

# Unit-Coefficient Simplex Decompositions under Weak Nondegeneracy

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## Abstract

Let  $P \subset \mathbb{R}^d$  be a finite union of compact convex polytopes, and let  $S$  contain its algebraic vertices. Suppose that every  $d + 2$  points of  $S$  affinely span  $\mathbb{R}^d$ . We prove that the unit-density measure on  $P$  is a sum of distinct simplex measures with coefficients in  $\{-1, 1\}$  and with all simplex vertices in  $S$ . Moreover, all the simplices may share any prescribed point of  $S$  as a vertex. The proof combines the signed-section and coning results of Akopyan, Bárány and Robins with an elementary coefficient-reduction lemma for a configuration having at most one affine dependence. This gives an affirmative answer to the coefficient question in the weakly nondegenerate setting of Pasechnik's 2013 Oberwolfach report. It does not resolve the later conjecture for arbitrary vertex configurations. We also obtain a coefficient bound by the largest facet jump for integer-valued polyhedral functions.

## 1 The precise setting

We interpret all indicator identities modulo Lebesgue-null sets. Write  $\mu_P = \mathbf{1}_P dx$ ; this is the *unit-density* measure, not the probability measure obtained by dividing by  $\text{vol}(P)$ . A generalized polytope is a finite union of compact convex polytopes. A polyhedral function is a finite real linear combination of indicators of convex polyhedra.

We use the algebraic vertex notion of [1]. A point is an algebraic vertex if its tangent-cone function is not a linear combination of line-cone indicators. A line-cone is a cone invariant in a nonzero line direction. These notions are taken modulo null sets. We denote the finite algebraic vertex set of  $f$  by  $V_f$ .

A finite set  $S \subset \mathbb{R}^d$  is *weakly nondegenerate* if every  $(d + 2)$ -element subset spans  $\mathbb{R}^d$  affinely. Equivalently, every affine hyperplane contains at most  $d + 1$  points of  $S$ . This allows dependent  $(d + 1)$ -element subsets and is weaker than general position.

**Theorem 1.** *Let  $d \geq 2$ , let  $P \subset \mathbb{R}^d$  be a generalized polytope of positive volume, and let  $S$  be a finite weakly nondegenerate set containing  $V_{\mathbf{1}_P}$ . For every  $p \in S$  there is a set  $\mathcal{T}_p$  of distinct nondegenerate  $d$ -simplices, each having  $p$  as a vertex and all vertices in  $S$ , such that*

$$\mathbf{1}_P = \sum_{T \in \mathcal{T}_p} \epsilon_T \mathbf{1}_T, \quad \epsilon_T \in \{-1, 1\}.$$

*In particular  $|\mathcal{T}_p| \leq \binom{|S|-1}{d}$ .*

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Theorem 2 and Question 2 on page 638 of [3] concern the coefficient problem under the same weak nondegeneracy assumption, with  $S$  the combinatorial vertex set. Algebraic vertices belong to every simplex dissection vertex set, by [1, Theorem 1], so Theorem 1 applies to that choice of  $S$ .

Two distinctions are important. First, the question concerns the *existence* of a unit-coefficient representation; it cannot mean that every representation has unit coefficients, since simplex relations can be added. Second, the report uses probability-measure terminology in a passage whose preceding formulas and triangulation identity use unit density. The later formulation [2, Conjecture 8] explicitly uses standard measures. We use that consistent normalization. For normalized probability measures our identity instead has coefficients  $\epsilon_T \text{vol}(T) / \text{vol}(P)$ .

The arbitrary-configuration version of [2, Conjecture 8] is strictly broader than Theorem 1. No conclusion about that version is claimed here. The signed-coning construction itself is known [1, §3]; in the simplicial three-dimensional setting it also appears in the discussion [4]. Our coefficient argument addresses facets with one additional, possibly dependent, vertex.

## 2 A one-relation coefficient lemma

**Lemma 2.** *Let  $m \geq 1$ , let  $S \subset \mathbb{R}^m$  be a spanning set with  $|S| \leq m + 2$ , and let*

$$g = \sum_{\Delta} c_{\Delta} \mathbf{1}_{\Delta}, \quad c_{\Delta} \in \mathbb{Z},$$

*where  $\Delta$  ranges over the full-dimensional simplices on  $S$ . If  $|g| \leq M$  almost everywhere, for an integer  $M \geq 0$ , then  $g$  has such a representation with all coefficients in  $\{-M, -M + 1, \dots, M\}$ .*

*Proof.* For  $|S| = m + 1$  there is only one full-dimensional simplex, so the claim is immediate. Suppose  $S = \{s_1, \dots, s_{m+2}\}$ . The augmented coordinate matrix has rank  $m + 1$ , and its kernel is generated by a vector  $\lambda = (\lambda_i)$  with

$$\sum_i \lambda_i = 0, \quad \sum_i \lambda_i s_i = 0.$$

Put  $I_+ = \{i : \lambda_i > 0\}$ ,  $I_- = \{i : \lambda_i < 0\}$  and  $I_0 = \{i : \lambda_i = 0\}$ . Both  $I_+$  and  $I_-$  are nonempty. The simplex  $\Delta_i = \text{conv}(S \setminus \{s_i\})$  is nondegenerate exactly when  $\lambda_i \neq 0$ . Thus we may write  $g = \sum_{i \in I_+ \cup I_-} c_i \mathbf{1}_{\Delta_i}$ .

The two circuit subdivisions give

$$\sum_{i \in I_+} \mathbf{1}_{\Delta_i} = \sum_{j \in I_-} \mathbf{1}_{\Delta_j} = \mathbf{1}_{\text{conv } S} \quad \text{almost everywhere.} \quad (1)$$

For completeness, the nonnegative barycentric representations of a point of  $\text{conv } S$  form an interval parallel to  $\lambda$ . For a point avoiding the finitely many simplex boundaries, one endpoint of that interval has exactly one zero coordinate indexed by  $I_+$ , and the other has exactly one zero coordinate indexed by  $I_-$ . They identify the unique simplex from each family containing the point. Coordinates indexed by  $I_0$  cause no exception away from those boundaries. This proves (1) also when the affine dependence has zero entries.

Every pair  $i \in I_+$ ,  $j \in I_-$  has an intersection of interiors  $\text{int } \Delta_i \cap \text{int } \Delta_j \neq \emptyset$ . Here is an explicit witness. Set

$$x_i = 0, \quad x_j = -\lambda_j, \quad x_k = \max(0, -\lambda_k) + 1 \quad (k \notin \{i, j\}), \quad y = x + \lambda.$$

Then  $x_k > 0$  for  $k \neq i$ , while  $y_j = 0$  and  $y_k > 0$  for  $k \neq j$ . Their coordinate sums agree and are positive, and their affine images agree. Division by the common sum therefore yields strict

barycentric coordinates in both simplices. Their interiors are open, so the intersection has positive  $m$ -dimensional measure.

On this intersection, away from boundaries and the null exceptional set for the bound on  $g$ , equation (1) gives  $g = c_i + c_j$ . Consequently

$$-M \leq c_i + c_j \leq M \quad (i \in I_+, j \in I_-). \quad (2)$$

Write

$$a = \min_{I_+} c_i, \quad A = \max_{I_+} c_i, \quad b = \min_{I_-} c_j, \quad B = \max_{I_-} c_j.$$

In particular  $a + b \geq -M$ ,  $A + B \leq M$ , and  $A - a \leq 2M$ ,  $B - b \leq 2M$ . By (1), adding  $t$  to the coefficients in  $I_+$  and subtracting  $t$  from those in  $I_-$  does not change  $g$ . The desired bound is equivalent to

$$t \in [-M - a, M - A] \cap [B - M, b + M]. \quad (3)$$

The within-interval endpoint inequalities follow from the two range bounds. The cross inequalities follow from  $a + b \geq -M \geq -2M$  and  $A + B \leq M \leq 2M$ . Thus the intersection is nonempty. Its endpoints are integers, so it contains an integer  $t$ . The shifted coefficients prove the lemma.  $\square$

*Remark 3.* The argument does not require the dependence to have full support. In particular it includes pyramids over a smaller affine circuit. For computation one can take  $t = \max(-M - a, B - M)$ .

### 3 Signed sections and coning

For an oriented affine hyperplane  $H$  with normal  $n_H$ , the signed section of  $f$  is the polyhedral function on  $H$  given almost everywhere by

$$f_H(x) = \lim_{\varepsilon \downarrow 0} f(x + \varepsilon n_H) - \lim_{\varepsilon \downarrow 0} f(x - \varepsilon n_H).$$

Only finitely many hyperplanes have nonzero signed section. We use two established results of Akopyan, Bárány and Robins: a bounded-support integer-valued polyhedral function is an integer combination of simplex indicators on its algebraic vertices [1, Theorem 1'], and

$$V_{f_H} \subseteq V_f \cap H \quad (4)$$

for every signed section [1, Lemma 3]. An integer-valued polyhedral function has an integer basic decomposition by refining its finitely many defining hyperplanes into cells, so the integrality hypothesis of their theorem applies.

**Theorem 4.** *Let  $f$  be an integer-valued polyhedral function of bounded support in  $\mathbb{R}^d$ ,  $d \geq 2$ . Suppose a finite weakly nondegenerate set  $S$  contains  $V_f$ , and fix  $p \in S$ . If every signed section satisfies  $|f_H| \leq M$  almost everywhere, where  $M$  is a nonnegative integer, then*

$$f = \sum_T b_T \mathbf{1}_T, \quad b_T \in \mathbb{Z}, \quad |b_T| \leq M,$$

*using distinct nondegenerate  $d$ -simplices on  $S$  having  $p$  as a vertex.*

*Proof.* If  $f = 0$  the empty sum suffices. For every nonzero signed section whose hyperplane  $H$  does not contain  $p$ , orient  $H$  toward  $p$ . Equation (4) gives  $V_{f_H} \subseteq S \cap H$ , and weak nondegeneracy gives  $|S \cap H| \leq d + 1$ . The integer simplex representation theorem in dimension  $m = d - 1$  represents  $f_H$  using simplices on  $S \cap H$ . If that set does not span  $H$ , the same theorem forces  $f_H = 0$  almost everywhere, so there is nothing to do. Otherwise Lemma 2 gives

$$f_H = \sum_B b_{H,B} \mathbf{1}_B, \quad b_{H,B} \in \mathbb{Z}, \quad |b_{H,B}| \leq M,$$

where the bases  $B$  are distinct nondegenerate  $(d - 1)$ -simplices on  $S \cap H$ .

The coning identity of [1, §3, equations (2)–(3)] now yields

$$f = \sum_{H: p \notin H} \sum_B b_{H,B} \mathbf{1}_{\text{conv}(\{p\} \cup B)}. \quad (5)$$

One can check this identity directly. For almost every  $x$ , follow the ray from  $p$  through  $x$  outward to infinity. At its successive intersections beyond  $x$  with the relevant hyperplanes, the oriented signed sections telescope from  $f(x)$  to zero. Each such intersection contributes exactly the value of the corresponding pyramid function at  $x$ . Hyperplanes through  $p$  do not meet a generic such ray away from  $p$  and contribute nothing. The excluded rays form a null set: they lie in the finitely many cones over facet boundaries or hyperplane intersections.

It remains to check that coefficients from different bases do not accumulate in (5). Each cone is nondegenerate and has  $p$  as a vertex. Its other  $d$  vertices determine its base simplex and the unique hyperplane containing that base. Hence two equal output simplices must come from the same  $H$  and the same  $B$ . They have already been collected in the local representation. Thus all coefficients in (5) retain the bound  $M$ .  $\square$

*Proof of Theorem 1.* Apply Theorem 4 to  $f = \mathbf{1}_p$  and  $M = 1$ . A signed section of an indicator takes values only in  $\{-1, 0, 1\}$ . Omit the zero coefficients. The number of possible vertex sets containing  $p$  is at most  $\binom{|S|-1}{d}$ .  $\square$

The corresponding assertion in dimension one follows by taking the signed jumps at interval endpoints and coning them to the chosen endpoint  $p$ . Distinct endpoints give distinct nondegenerate intervals, with coefficients of absolute value at most the largest jump. Thus the unit-coefficient conclusion also covers that case.

## 4 Scope, attribution, and reproducibility

The general integer-decomposition theorem and the signed-section machinery are established results of [1], not claims of this note. The one-dependence structure of weakly nondegenerate configurations also appears in [2, Proposition 23]. The additional observation here is the coefficient reduction in Lemma 2 and its application to the bounded jumps of facets. It produces a complete proof under the stated hypothesis. A hyperplane containing  $d + 2$  or more allowed vertices can have several independent affine relations; the one-parameter argument does not cover it. We do not remove that obstruction by a limiting or generic perturbation argument.

The source package includes a Python/SymPy regression checker. It enumerates 6 267 982 small integer coefficient configurations, of which 69 426 meet the pairwise bounds, and checks their gauge reduction. It also verifies 107 strict intersection witnesses for 23 rational configurations in dimensions one through five, including seven configurations with zero relation entries. There are 958 generic rational triangulation checks and 2 394 exact three-dimensional pointwise coning checks,

together with volume and first-moment identities. These finite tests are implementation checks, not the proof of either theorem.

This unrefereed preprint was developed with AI assistance for mathematical exploration, literature retrieval, drafting and exact computation. The written argument and source comparisons were self-audited; no independent expert review, proof-assistant certification or exhaustive priority determination is asserted.

## References

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