

An Arc Hilbert–Poincaré Criterion for Rational Double Points

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Abstract

For a normal complex algebraic surface germ, we prove that its focused arc Hilbert–Poincaré series equals $(1 - q)^{-3} \prod_{j \geq 2} (1 - q^j)^{-2}$ if and only if the germ is a rational double point. The known forward formula is complemented by a converse: the coefficients of weights one and two recover a hypersurface presentation of multiplicity two, and a graded annihilator argument forces every finite prefix of its jet equations to be a regular sequence. The resulting fiber dimension bound gives irreducibility of all jet schemes, to which Mustață’s criterion applies. The same argument gives a converse to the known formula for isolated normal hypersurface singularities of maximal canonical multiplicity. Normality is essential for the surface statement: the known normal-crossing example $xy = 0$ has the same series. We keep this unrestricted negative observation separate from the normal-surface criterion.

1 The surface criterion and its scope

Let (X, o) be a germ of an algebraic variety over $\mathbb{C}$ at a closed point. Write X_m for its m th jet scheme, $\pi_m : X_m \rightarrow X$ for the center map, and $X_m^o = \pi_m^{-1}(o)$. The coordinate ring B of the scheme of arcs centered at o carries the positive grading induced by reparametrization. We use the scheme-theoretic ring, including its nilpotents, and set

$$\text{AHP}_{X,o}(q) = \text{Hilb}_B(q) = \sum_{h \geq 0} \dim_{\mathbb{C}} B_h q^h.$$

The grading assigns weight j to a coordinate of the coefficient of t^j . Each homogeneous component is finite dimensional.

Mourtada asks whether this series characterizes rational double points [2, Question 4.5]; the question appears again in [3, p. 372, Question 2]. The explicit category matters: without a normality hypothesis the normal-crossing formula of Bruscek, Mourtada and Schepers already supplies a negative answer. The following theorem gives the characterization within normal surfaces.

Theorem 1. *Let (X, o) be a normal complex algebraic surface germ. Then o is a rational double point, equivalently an ADE singularity, if and only if*

$$\text{AHP}_{X,o}(q) = \frac{1}{(1 - q)^3} \prod_{j \geq 2} \frac{1}{(1 - q^j)^2}. \quad (1)$$

The forward implication is the formula in [1, Corollary 3.8] and [2, Proposition 4.4]. Our argument for the converse uses the elementary graded lemma below and the jet-scheme theorems of Mustață [4]. All assertions about infinite series are identities of formal power series; no analytic convergence or reduced-arc replacement is involved. We make no absolute priority claim.

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2 Detecting regular sequences from a formal Hilbert series

Let $R = \bigoplus_{h \geq 0} R_h$ be a graded $\mathbb{C}$ -algebra with $R_0 = \mathbb{C}$ and $\dim R_h < \infty$. It need not be Noetherian. Suppose $f_1, f_2, \dots$ are homogeneous of positive degrees d_i with $d_i \rightarrow \infty$. Write

$$A_i = R/(f_1, \dots, f_i), \quad A_0 = R, \quad C_i = (0 :_{A_{i-1}} f_i).$$

Every ideal here is proper because it is generated in positive degree.

Lemma 2. *With this notation, the identity*

$$\text{Hilb}_{R/(f_1, f_2, \dots)}(q) = \text{Hilb}_R(q) \prod_{i \geq 1} (1 - q^{d_i}) \quad (2)$$

holds if and only if $f_1, \dots, f_N$ is a regular sequence in R for every N . If some C_i is nonzero, the first nonzero coefficient of the left side of (2) minus its right side is positive.

Proof. The multiplication exact sequence gives

$$\text{Hilb}_{A_i} = (1 - q^{d_i}) \text{Hilb}_{A_{i-1}} + q^{d_i} \text{Hilb}_{C_i}.$$

Iteration through N gives

$$\text{Hilb}_{A_N} - \text{Hilb}_R \prod_{i=1}^N (1 - q^{d_i}) = \sum_{i=1}^N q^{d_i} \text{Hilb}_{C_i} \prod_{j=i+1}^N (1 - q^{d_j}). \quad (3)$$

For any fixed weight, all expressions stabilize as $N \rightarrow \infty$: only finitely many generators have degree below that weight. Thus the infinite analogue of (3) is legitimate coefficient by coefficient.

For a nonzero graded space C let $\nu(C)$ be its least nonzero degree. If at least one C_i is nonzero, the integer

$$\lambda = \min_{C_i \neq 0} (d_i + \nu(C_i))$$

exists. Each factor on the right of (3) following Hilb_{C_i} has constant term one and only nonnegative exponents. There are no contributions below λ ; at weight λ the coefficient is

$$\sum_{\substack{C_i \neq 0 \\ d_i + \nu(C_i) = \lambda}} \dim(C_i)_{\nu(C_i)} > 0.$$

This finite sum cannot be canceled by later generators. It contradicts (2). Conversely, if all C_i vanish, the multiplication maps are injective and (3) has zero right side for every N . $\square$

The last assertion is a statement about the *first* difference. We do not assert a coefficientwise inequality at every later weight.

3 From arc coefficients to jet dimensions

A minimal completed presentation of a germ of embedding dimension e has the form

$$\widehat{O}_{X,o} = \mathbb{C}[[x_1, \dots, x_e]]/I, \quad I \subset (x_1, \dots, x_e)^2.$$

For centered arcs substitute $x_a(t) = \sum_{j \geq 1} x_{a,j} t^j$. The coefficients of the substituted relations are homogeneous of weight equal to their t -degree. A formal relation contributes only finitely many terms in any fixed weight, so this gives a locally finite graded presentation of B . The focused arc and jet algebras depend only on the completed local ring [1, Propositions 3.2–3.3].

In particular B_1 has dimension e : no relation has weight one. For a hypersurface $I = (f)$ of order μ , its first nonzero arc equation is f_μ , a nonzero polynomial of weight μ . There are no relations of lower weight. This also explains the multiplicity test in [1, Proposition 3.6].

Proof of Theorem 1. Suppose (1) holds. Its coefficient of q is three, so $e = 3$. Normality and dimension two give a hypersurface presentation: the kernel of the completed minimal presentation is a height-one prime of the regular local UFD $\mathbb{C}[[x_1, x_2, x_3]]$, and hence is principal. Normality of the completion holds for the complex algebraic local ring. Equivalently, the germ is a local complete intersection of codimension one in a smooth three-dimensional germ.

Set

$$S = \mathbb{C}[x_{a,j} : 1 \leq a \leq 3, j \geq 1], \quad \deg x_{a,j} = j.$$

Its coefficient of weight two is $6 + 3 = 9$. The right side of (1) has coefficient eight, so the hypersurface equation has order exactly two. Write

$$f(x_1(t), x_2(t), x_3(t)) = \sum_{j \geq 2} f_j t^j.$$

Then

$$B = S/(f_2, f_3, \dots), \quad \text{Hilb}_S(q) = \prod_{j \geq 1} (1 - q^j)^{-3}.$$

Consequently (1) is precisely

$$\text{Hilb}_B = \text{Hilb}_S \prod_{j \geq 2} (1 - q^j).$$

Lemma 2 implies that every prefix $f_2, \dots, f_m$ is regular in S .

For $m \geq 1$ put $S_m = \mathbb{C}[x_{a,j} : 1 \leq a \leq 3, 1 \leq j \leq m]$. All equations through f_m lie in S_m , and polynomial extension $S_m \rightarrow S$ is faithfully flat. Thus regularity descends to S_m . The fiber X_m^o is the spectrum of $S_m/(f_2, \dots, f_m)$, and therefore

$$\dim X_m^o = 3m - (m - 1) = 2m + 1. \quad (4)$$

For $m = 1$ the sequence is empty and the same formula holds.

Shrink X to an integral affine neighborhood on which it is a local complete intersection and has no singular point other than o . This is possible because a normal surface has isolated singularities. The jets over its smooth part form an irreducible open subset of dimension $2(m+1)$. Each component of X_m has dimension at least $2(m+1)$, since locally the jet scheme is cut out by the expected number of equations in a smooth ambient jet scheme. Equation (4) excludes a component lying entirely over o . Equivalently, Mustařa's Proposition 1.4 applies and shows that X_m is irreducible for every $m \geq 1$. His Theorem 4.4 now implies that o is a rational double point. This proves the converse. The forward implication is the cited known formula. $\square$

4 An isolated hypersurface extension

For $d \geq 2$ define

$$E_d(q) = \prod_{j=1}^{d-1} (1 - q^j)^{-(d+1)} \prod_{j \geq d} (1 - q^j)^{-d}.$$

Corollary 3. *Let (X, o) be an isolated normal hypersurface singularity of dimension $d \geq 2$ over $\mathbb{C}$, of multiplicity d . Then $\text{AHP}_{X,o} = E_d$ if and only if (X, o) is canonical.*

Proof. The canonical implication is [1, Proposition 3.7]. For the converse the focused arc equations have one equation f_j of weight j for every $j \geq d$. The asserted formula is exactly the regular-sequence product. Lemma 2 makes each finite prefix regular. For $m \geq d - 1$ the jet fiber consequently has dimension

$$(d + 1)m - (m - d + 1) = d(m + 1) - 1.$$

For $1 \leq m < d - 1$ no equation occurs and the dimension is $(d + 1)m < d(m + 1)$. Since the singularity is isolated, Proposition 1.4 and Theorem 0.1 of [4] imply that a neighborhood of o has rational singularities. For a hypersurface these are equivalent to canonical singularities, as recalled in that paper. $\square$

5 Scope checks and finite diagnostics

Normality cannot be removed from Theorem 1. For $X = V(xy) \subset \mathbb{A}^3$, the focused equations are the coefficients of $x(t)y(t)$. Their zero set through weight m is the union, for $1 \leq a \leq m$, of the linear spaces

$$x_1 = \cdots = x_{a-1} = 0, \quad y_1 = \cdots = y_{m-a} = 0.$$

This follows by considering the orders of the two truncated series, with the order of zero taken to be $m + 1$. Each space has codimension $m - 1$. Thus the $m - 1$ equations form a regular sequence in the polynomial ring, and their Hilbert series gives (1). The surface is reducible and not normal, so it is not ADE. This is the case $d = e = 2$ of the already known normal-crossing formula [1, Proposition 3.9], not a new counterexample.

For a different check, consider the isolated normal surface $f = x^2 + y^3 + z^6$. Its first equations are

$$\begin{aligned} f_2 &= x_1^2, \\ f_3 &= 2x_1x_2 + y_1^3, \\ f_4 &= 2x_1x_3 + x_2^2 + 3y_1^2y_2, \\ f_5 &= 2x_1x_4 + 2x_2x_3 + 3y_1^2y_3 + 3y_1y_2^2. \end{aligned}$$

The radical of (f_2, f_3, f_4) is (x_1, y_1, x_2) , so this prefix is a complete intersection of height three. The next equation belongs to its minimal prime and is a zero divisor. Lemma 2 therefore rules out (1) without any infinite computation.

As a supplementary diagnostic, the accompanying script constructs every monomial multiple of the arc equations in each weight through ten and computes their matrix ranks over $\mathbb{Q}$. The target coefficients are

$$1, 3, 8, 18, 38, 74, 139, 249, 434, 734, 1215.$$

They agree in all eleven checked weights for $xy - z^2$, $x^2 + y^3 + z^5$, and xy . For $x^2 + y^3 + z^6$, the first difference is in weight ten, where the coefficient is 1216. The nonreduced example x^2 first differs in weight four, and the cubic cone $x^3 + y^3 + z^3$ already differs in weight two. These finite rational computations check conventions and examples; the theorems follow from the written arguments.

The paper establishes the stated normal and isolated categories. It does not claim a characterization among arbitrary nonnormal or nonreduced germs, nor does its finite diagnostic table certify an infinite identity. The source question and all theorem dependencies are explicitly attributed. AI assistance was used for research, drafting, symbolic checks, and self-audit. The author is responsible for the manuscript. This preprint has not undergone independent peer review or formal proof-assistant verification.

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