q_T, \circ) = \mathbb{Z}^3$. Both sides of the left distributive law for (a, k) , (a', k') , (a'', k'') are equal to $(a + a' + A^{k'} a'', k + k' + k'')$; so Q_T is a skew brace (an instance of [GV17, Example 1.5]), with $\lambda_{(a, k)}(a', k') = (A^{-k} a', k')$. For $q = (a, k)$ and $y = (a', k')$,

$$q * y = ((A^{-k} - 1)a', 0), \quad y + q - y = (A^{k'} a + (1 - A^k)a', k), \quad y \circ q \circ y^{-1} = q.$$

As $A^{\pm 1} - 1$ has determinant 3, the element q is an (s)-element if and only if $3 \mid k$. So $S(Q_T) = \mathbb{Z}^2 \times 3\mathbb{Z}$, an ideal, in accordance with Theorem 1.4(a): Q_T is two-sided, because its multiplicative group is abelian.

The map $e(a, k) = (-1)^{a_1}$, where $a = (a_1, a_2)$, is a homomorphism $(Q_T, \circ) \rightarrow \{\pm 1\} = \text{Aut}(\mathbb{Z})$. Let $B_T = \mathbb{Z} \rtimes_e Q_T$. By Lemma 6.1,

$$S(B_T) = \{(v, (a, k)) : a_1 \text{ even}, 3 \mid k\}.$$

This is a subgroup of index 6 of $(B_T, +) = \mathbb{Z} \times (\mathbb{Z}^2 \rtimes_A \mathbb{Z})$, a torsion-free, finitely generated group. It is neither normal in $(B_T, +)$ nor λ -invariant: for $x = (0, ((0, 1), 0)) \in S(B_T)$ and $t = (0, ((0, 0), 1))$ we have $t + x - t = (0, ((-1, -1), 0))$ and $\lambda_t(x) = (0, ((1, 0), 0))$, and in both the coordinate a_1 is odd.

Example 6.6 (A centrally nilpotent brace of exponent 4). Let V be the $\mathbb{F}_2$ -vector space with basis e_i, f_i ($i = 1, 2, \dots$), E the span of the e_i , and β the automorphism of V with $\beta(e_i) = e_i$ and $\beta(f_i) = f_i + e_i$. Then $\beta^2 = 1$, $(\beta - 1)V = E$ and $(\beta - 1)E = 0$. Let $\alpha_a = 1$ for $a \in \{0, 1\}$ and $\alpha_a = \beta$ for $a \in \{2, 3\}$; by (A2) this is a homomorphism of $(\mathbb{Z}/4, \circ)$. Let $B_U = V \rtimes_{\alpha} Q_4$. By Lemma 6.1, $S(B_U) = V \times \{0, 1\}$, because $(\alpha_a - 1)V$ is the infinite group E for $a \in \{2, 3\}$ and is zero for $a \in \{0, 1\}$, and all orbits $\alpha(Q_4)v$ have at most two elements. Again $(0, 1) \in S(B_U)$, while $(0, 1) + (0, 1) = (0, 2)$ and $\lambda_{(0, 1)}(0, 1) = (0, 3)$ are not (s)-elements.

By (5), $x * y = ((\alpha_a - 1)w, 2ab)$ for $x = (v, a)$, $y = (w, b)$. In the notation of Remark 3.3 this gives $B^2 = B^{(2)} = E \times \{0, 2\}$, then $B^3 = B * B^2 = 0$, and $B^{(3)} = E \times \{0\}$, $B^{(4)} = 0$: the brace B_U is left and right nilpotent. Moreover $\text{Ker}(\lambda) = V \times \{0\}$ and $Z(B_U, \circ) = E \times \{0, 1\}$, so the annihilator $\text{Ann}(B_U) = \text{Soc}(B_U) \cap Z(B_U, \circ)$ is $E \times \{0\}$; the quotient by this ideal is the direct product of the trivial brace V/E and Q_4 , with annihilator $(V/E) \times \{0, 2\}$, and the next quotient is a trivial brace of order 2. So the annihilator series $0 \subset E \times \{0\} \subset V \times \{0, 2\} \subset B_U$ reaches B_U : this brace is centrally nilpotent. Both of its groups have exponent 4.

Table 1 summarises the examples. By Theorem 1.3, a λ -invariant set $S(B)$ is a subgroup of $(B, +)$; so, apart from the case of an ideal, the last three lines show all combinations that can occur when $S(B)$ is a subgroup of $(B, +)$.

7 A solution of the Yang–Baxter equation

In this section we prove Theorem 1.6. We use the conventions of [GV17, CFT25]. For a skew brace B the map

$$r_B(a, b) = (\lambda_a(b), \lambda_a(b)^{-1} \circ a \circ b) = (u, \lambda_u^{-1}(-u + a + u)), \quad u = \lambda_a(b),$$

is a non-degenerate solution of the Yang–Baxter equation [GV17, Theorem 3.1]; the second expression follows from $u \circ z = u + \lambda_u(z)$ and $a \circ b = a + u$. If $X \subseteq B$ is invariant under all maps λ_a and under all conjugations of $(B, +)$, then r_B restricts to a non-degenerate solution on X [GV17, Corollary 3.3]. For a non-degenerate solution (X, r) , $r(x, y) = (\sigma_x(y), \tau_y(x))$, the *structure group* is $G(X, r) = \langle X \mid x \circ y = \sigma_x(y) \circ \tau_y(x) \rangle$, with canonical map $\iota: X \rightarrow G(X, r)$.

Theorem 7.1 ([GV17, Theorem 3.9]). *There is a unique skew brace structure on $G = G(X, r)$, with multiplicative group $G(X, r)$, such that $r_G(\iota \times \iota) = (\iota \times \iota)r$. If B is a skew brace and $f: X \rightarrow B$ a map with $(f \times f)r = r_B(f \times f)$, then there is a unique homomorphism $\phi: (G, \circ) \rightarrow (B, \circ)$ with $\phi\iota = f$ and $(\phi \times \phi)r_G = r_B(\phi \times \phi)$.*

The homomorphism ϕ of Theorem 7.1 is a homomorphism of skew braces. Indeed, the first coordinates of the last identity give $\phi(\lambda_a(b)) = \lambda_{\phi(a)}(\phi(b))$, and therefore $\phi(a + b) = \phi(a \circ \lambda_{a^{-1}}(b)) = \phi(a) \circ \lambda_{\phi(a)^{-1}}(\phi(b)) = \phi(a) + \phi(b)$ by (1).

Proof of Theorem 1.6. Step 1: the solution. In the skew brace $B_C = \mathbb{Z}[\omega] \rtimes_{\alpha} Q_6$ of Example 6.4 let

$$a_k = (0, r^k s), \quad b_k = (\omega^k, 1) \quad (k \in \mathbb{Z}/3).$$

By (5), $\lambda_{(v, r^i s^j)}(w, p) = (\omega^i w, s^j p s^j)$ and $(w, p) + (v, q) - (w, p) = (v, p q p^{-1})$. So the set X of these six elements is invariant under all λ_a and all conjugations of $(B_C, +)$: reflections are mapped to reflections, and the elements b_k are central in $(B_C, +)$ and are permuted by the maps λ_a . Hence r_{B_C} restricts to a non-degenerate solution on X , which is bijective, as X is finite. We compute it. First, $\lambda_{a_k}(a_l) = (0, s r^l s s) = a_{-l}$, $\lambda_{a_k}(b_l) = b_{k+l}$ and $\lambda_{b_k} = \text{id}$. For the pair (a_k, a_l) we get $u = a_{-l}$, $-u + a_k + u = a_{l-k}$ and $\lambda_u^{-1}(a_{l-k}) = a_{k-l}$; for (a_k, b_l) , $u = b_{k+l}$ is central in $(B_C, +)$ and $\lambda_u = \text{id}$; for (b_k, a_l) , $u = a_l$ and $\lambda_{a_l}^{-1}(b_k) = b_{k-l}$; for (b_k, b_l) , $u = b_l$. This is the map r of the theorem. It is not involutive: $r^2(a_0, a_1) = r(a_2, a_2) = (a_1, a_0)$. We note three facts:

$$\sigma_u(\tau_y(x)) = -u + x + u \quad \text{for } u = \sigma_x(y), \quad \sigma_{a_0}(x) = -a_0 + x + a_0, \quad r(x, a_0) = (a_0, x) \quad (6)$$

for all $x, y \in X$, where the sums are computed in B_C . The first holds because $\tau_y(x) = \lambda_u^{-1}(-u + x + u)$ and $\sigma_u = \lambda_u$ on X ; the other two are read off from the formulas for r , as $-a_0 + a_l + a_0 = a_{-l}$. Finally X generates $(B_C, +)$, since $b_0 = (1, 1)$ and $b_1 = (\omega, 1)$ generate $\mathbb{Z}[\omega] \times \{1\}$ and the reflections generate S_3 .

Step 2: a surjection. By Theorem 7.1 and the remark after it, applied to the inclusion $f: X \rightarrow B_C$, there is a homomorphism of skew braces $\pi: G \rightarrow B_C$ with $\pi(\iota(x)) = x$. It is surjective, because its image is a subgroup of $(B_C, +)$ containing X .

Step 3: the additive group of G . Let $x, y \in X$ and $u = \sigma_x(y)$. As $\lambda_{\iota(x)}(\iota(y)) = \iota(\sigma_x(y))$, the defining relation $\iota(x) \circ \iota(y) = \iota(u) \circ \iota(\tau_y(x))$ reads $\iota(x) + \iota(u) = \iota(u) + \iota(\sigma_u(\tau_y(x)))$. Since σ_x is a bijection of X , (6) gives, for all $x, u \in X$,

$$-\iota(u) + \iota(x) + \iota(u) = \iota(-u + x + u). \quad (7)$$

Every $\lambda_{\iota(x)}$ maps the finite set $\iota(X)$ injectively into itself, hence permutes it; so $\lambda_g(\iota(X)) = \iota(X)$ for all $g \in G$. Then the subgroup of $(G, +)$ generated by $\iota(X)$ is λ -invariant, hence a subgroup of $(G, \circ)$

which contains the generators of $(G, \circ)$: the set $\iota(X)$ generates $(G, +)$. By (7), conjugation by $\iota(u)$ in $(G, +)$ maps $\iota(X)$ injectively into itself, hence permutes it; so $\iota(X)$ is invariant under all conjugations of $(G, +)$.

Step 4: $g_0 = \iota(a_0)$ is an (s)-element. By Step 3 the sets $[g_0]_+$ and $[g_0]_\lambda$ lie in the finite set $\iota(X)$. By (6) and (7), $\lambda_{g_0}(\iota(x)) = \iota(\sigma_{a_0}(x)) = -g_0 + \iota(x) + g_0$ for $x \in X$. Two automorphisms of $(G, +)$ which agree on generators are equal, so $\lambda_{g_0}(a) = -g_0 + a + g_0$ for all $a \in G$, and $g_0 * a = -g_0 + (a + g_0 - a) \in -g_0 + \iota(X)$: the set T_{g_0} is finite. Finally, by (6) the defining relations of $G(X, r)$ contain $x \circ a_0 = a_0 \circ x$ for all $x \in X$; so g_0 is central in $(G, \circ)$ and $[g_0]_\circ = \{g_0\}$. By Lemma 2.1, $g_0 \in S(G)$.

Step 5: $z = -\iota(a_1) + g_0 + \iota(a_1)$ is not an (s)-element. By (7), $z = \iota(-a_1 + a_0 + a_1) = \iota(a_2)$. As π is a surjective homomorphism of skew braces, the four families of $\pi(z) = a_2$ in B_C are the images of those of z . If z were an (s)-element, then $a_2 = (0, r^2s)$ would be one, which contradicts $S(B_C) = \mathbb{Z}[\omega] \times \{1, s\}$. $\square$

This agrees with [CFT25, Theorem 4.5], by which the structure skew brace of a finite solution with indecomposable derived solution has a uniform version of property (S): by (6) the derived solution of (X, r) is $(x, u) \mapsto (u, -u + x + u)$, and it is decomposable, with the orbits $\{a_0, a_1, a_2\}$, $\{b_0\}$, $\{b_1\}$, $\{b_2\}$.

8 Additive groups generated by finitely many (s)-elements

By Theorem 1.3, a skew brace whose multiplicative group is generated by (s)-elements has property (S). For the additive group this is not so: $(B_A, +)$ is generated by the (s)-elements $(1, 0)$ and $(0, 1)$. The following statement is what remains true. Recall that $\text{Ann}(B) = \text{Soc}(B) \cap Z(B, \circ)$ is an ideal of B .

Proposition 8.1. *Let B be a skew brace whose additive group is generated by finitely many elements $x_1, \dots, x_n$. Assume that the sets $[x_i]_+$ and $[x_i]_\lambda$ are finite for $i = 1, \dots, n$, and that T_t is finite for every $t \in B$. Then $B/\text{Ann}(B)$ is finite. In particular this holds*

- (a) *if B has property (S) and $(B, +)$ is finitely generated;*
- (b) *if $(B, +)$ is finitely generated and both $B^{(2)}$ and the commutator subgroup $[B, B]_+$ of $(B, +)$ are finite.*

Proof. Let $K = \text{Ker}(\lambda)$. An automorphism of $(B, +)$ which fixes $x_1, \dots, x_n$ is the identity; so $K = \bigcap_i \text{Fix}^l(x_i)$, and K has finite index in $(B, \circ)$ by Lemma 2.1. Likewise $Z(B, +) = \bigcap_i C_{x_i}^+$ has finite index in $(B, +)$. Let $t_1, \dots, t_m$ be representatives of the cosets of K in $(B, \circ)$. Every map λ_c is one of the maps λ_{t_j} , so

$$F := \{b \in B : \lambda_c(b) = b \text{ for all } c \in B\} = \bigcap_j \text{Fix}^r(t_j)$$

is a subgroup of finite index of $(B, +)$, the sets T_{t_j} being finite. Hence $N_0 = Z(B, +) \cap F$ has finite index in $(B, +)$. For $a, b \in N_0$ we have $a \circ b = a + \lambda_a(b) = a + b$ and $a^{-1} = \lambda_{a^{-1}}(-a) = -a$; so N_0 is a subgroup of $(B, \circ)$ on which the two operations coincide. Therefore $K \cap N_0$, which has finite index in $(N_0, \circ)$, has finite index in $(N_0, +)$ and hence in $(B, +)$. Finally $K \cap N_0 = \text{Ann}(B)$: an element $b \in K \cap Z(B, +)$ satisfies $b \circ c = b + c$ and $c \circ b = c + \lambda_c(b)$ for all c , so it is central in $(B, \circ)$ if and only if $b \in F$.

In case (a) all the sets in question are finite. In case (b), $T_t \subseteq B^{(2)}$, $[x]_\lambda \subseteq B^{(2)} + x$ and $[x]_+ \subseteq [B, B]_+ + x$ for all $t, x \in B$. $\square$

Here $B^{(2)}$ is the subgroup of $(B, +)$ generated by all $a * b$, as in Remark 3.3. Statement (a) is [CFT25, Corollary 3.14], which is derived there from the lemma discussed in Remark 8.2; the proof above does not use that lemma. It is also contained in [DF26, Corollary 5.8]. Both are stated for skew braces with property (S) that are finitely generated as skew braces; these have a finitely generated additive group by [CFT25, Theorem 3.13]. Statement (b) is the second half of [JKVV21, Theorem 5.4].

Remark 8.2 (On Lemma 3.12 of [CFT25]). Lemma 3.12 of [CFT25] (p. 178 of the journal version; Lemma 3.8 in the first arXiv version) asserts the conclusion of Proposition 8.1, that $B/\text{Ann}(B)$ is finite, for every skew brace B whose additive group is generated by finitely many (s)-elements $x_1, \dots, x_n$; nothing is assumed about the other elements of B . In this form the statement does not hold: the brace B_A satisfies the hypothesis and not the conclusion.

The verification. The group $(B_A, +) = \mathbb{Z} \times \mathbb{Z}/4$ is generated by $x_1 = (1, 0)$ and $x_2 = (0, 1)$, which are (s)-elements by Theorem 1.2. By Remark 3.3, $\text{Soc}(B_A) = \mathbb{Z} \times \{0\}$. An element (m, a) is central in $(B_A, \circ)$ if and only if $m + \varepsilon(a)n = n + \varepsilon(b)m$ for all (n, b) , that is, if and only if $\varepsilon(a) = 1$ and $m = 0$ (take $b = 0$, then $b = 2$). So $Z(B_A, \circ) = \{(0, 0), (0, 1)\}$, $\text{Ann}(B_A) = 0$, and the quotient $B_A/\text{Ann}(B_A) = B_A$ is infinite. The same holds for the sub-brace C of Theorem 1.2: the group $(C, +)$ is generated by the (s)-elements $(1, 1)$ and $(3, 1)$, since $(3, 1) - 3(1, 1) = (0, 2)$, and $\text{Ann}(C) = 0$, because $(C, \circ) \cong D_\infty$ has trivial centre.

The step of the proof. The proof in [CFT25] considers the map f which sends the coset $b + \text{Ann}(B)$ to the tuple of the $3n$ elements $b * x_i$, $x_i * b$ and $b + x_i - b - x_i$, and refers for its injectivity to the second half of the proof of [JKVV21, Theorem 5.4]. For B_A this map is not injective: by (2) its value at (m, a) depends only on a , so that, for instance, f takes the same value at $(1, 0)$ and at $(0, 0)$, although $(1, 0) \notin \text{Ann}(B_A)$. The argument referred to, which is the same in the journal version and in the arXiv version of [JKVV21], runs as follows: if f takes the same value at a and at b , then $x_i * (-a + b) = 0$ for all i , and from this it is concluded that $d = -a + b$ is fixed by all maps λ_c . This inference is valid if the x_i generate $(B, \circ)$, because $c \mapsto \lambda_c$ is a homomorphism of $(B, \circ)$; for generators of $(B, +)$ it needs an additional hypothesis, since the set $\{c : c * d = 0\} = \text{Fix}^l(d)$ is a subgroup of $(B, \circ)$ and in general not of $(B, +)$. In B_A , for $d = (1, 0)$, we have $x_1 * d = x_2 * d = 0$, but $(x_2 + x_2) * d = (0, 2) * (1, 0) = (-2, 0)$.

What holds, and the uses of the lemma. By Proposition 8.1(a) the conclusion of the lemma holds for skew braces with property (S). This is the case in which the lemma is applied in [CFT25]. It is cited there in the proof of Theorem 3.13, in the proof of Corollary 3.14 (which is repeated as Corollary 3.17) and in the proof of Corollary 3.24; in these places it is applied to a skew brace with property (S) whose additive group is finitely generated (in the last one, to a finitely generated sub-skew brace of a skew brace with property (S), to which [CFT25, Theorem 3.13] applies). In the proof of Theorem 3.20 it is cited for a skew brace C which is finite over $\text{Ann}(C)$, in order to conclude that $C^{(2)}$ is finite; this is the first half of [JKVV21, Theorem 5.4], whose proof does not involve the map f . The lemma is also mentioned, without being used, before Theorem 3.37. These are all the places where the lemma occurs in [CFT25]; they were checked in the journal version and in the second arXiv version, and nothing else in [CFT25] was checked with regard to this lemma. So, as far as checked, the results of [CFT25] which rely on the lemma are not affected.

The theorem of [JKVV21]. In [JKVV21] the inference discussed above occurs in the proof of the second half of Theorem 5.4 (pp. 16–17 of the journal version; the arXiv version has the same text), which states that $A/\text{Ann}(A)$ is finite if $A^{(2)}$ and $[A, A]_+$ are finite and $(A, +)$ is finitely generated. The theorem itself is not affected: its second half is Proposition 8.1(b), and the proof of its first half is independent of this step.

9 Computations and reproducibility

All theorems, propositions and lemmas of this note are proved without computation; the verifications (A1)–(A6) are the proof of Theorem 1.2. The computations are tests. They were made with four independent sets of programs, which form the reproducibility package together with their recorded outputs: the original programs (`original/`) and those of the verification runs A, B and 2 described below (`verification_run_A/`, `verification_run_B/`, `independent_run_2/`). Every number below is taken from these outputs. A skew brace structure on a fixed group is called *labelled*; isomorphic structures on the same group are counted separately.

Enumeration. Each set contains its own enumeration of small skew braces, and they agree: there are 477 labelled skew braces on the groups of order 2 to 12, in 1, 1, 4, 1, 6, 1, 47, 4, 6, 1, 38 isomorphism classes for the orders 2, $\dots$, 12, of which 1, 1, 4, 1, 2, 1, 27, 4, 2, 1, 10 are of abelian type. The programs of run B also find the 498 labelled skew braces of order at most 15, in 119 classes, and the 357 classes of braces of order 16. The numbers of classes are those of the tables in [GV17, Section 5].

Identities. The original programs test the identities of Lemmas 2.2 and 2.3 and the two inclusions behind the last step of the proof of Theorem 1.3 on the 477 labelled skew braces, and Lemma 5.1, the displays in Steps 2 and 4 of the proof of Theorem 1.4(a) and the inclusions of Steps 1, 2 and 4 on the 313 two-sided ones, of which 110 have a non-abelian additive group. As a control, these inclusions fail on 130 of the 164 labelled skew braces that are not two-sided. The programs of run A repeat these tests with a second library, and test the identities used in the proofs of Proposition 4.3 and Theorem 1.4(c); of the 477 labelled skew braces 259 are bi-skew braces, and 235 are both bi-skew and two-sided. The programs of run B test the identities, Lemmas 2.1 and 2.3 (with the conjugacy classes of G computed by brute force) and quantitative forms of the last step of the proof of Theorem 1.3 on the $498 + 357 = 855$ skew braces mentioned above; as a control, the equality which would make S_1 normal in $(B, \circ)$ fails on 147 of the 498 and on 42 of the 357. Of these skew braces $333 + 221$ are two-sided; Lemma 5.1 and the inclusions hold on them, and (T1), (T2) fail on each of the others. Run 2 repeats the tests on the 477 labelled skew braces with a fourth library (the same numbers 313, 110, 259, 235) and tests the identities of the proof of Proposition 8.1.

The examples. For B_A , B_B and B_C the original programs and those of run A verify the axioms as identities between integer linear forms for all 64, 512 and 216 triples of Q -coordinates, which is an exact verification, and determine exactly the set of q for which (v, q) is an (s)-element. Run A treats B_T in the same way (identities between affine forms for all $12^3 = 1728$ triples of residue classes of (a_1, a_2, k) modulo $(2, 2, 3)$), computes the series of Example 6.6 in truncations of B_U of order 16, 64, 256 and 1024, and evaluates the map f of Remark 8.2 in the braces $\mathbb{Z}/N \times \mathbb{Z}/4$ for $N = 3, 5, 9, 15$: it takes four values, while the annihilator is zero. Run B verifies the axioms of B_A symbolically, determines the four subgroups and the two indices of Definition 1.1 exactly for all types of elements, and computes the indices in the quotients $\mathbb{Z}/N \times \mathbb{Z}/4$ for 28 values of N up to 60. Run 2 repeats these verifications exactly, with programs of its own, for B_A , C , B_B , B_C and B_T , recomputes the statements of Remarks 3.1 to 3.3 and 8.2 about B_A and C , and treats Example 6.6 in truncations of order 16, 64, 256. All four sets verify the finite data of Step 1 of the proof of Theorem 1.6: the invariance of X , the table of r , the braid relation on the 216 triples, and (6).

The pairs (Q, M) . All four sets list the pairs of a labelled skew brace Q of order 2 to 12 and a normal subgroup M of $(Q, \circ)$: there are 3139 pairs, and in 962 of them M is not an ideal of Q . For the 2271 pairs with $|Q| \leq 8$ the original programs and those of run A build $\mathbb{Z}[\Gamma] \rtimes Q$ and confirm $S = \mathbb{Z}[\Gamma] \times M$ by an exact computation. Runs B and 2 decide membership in $S(\mathbb{Z}[\Gamma] \rtimes Q)$ exactly for all 3139 pairs, and run B also for the 5801 pairs coming from the 357 braces of order 16, in 2274 of which M is not an ideal: in every case (v, q) is an (s)-element exactly if $q \in M$. In B_A and in the 3139 semidirect products run 2 also determines the set FCI of Remark 4.4; it is the set of (s)-elements. According to these surveys, a pair in which M is not a subgroup of $(Q, +)$ first occurs for order 4,

and there only for $Q = Q_4$ with $M = \{0, 1\}$ or $\{0, 3\}$; a left ideal M that is not normal in $(Q, +)$, and a subgroup M of $(Q, +)$ that is neither normal nor λ -invariant, first occur for order 6; and a normal subgroup M of an abelian group $(Q, +)$ that is not λ -invariant first occurs for order 8. So Examples 6.2 to 6.4 use finite skew braces of the smallest possible order.

Runs A and B ran the original programs again, and all programs were run again from the folder layout of the package, the last time by run 2 from an extracted copy of the archive; the outputs were identical to the recorded ones, apart from lines which record running times.

Verification. The results were first obtained with complete proofs and with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. Runs A and B examined the first written version of the results, and neither used the report of the other. Run 2 examined the text of this note.

Run A compared the question, the definitions and the numbering with [Col23] and with the two arXiv versions and the journal version of [CFT25]; re-derived the proof of Theorem 1.2 from the definitions and examined the conventions of Remark 3.2; re-derived line by line the proofs of Lemmas 2.1 to 2.3, of Theorems 1.3, 1.5 and 1.6 and of Theorem 1.4(a), (b); and compared the cited statements of [GV17] with the arXiv version. It found no error. It required one correction, namely the comparison with [CFT25, Lemma 3.12], which was missing. It contributed Theorem 1.4(c), (d) with the proof of Proposition 4.3 given here, Examples 6.5 and 6.6, and the statements on nilpotency and solubility in Remark 3.3.

Run B re-derived Theorem 1.2 from its statement, with exact programs of its own, before it read the original programs; re-derived the other proofs; made the computations described above; and examined the literature. Independently of run A it found that B_A does not satisfy the conclusion of [CFT25, Lemma 3.12], and it proved Proposition 8.1(a). It found the derivation of Remark 5.2. Like run A it required the comparison with that lemma; it also required that the relation of Theorem 1.3 and of Theorem 1.4(a) to [CPV26, DF26] and the limits of the literature search be stated, and two small corrections of the programs. All corrections required by the two runs were made.

When the note was written, four things were added after these two runs: the proof of Lemma 4.2; the formulation of Proposition 8.1 with hypotheses on the generators and on the sets T_t , and its case (b); the first paragraph of Section 8; and, in Remark 8.2, the list of the places where [CFT25] cites its Lemma 3.12. All proofs were written out again for the note.

Run 2 examined the text of the note as a whole, and these four additions first. It fetched the sources again and compared the question, the definitions and every statement quoted from [Col23, CFT25, CPV26, DF26, GV17, JKVV21, MY27] with them; read the second arXiv version of [CFT25] in full; and re-derived all proofs line by line, among them those of Lemma 4.2, Proposition 4.3, Theorem 1.4(c), (d) and Proposition 8.1 and the computations of Examples 6.5 and 6.6, which run B had not examined. It checked every part of Remark 8.2: the lemma and its proof in the journal version and in both arXiv versions of [CFT25], the verification for B_A and C , the step of the proof in the arXiv version of [JKVV21], and each place where [CFT25] cites the lemma. It recomputed the finite claims with programs of its own and repeated the literature search. It found no wrong theorem, proposition, lemma or example, and no gap in a proof. It required ten corrections, which were made. They concern the relation to the literature, the wording and the package: the comparison with [MY27]; the remark that the proof of Proposition 4.3 is a variant of the one in [DF26]; the statements on the literature search, since this run obtained the list of the documents citing [CFT25] in zbMATH; and six sentences which were inexact or ambiguous. The equality $\text{FCI}(B) = S(B)$ of Remark 4.4 was found and proved by run 2 and has not been examined by another run; it was tested as described above. After run 2 the proof of Theorem 5.4 in the journal version of [JKVV21] was compared with the arXiv version, with which it agrees, and the sentences on this paper in Remark 8.2

and in the paragraphs on the sources were changed accordingly; these changes of wording were not examined by a verification run. No proof depends on a computation.

Scope and priority. The note answers Question 1 of [Col23] in the negative, for skew braces and already for braces, and gives a positive answer in the classes of Theorem 1.4. Questions 2 and 3 of [Col23] are not treated. We do not know whether the answer is positive for finitely generated left nilpotent skew braces; we do not treat λ -homomorphic skew braces; and we did not determine the set $S(G(X, r))$ in Theorem 1.6.

The question, the notion of an (s)-element and Theorem 1.4(a) for braces are due to Colazzo, Ferrara and Trombetti [Col23, CFT25]; the brace Q_4 is [CFT25, Example 3.8]. The semidirect product (4) is a known construction ([CPV26, Definition 2.11], [DF26, Section 4.3]; [GV17, Example 1.4] for trivial Q). That $S(B)$ is a subgroup of $(B, \circ)$ follows at once from [DF26, Corollary 5.4] with [CPV26, Proposition 5.8], and also from Proposition 2.3 of [MY27] with the equality of Remark 4.4; Theorem 1.4(a) for skew braces of non-abelian type is a short consequence of the same corollary with [CPV26, Proposition 5.9 and Remark 2.4] (Remark 5.2). The second half of Theorem 1.4(b) is [CPV26, Theorem 5.22 and Corollary 5.23]; Proposition 4.3 is [DF26, Corollary 5.4]; and Proposition 8.1 is contained in [CFT25, Corollary 3.14], [DF26, Corollary 5.8] and [JKVV21, Theorem 5.4]. The proofs of these statements are included for completeness. The paper [DF26] is a preprint. In the sources listed below we did not find an answer to the question, the brace B_A or another example, the normality of $S(B)$ in $(B, \circ)$, the statements of Theorem 1.4(c), (d) (but see Remark 4.4 for bi-skew braces) or of Theorems 1.5 and 1.6, the equality of Remark 4.4, or a discussion of [CFT25, Lemma 3.12].

What was read. The abstract [Col23], in the publisher's file. Of [CFT25]: the second arXiv version in full; the journal version in Sections 1 to 3 and at every place where Lemma 3.12 is cited, with the definitions and statements of Section 4; the first arXiv version for the numbering and for the passages on (s)-elements; its proofs were not checked, apart from the points discussed in Remark 8.2. Of [CPV26]: Section 2 and Section 5 up to Corollary 5.23 in the second arXiv version, whose numbering we use; the journal version is not open access and was not read. Of [DF26], in its only version: Sections 1 to 5, with the proofs of Proposition 5.1 and Theorem 5.2. Of [GV17]: Sections 1 to 3 and the tables of Section 5 in the third arXiv version, whose numbering we use; the journal version was not compared, and the proof of Theorem 3.9, which rests on a construction of Lu, Yan and Zhu, was not checked. Of [JKVV21]: Theorem 5.4 and its proof, in the journal version and in the arXiv version, which agree in this passage. Of [MY27], in the arXiv version: Section 2 from the definition of the brace centraliser to Proposition 2.4; of the statements quoted in Remark 4.4 only the proof of the first was checked. We did not read [Neu54, Chi19]; the lemma of Neumann is proved in the text, and from [Chi19] only the definition is used. The bibliographic data of all references were checked with Crossref, the arXiv listing or zbMATH.

The searches were made on 10 and 11 October 2026. When the results were first obtained: queries to the arXiv listing, the lists of works citing [CFT25] in OpenAlex (7 works) and in Semantic Scholar (21), two queries to zbMATH and two web searches; the sources of [CPV26, DF26, MY27] and of eight further recent preprints were searched for the relevant terms. Run A made three web searches, one of them for an erratum to [CFT25]. Run B searched the sources of 19 arXiv papers (the 16 citing papers with an arXiv identifier and 3 related ones) and of five later papers of the authors of [CFT25] for the relevant terms and read every hit, went through the titles of the 150 most recent arXiv entries on skew braces, read the zbMATH review of [CFT25], and made three web searches; one more was made at the writing. Run 2 repeated the queries to the arXiv listing (among them the 22 entries on skew braces since July 2026), to OpenAlex and to Semantic Scholar, with the same lists of citing works; obtained the list of the 9 documents citing [CFT25] in zbMATH, all of them papers whose sources were searched; and made one web search. No source that answers the question was found, and

no erratum to [CFT25]. The limits are these: the citing paper [BEF⁺26], on the word and conjugacy problems, was seen in abstract only, because its full text was not accessible to us; one item of the Semantic Scholar list, lecture notes without an identifier, could not be located; the journal versions of [GV17, CPV26, MY27] were not read; and preprints not yet indexed may have been missed. A search that finds nothing is not a proof of novelty; no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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