

A Kupisch Criterion for Higher Auslander Koszul Duals

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Abstract

Let B be a connected nonsemisimple split basic Nakayama algebra, with projective lengths c_i and injective lengths d_i indexed by the same simple modules. For its Auslander algebra A and Koszul dual E , we prove

$$\mathrm{gldim} E = \max_i (c_i + d_i - 2), \quad \mathrm{domdim} E = \min_i (c_i + d_i - 2).$$

Consequently, the higher Auslander condition is exactly constancy of $c_i + d_i$, apart from the dimension-one convention. The proof combines an explicit graded morphism calculation with the Auslander–Koszul correspondence. For connected linear Nakayama algebras, constancy holds precisely in the hereditary case. For cyclic algebras it also holds in nonselfinjective families, including the two-simple series $(2r + 1, 2r + 2)$, $r \geq 1$. In particular the series $(3, 4)$ gives global and dominant dimension 5; we explain a discrepancy with an example in an earlier preprint. This classifies the Nakayama subclass of the broader representation-finite question, which remains open outside this subclass.

1 Statement and conventions

All algebras and modules are finite dimensional over a field k , and modules are right modules. We work with split basic Nakayama algebras presented by monomial relations on a linearly oriented line or cycle. In particular, this covers basic connected Nakayama algebras over an algebraically closed field. Write $J = \mathrm{rad} B$, $P_i = e_i B$, $S_i = P_i / P_i J$, and $I_i = D(Be_i)$, where $D = \mathrm{Hom}_k(-, k)$. Our orientation convention is that the successive composition factors of P_i are $S_i, S_{i+1}, \dots, S_{i+c_i-1}$. Put

$$c_i = \mathrm{LL}(P_i), \quad d_i = \mathrm{LL}(I_i).$$

In the cyclic case, indices are modulo the number n of simples. In the linear case, indices range from 1 to n , $c_n = 1$, and $c_i \geq 2$ for $i < n$. In both cases $c_{i+1} \geq c_i - 1$ whenever relevant.

Let

$$M = \bigoplus_{i=1}^n \bigoplus_{l=1}^{c_i} M(i, l), \quad M(i, l) = P_i / P_i J^l, \quad A = \mathrm{End}_B(M).$$

We use the convention in which $P_X = \mathrm{Hom}_B(M, X)$ is a right projective A -module and $I_X = D\mathrm{Hom}_B(X, M)$ is the corresponding right injective. Let E be the Koszul dual of A for its radical grading. Changing the convention for multiplication replaces a dual by its opposite and does not change the dimensions below. Dominant dimension is the first index of a nonprojective term in a minimal injective coresolution of the regular module. A q -Auslander algebra satisfies $\mathrm{gldim} E \leq q + 1 \leq \mathrm{domdim} E$.

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Theorem 1. *For the connected nonsemisimple algebra B above, A is Koszul. Set $t_i = c_i + d_i - 2$. Then*

$$\text{gldim } E = \max_i t_i, \quad \text{domdim } E = \min_i t_i.$$

Thus $\text{gldim } E = \text{domdim } E$ if and only if $c_i + d_i$ is independent of i . If the common value is $h \geq 4$, then E is an $(h - 3)$ -Auslander algebra. The common value $h = 3$ gives global and dominant dimension 1.

The original question asks which representation-finite B have higher Auslander Koszul duals of their Auslander algebras [2, p. 463]. Chan–Iyama–Marcinzik formulate it in terms of stiffness [1, Question 4.21]. An algebra is ℓ -stiff when its Loewy length is ℓ and its indecomposable injectives of length ℓ are exactly its projective-injectives. Their general correspondence is an input here, not a new claim. The contribution of this note is the explicit Nakayama criterion, its dimension formula and its consequences. No classification of all representation-finite algebras is claimed.

2 Degrees of morphisms

Choose the standard path basis $x_0, \dots, x_{l-1}$ of $M(i, l)$. A basis for $\text{Hom}_B(M(i, l), M(j, m))$ consists of maps f_s with

$$f_s(x_0) = y_s, \quad 0 \leq s < m, \quad s + l \geq m, \quad j + s = i. \quad (1)$$

The last equality is modulo n in the cyclic case and an ordinary equality in the linear case. The map sends x_r to y_{s+r} when $s + r < m$, and otherwise to zero. Formula (1) follows by choosing the image of the top generator and imposing its annihilating path relation.

Lemma 2. *Define $\deg f_s = l - m + 2s$. The nonzero composite of two basis maps has the sum of their degrees. The degree is nonnegative, and it vanishes only for an identity. Moreover, a degree- a basis map is a composite of exactly a degree-one maps. Consequently, the r th categorical radical is spanned by basis maps of degree at least r .*

Proof. Composition adds shifts, so the degree formula is additive. Write $q = m - s$ for the image length. The map factors as the canonical quotient of $M(i, l)$ onto $M(i, q)$, followed by the inclusion of that module as $\text{rad}^s M(j, m)$. The quotient factors into $l - q$ quotients with simple kernel. The inclusion factors into s inclusions with simple cokernel. All intermediate modules are submodules or quotients of the stated modules, hence exist. Each step has degree one, and the number of steps is $l - q + s = l - m + 2s$. These quantities are nonnegative, with degree zero possible only for the identity. Thus the span of positive-degree maps is the categorical radical; a product of r such maps has degree at least r . The displayed factorization proves the reverse inclusion for its powers. $\square$

Lemma 3. *For every indecomposable $X = M(i, l)$,*

$$\text{LL}(\text{I}_X) = l + d_i - 1.$$

In particular,

$$\text{LL}(A) = \max_i (c_i + d_i - 1), \quad \text{LL}(\text{I}_{P_i}) = c_i + d_i - 1.$$

Proof. The Loewy length of $D \text{Hom}_B(X, M)$ is one plus the largest degree of a nonzero map from X to an indecomposable module. For a map (1), the quotient of its target of length $s + 1$ has socle S_i . It embeds into I_i , so $s + 1 \leq d_i$. Because $m - s \geq 1$,

$$\deg f_s = l + s - (m - s) \leq l + d_i - 2.$$

Equality is attained by the composite $X \twoheadrightarrow S_i \hookrightarrow I_i$, whose shift is $d_i - 1$ and whose image length is one. Lemma 2 computes its radical degree. Taking the maximum over $l \leq c_i$ proves the remaining assertions. $\square$

For completeness, the coKupisch lengths are directly computable:

$$d_i = \max\{t \geq 1 : c_{i-t+1} \geq t\}. \quad (2)$$

In a line, the set is restricted to $1 \leq i - t + 1 \leq n$; in a cycle, the subscripts are periodic. The maximum is bounded by $\max c_i$. Indeed, these are precisely the possible lengths of uniserial modules with socle S_i .

3 Koszulity and the dimension formula

Lemma 4. *The grading of Lemma 2 makes A Koszul. Its projective-injective modules are precisely $\mathsf{I}_{P_i} \simeq \mathsf{P}_{I_i}$.*

Proof. For a nonprojective $X = M(i, l)$, the almost split sequence is

$$0 \longrightarrow M(i+1, l) \longrightarrow M(i, l+1) \oplus M(i+1, l-1) \longrightarrow M(i, l) \longrightarrow 0,$$

where a length-zero summand is omitted. Every component map is a degree-one quotient or inclusion. Applying $\mathrm{Hom}_B(M, -)$ gives the minimal projective resolution of the simple A -module indexed by X , of length two. For $X = P_i$, the resolution is

$$0 \longrightarrow \mathsf{P}_{\mathrm{rad} P_i} \longrightarrow \mathsf{P}_{P_i} \longrightarrow \mathsf{S}_{P_i} \longrightarrow 0,$$

with its first term omitted if $\mathrm{rad} P_i = 0$. Thus all simples have linear resolutions. The description of the projective-injectives is the usual Auslander correspondence: the natural duality $D \mathrm{Hom}_B(P_i, -) \simeq \mathrm{Hom}_B(-, I_i)$ gives the displayed isomorphism, and these exhaust the projective-injectives. $\square$

We record explicitly the established transfer statement being used. For a Koszul Auslander algebra A and its dual E , the results of [1, Theorem 4.1 and Proposition 4.8] give

$$\mathrm{pdim}_E \mathsf{S}_X^! = \mathrm{LL}(\mathsf{I}_X) - 1, \quad (3)$$

$$\mathsf{I}_X^! \text{ is nonprojective} \iff \mathsf{I}_X \text{ is projective}. \quad (4)$$

For the indices in (4), the simple $\mathsf{S}_X^!$ has $\mathrm{Ext}_E^r(\mathsf{S}_X^!, E)$ nonzero in exactly the degree $\mathrm{LL}(\mathsf{I}_X) - 1$. These facts arise from graded Koszul duality and the Auslander–Gorenstein condition.

Proof of Theorem 1. Koszulity was proved in Lemma 4. Finite dimensionality of E also follows from $\mathrm{gldim} A = 2$. Equation (3) and Lemma 3 give

$$\mathrm{pdim}_E \mathsf{S}_{M(i,l)}^! = l + d_i - 2.$$

Taking the maximum over the simples gives the global dimension.

For dominant dimension, in a minimal injective coresolution of the regular E -module, the multiplicity of $\mathsf{I}_X^!$ in degree r is $\dim_k \mathrm{Ext}_E^r(\mathsf{S}_X^!, E)$. By (4) the only nonprojective injectives have indices $X = P_i$. For each such index, the preceding transfer statement says that its first and only occurrence is in degree $\mathrm{LL}(\mathsf{I}_{P_i}) - 1 = t_i$. Hence the first nonprojective term occurs in degree $\min_i t_i$. The remaining claims follow immediately. $\square$

Remark 5 (Version-specific dependency). Reference [1] means *version 2*, dated 31 October 2022. The current version 3 has a different title and omits the Koszul duality section; its arXiv comments state that those results will appear separately. We do not attribute the transfer statement to a section of version 3 that is absent. The elementary morphism and Loewy-length calculations above are independent of that transfer theorem. The small quadratic-dual resolutions in Section 5 provide a separate check of its application here.

4 Consequences

Corollary 6. *For a connected linear Nakayama algebra with $n \geq 2$ simples, the two dimensions are equal exactly when B is hereditary, that is, the path algebra of $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$. Their common value is $n - 1$.*

Proof. Let $r_i = i + c_i - 1$ and let a_i be the least index j for which $r_j \geq i$. Admissibility makes (r_i) nondecreasing, and $d_i = i - a_i + 1$. Constancy of $c_i + d_i$ is therefore equivalent to constancy of $r_i - a_i$. For $i = 1$ this value is $r_1 - 1$. For every $1 \leq i \leq r_1$, we have $a_i = 1$, so constancy forces $r_i = r_1$. If $r_1 < n$, taking $i = r_1$ gives $c_i = 1$ at a nonsink vertex, contradicting connectedness. Thus $r_1 = n$, and every $r_i = n$. This is exactly the hereditary Kupisch series $(n, n - 1, \dots, 1)$. Conversely that series has $d_i = i$ and $c_i + d_i = n + 1$. $\square$

Corollary 7. *For a connected cyclic Nakayama algebra with two simples, the dual E is higher Auslander exactly for*

$$(c_1, c_2) = (a, a) \quad (a \geq 2), \quad \text{or} \quad (c_1, c_2) = (2r + 1, 2r + 2) \quad (r \geq 1),$$

up to cyclic rotation. The respective global and dominant dimensions are $2a - 2$ and $4r + 1$.

Proof. Admissibility says that the two lengths differ by at most one. Equal lengths give $d_i = a$. For unequal lengths write $c = (a, a + 1)$. Formula (2) gives $d = (a, a + 1)$ when a is even and $d = (a + 1, a)$ when a is odd. Only the odd case makes the sums constant. $\square$

For another nonselfinjective example, $c = (2, 2, 3, 3)$ has $d = (3, 3, 2, 2)$, so both dimensions are 3. No selfinjectivity assumption is needed for the criterion.

5 The $(3, 4)$ example and exact checks

The paragraph after Question 4.21 in [1] lists Kupisch series $(3, 4)$ as a negative example for stiffness. With the conventions explicitly fixed above, it instead gives

$$d = (4, 3), \quad (\text{LL}(\text{I}_{M(1,1)}), \text{LL}(\text{I}_{M(1,2)}), \text{LL}(\text{I}_{P_1})) = (4, 5, 6),$$

and lengths $(3, 4, 5, 6)$ for the four modules with top S_2 . Exactly the two indices P_1, P_2 give length 6. Thus A is 6-stiff and Theorem 1 gives $\text{gldim } E = \text{domdim } E = 5$. This discrepancy concerns that explicit example, not a claimed disproof of the general transfer theorem. The series $(2, 3)$, by contrast, gives dimensions 4 and 2 and is negative.

We also computed minimal resolutions over the quadratic dual directly over $\mathbb{Q}$. This calculation uses kernels of quadratic multiplication and rational linear algebra, without the stiffness theorem. For the diagnostic table, choose the dual presentation C whose arrows have the direction of the maps in Lemma 2; the alternative multiplication convention gives its opposite. Both homological dimensions are invariant under taking opposites. Index its seven vertices by

$$(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4),$$

and number them $0, \dots, 6$ in that order. The simple projective dimensions are

$$(3, 4, 5, 2, 3, 4, 5).$$

Only P_5, P_6 are noninjective. Their minimal injective coresolutions have the following successive terms:

	J^0	J^1	J^2	J^3	J^4	J^5
P_5	I_4	I_0	I_3	I_4	I_5	I_6
P_6	I_5	I_1	I_3	I_0	I_1	I_2

The nonprojective injectives are exactly I_2, I_6 ; all earlier terms are projective. The supplementary files specify the multiplication and construct the maps and kernels exactly.

The bounded verification suite checks all 196 connected linear Kupisch series with $2 \leq n \leq 7$, and all 543 cyclic series with $1 \leq n \leq 5$ and entries from 2 through 6. It verifies the Loewy formulas for all 739 algebras. A separate minimal-resolution calculation verifies both homological dimensions for 97 of these algebras, checking that the quadratic dual has no surviving cubic paths. These checks supplement, and do not replace, the general proof.

The categorical radical viewpoint is classical; related general nilpotency formulas and references are collected in [3]. The present note makes no absolute-priority claim. AI assistance was used for literature searches, symbolic computations, drafting and revision. The author is responsible for the mathematical statements and their presentation. This preprint has not undergone peer review.

References

- [1] A. Chan, O. Iyama and R. Marczinzik, *Dominant Auslander–Gorenstein algebras and Koszul duality*, arXiv:2210.06180v2 (2022), especially Section 4. arxiv.org/abs/2210.06180v2. Version 3 (2024) is titled *Dominant Auslander–Gorenstein algebras and mixed cluster tilting* and omits Section 4 of v2.
- [2] R. Marczinzik, joint work with A. Chan and O. Iyama, *Auslander regular algebras and Koszul duality*, in *Representation Theory of Quivers and Finite-Dimensional Algebras*, Oberwolfach Reports 7/2023, p. 463. [doi:10.4171/owr/2023/7](https://doi.org/10.4171/owr/2023/7).
- [3] C. Chaio, V. Guazzelli and P. Suarez, *On the nilpotency index of the radical of a module category*, arXiv:2003.04189 (2020), Section 1. arxiv.org/abs/2003.04189.