

A Closed Curve with Finite Critical Menger Curvature and No Tangent at One Point

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25 October 2026

Abstract

We give an explicit embedded planar loop of length one with finite integral Menger curvature $\mathcal{M}_3$ whose arclength parametrization is not differentiable at one point. The curve is smooth everywhere else. Its local graph has a slope of the form $a \sin(\log \log(1/|x|))$ plus a smooth term. A direct three-point estimate bounds the critical Menger energy of a Lipschitz graph by the $W^{1/3,3}$ seminorm of its slope. The oscillating slope has finite seminorm, while its secants have two distinct limiting directions. Replacing a small arc of a circle by this graph closes the example without introducing additional singularities. This answers negatively the everywhere-differentiability question for $\mathcal{M}_3$ posed in the 2013 Geometric Knot Theory report and in Problem 5.3 of Strzelecki and von der Mosel's survey. The related nondifferentiability phenomenon for Möbius energy and the Sobolev approach to curvature energies are prior work.

1 Statement and scope

For three distinct points $P, Q, R \in \mathbb{R}^d$, let $c(P, Q, R)$ be the reciprocal of their circumradius, with value zero for collinear points. For a rectifiable embedded curve Γ define

$$\mathcal{M}_3(\Gamma) = \int_{\Gamma} \int_{\Gamma} \int_{\Gamma} c(P, Q, R)^3 d\mathcal{H}^1(P) d\mathcal{H}^1(Q) d\mathcal{H}^1(R).$$

Values on coincident triples are immaterial. This energy is unchanged by dilation.

The 2013 Oberwolfach report [1, p. 1353, question (2)] asks whether finite $\mathcal{M}_3$ forces an arclength parametrization to be differentiable everywhere. The same question appears in [2, Problem 5.3], for unit loops whose image length equals their traversed length. Thus multiple coverings and backtracking parametrizations are excluded.

Theorem 1. *There is an embedded arclength parametrized loop $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^2$ with*

$$\mathcal{H}^1(\gamma(\mathbb{R}/\mathbb{Z})) = 1, \quad \mathcal{M}_3(\gamma) < \infty,$$

which is smooth away from $\gamma(0)$ and has no geometric tangent there. In particular, $\gamma'(0)$ does not exist.

The proof is elementary and given below. Its analytic mechanism is the failure of continuity at a critical fractional Sobolev exponent. Blatt's characterization of finite Menger energy [3] uses fractional Sobolev spaces in the supercritical range $p > 3$. Our graph estimate proves the sufficient endpoint bound directly; it does not invoke that theorem at an unstated exponent.

Nondifferentiable curves of finite Möbius energy were already constructed by Blatt and Reiter [4, Theorem 1.1 and Corollary 4.2]. We do not claim that critical knot energies failing

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to force differentiability, or slowly changing directions across scales, are new phenomena. The purpose here is a concrete $\mathcal{M}_3$ example with its entire endpoint estimate written out. Scholtes [5] proves weak approximate tangency results for finite geometric curvature energies and constructs examples for general sets; weak approximate tangency does not assert the existence of a classical tangent.

The other questions in the Oberwolfach paragraph concern circle minimization and the regularity of stationary points. This note does not settle them. Dohmen and Scholtes [7] recently proved the circle minimum in the embedded C^1 class for $p \geq 2$, and for the full rectifiable class specified there when $p > 3$. Their supercritical extension does not apply to the nonsmooth critical loop constructed here.

2 An endpoint estimate for graphs

For an interval I and a measurable function g write

$$[g]_{1/3,3;I}^3 = \int_I \int_I \frac{|g(s) - g(t)|^3}{|s - t|^2} ds dt,$$

allowing the value $+\infty$.

Lemma 2. *Let $f : I \rightarrow \mathbb{R}$ be Lipschitz, put $g = f'$ almost everywhere, and suppose $\|g\|_\infty \leq B$. For the graph $\Gamma_f = \{(x, f(x)) : x \in I\}$,*

$$\mathcal{M}_3(\Gamma_f) \leq 12(1 + B^2)^{3/2} [g]_{1/3,3;I}^3. \quad (1)$$

Proof. For $x < y < z$ put

$$m_{xy} = \frac{f(y) - f(x)}{y - x},$$

and define m_{yz} and m_{xz} similarly. The triangle-area formula for circumradius gives the exact identity

$$c((x, f(x)), (y, f(y)), (z, f(z))) = \frac{2|m_{yz} - m_{xy}|}{(z - x)\sqrt{1 + m_{xy}^2}\sqrt{1 + m_{yz}^2}\sqrt{1 + m_{xz}^2}}. \quad (2)$$

Indeed the determinant of the two successive chords is $(y - x)(z - y)(m_{yz} - m_{xy})$, and the three chord lengths supply the denominator. The identity also holds when the determinant vanishes.

By absolute continuity and Jensen's inequality,

$$|m_{yz} - m_{xy}|^3 \leq \frac{1}{(y - x)(z - y)} \int_x^y \int_y^z |g(s) - g(t)|^3 dt ds. \quad (3)$$

To integrate this estimate, fix $s < t$ and set $d = t - s$. Tonelli's theorem applies because all integrands are nonnegative. Enlarging the possible x and z to half-lines, and writing $x = s - u$, $z = t + v$, $y = s + q$, gives the kernel bound

$$\begin{aligned} K(d) &= \int_0^\infty \int_0^\infty \int_0^d \frac{dq dv du}{(d + u + v)^3 (q + u)(d - q + v)} \\ &= \int_0^\infty \int_0^\infty \frac{\log(1 + d/u) + \log(1 + d/v)}{(d + u + v)^4} dv du \\ &= \frac{2}{3} \int_0^\infty \frac{\log(1 + d/u)}{(d + u)^3} du = \frac{2}{3d^2} \int_0^1 -\tau \log(1 - \tau) d\tau = \frac{1}{2d^2}. \end{aligned} \quad (4)$$

For the second line, use $1/(AB) = (1/A + 1/B)/(A+B)$ with $A = q+u$ and $B = d-q+v$. For the fourth, substitute $\tau = d/(d+u)$. Endpoint singularities are integrable; the last integral is $3/4$.

Dropping the three square-root factors in (2), the ordered parameter integral of c^3 is therefore at most

$$8 \int_{s < t} |g(s) - g(t)|^3 K(t-s) ds dt = 4 \int_{s < t} \frac{|g(s) - g(t)|^3}{(t-s)^2} ds dt.$$

The full parameter cube has six orderings, up to a null set. Each arclength density is at most $\sqrt{1+B^2}$. Finally the integral over $s < t$ is half the symmetric seminorm. These factors give (1). $\square$

3 A bounded oscillating slope

Fix

$$r = \frac{1}{16}, \quad a = \frac{1}{4}.$$

For $0 < |x| < r$ set

$$g(x) = a \sin(\log \log(1/|x|)), \quad f(x) = \int_0^x g(t) dt. \quad (5)$$

Assign any value to $g(0)$. The function g is even, and f is odd, Lipschitz, and smooth off zero.

Lemma 3. *The function g in (5) satisfies $[g]_{1/3,3;(-r,r)} < \infty$.*

Proof. First consider $0 < x < y < r$. For $y \leq 2x$, put $h = y - x$. Since $u \log(1/u)$ is increasing for $0 < u < r < e^{-1}$,

$$|g(y) - g(x)| \leq \frac{ah}{x \log(1/x)}.$$

Consequently,

$$\int_0^r \int_x^{\min(2x,r)} \frac{|g(y) - g(x)|^3}{(y-x)^2} dy dx \leq \frac{a^3}{2} \int_0^r \frac{dx}{x \log^3(1/x)} < \infty.$$

For $x \leq y/2$, set $x = ye^{-s}$ and $L = \log(1/y)$. Then $s \geq \log 2$, and the sine function is 1-Lipschitz, so

$$|g(x) - g(y)| \leq a \log(1 + s/L) \leq \frac{as}{L}.$$

As $y - x \geq y/2$ and $dx = ye^{-s} ds$ in absolute value,

$$\int_0^r \int_0^{y/2} \frac{|g(x) - g(y)|^3}{(y-x)^2} dx dy \leq 4a^3 \left(\int_0^r \frac{dy}{y \log^3(1/y)} \right) \left(\int_{\log 2}^\infty s^3 e^{-s} ds \right) < \infty.$$

This proves finiteness on the positive quadrant and, by symmetry, on the negative quadrant. For opposite signs, write $x = -u$, $y = v$ with $u, v > 0$. Evenness gives

$$\frac{|g(-u) - g(v)|^3}{(u+v)^2} \leq \frac{|g(u) - g(v)|^3}{|u-v|^2} \quad (u \neq v).$$

Thus the cross terms are finite as well. $\square$

Lemma 4. For $0 < x < r$ and $L = \log(1/x)$,

$$\left| \frac{f(x)}{x} - g(x) \right| \leq \frac{a}{L}. \quad (6)$$

In particular, the graph of f has no tangent at the origin.

Proof. With $t = xu$ and the same Lipschitz estimate as above,

$$\left| \frac{f(x)}{x} - g(x) \right| \leq a \int_0^1 \log \left(1 + \frac{\log(1/u)}{L} \right) du \leq \frac{a}{L} \int_0^1 \log(1/u) du = \frac{a}{L}.$$

For all sufficiently large integers k , the numbers

$$x_k = \exp(-\exp(\pi/2 + 2\pi k)), \quad y_k = \exp(-\exp(3\pi/2 + 2\pi k))$$

belong to $(0, r)$ and tend to zero. We have $g(x_k) = a$ and $g(y_k) = -a$, so their secant slopes tend respectively to a and $-a$. The corresponding unoriented lines are distinct. $\square$

4 Closing the curve and estimating all triples

Consider the circle of radius one centered at $(0, 1)$. Its lower arc over $[-r, r]$ is the graph of

$$b(x) = 1 - \sqrt{1 - x^2}.$$

Choose an even smooth cutoff χ with $0 \leq \chi \leq 1$, equal to one on $[-r/2, r/2]$, and zero for $|x| \geq 3r/4$. Replace this lower circle arc by the graph of

$$F(x) = b(x) + \chi(x)f(x), \quad -r \leq x \leq r. \quad (7)$$

Denote the resulting closed curve by Γ .

The replacement is a Lipschitz graph and equals the old arc on neighborhoods of both endpoints. It is smooth except at zero. It is also disjoint from the rest of the circle except at its joining endpoints: points of the remaining lower arc have $|x| \geq r$, whereas the upper arc over $[-r, r]$ has height at least $1 + \sqrt{1 - r^2} > 1$. On the replacement graph,

$$|F(x)| \leq b(r) + ar < 1.$$

The graph itself is injective because its first coordinate is x . Thus Γ is a simple closed rectifiable curve, smooth away from the origin.

On $I = (-r/2, r/2)$ its graph slope is $g + b'$. The smooth function b' has bounded derivative on $[-r, r]$, so

$$[b']_{1/3,3;I}^3 \leq \|b''\|_\infty^3 \int_{I^2} |s - t| ds dt < \infty.$$

The inequality $|u + v|^3 \leq 4(|u|^3 + |v|^3)$ and Lemma 3 show that $[g + b']_{1/3,3;I}$ is finite. Lemma 2 therefore controls every triple in this singular graph neighborhood.

Here are details that also control mixed triples. Cover Γ by the graph neighborhood just used and finitely many open smooth graph arcs whose closures avoid the origin. Such a finite cover exists by compactness, after first choosing a smaller neighborhood of the origin. Each smooth graph arc has bounded slope and Lipschitz derivative, so Lemma 2 gives finite $\mathcal{M}_3$ on each member of the cover.

Let $\gamma_0 : \mathbb{R}/L_\Gamma\mathbb{Z} \rightarrow \Gamma$ be an arclength parametrization. It is a homeomorphism, where $L_\Gamma = \mathcal{H}^1(\Gamma)$. Pull the finite cover back to the parameter circle and choose a Lebesgue number

$\delta > 0$. Every parameter triple of intrinsic diameter less than δ lies in one of these arcs. The integral over all such triples is bounded by the sum of the finitely many local energies.

For every remaining triple, some pair of parameters has intrinsic distance at least δ . Injectivity and compactness imply

$$m_\delta = \min_{\substack{s, t \in \mathbb{R}/L_\Gamma \mathbb{Z} \\ d_{L_\Gamma}(s, t) \geq \delta}} |\gamma_0(s) - \gamma_0(t)| > 0.$$

The circumradius of a triangle is at least half its longest side. Hence $c \leq 2/m_\delta$ on these remaining triples, and their energy is at most $(2/m_\delta)^3 L_\Gamma^3$. This proves $\mathcal{M}_3(\Gamma) < \infty$ with no omitted interaction between the inserted arc and the rest of the loop.

5 Failure of arclength differentiability

Take the origin as parameter zero and orient the local graph in the increasing x direction. Its right-hand arclength is

$$\ell(x) = \int_0^x \sqrt{1 + (g(t) + b'(t))^2} dt \quad (0 < x < r/2).$$

The function $v \mapsto \sqrt{1 + v^2}$ is 1-Lipschitz. Therefore, by the proof of (6) and $b'(t) = O(t)$,

$$\frac{\ell(x)}{x} = \sqrt{1 + g(x)^2} + O\left(\frac{1}{\log(1/x)} + x\right). \quad (8)$$

Also $F(x)/x = g(x) + O(1/\log(1/x) + x)$. Along the two sequences from Lemma 4, the difference quotients

$$\frac{\gamma_0(\ell(x)) - \gamma_0(0)}{\ell(x)} = \frac{(x, F(x))}{\ell(x)}$$

thus tend to the distinct vectors

$$\frac{(1, a)}{\sqrt{1 + a^2}} \quad \text{and} \quad \frac{(1, -a)}{\sqrt{1 + a^2}}.$$

Even the right derivative fails to exist. Finally define

$$\gamma(s) = L_\Gamma^{-1} \gamma_0(L_\Gamma s), \quad s \in \mathbb{R}/\mathbb{Z}.$$

This is an embedded unit-length arclength loop through the origin. Dilation preserves finiteness of $\mathcal{M}_3$ and does not remove the two different limiting directions. This proves Theorem 1.

6 Relation to earlier constructions

The critical graph bound is a sufficient estimate; it is not a characterization of all finite- $\mathcal{M}_3$ loops. The proof uses the same chord-difference and fractional-Sobolev philosophy as [3, 6], with the endpoint kernel evaluated explicitly. The construction has a geometric singularity, rather than a nonsmooth multiple-covering parametrization of a smooth image.

The same split of the slope seminorm works with the square in the numerator: $\iint |g(s) - g(t)|^2 / |s - t|^2 ds dt$ is finite, with $\log^2(1/x)$ in place of $\log^3(1/x)$. Thus the example is closely related to critical Möbius-energy constructions, rather than suggesting a new smoothing threshold for that energy. The earlier nondifferentiable finite-energy examples of [4] must be credited. A bounded review of the cited primary sources did not locate this explicit $\mathcal{M}_3$ loop and estimate; it does not certify priority for consequences of existing results.

AI assistance was used for literature retrieval, derivation and checking of the estimates, diagnostic computations, and preparation of this manuscript. This preprint is self-audited and unrefereed; no independent review or proof-assistant verification is claimed.

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