

The Gaussian Gabor System over the Complex Lattice Λ_3 of Gröchenig and Lyubarskii Is a Frame: A Computer-Assisted Proof

Alper Ferudun*

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Abstract

Gröchenig and Lyubarskii (Oberwolfach Report 11/2013) considered the Gabor system $\mathcal{G}(g, \Lambda_3)$ of time-frequency shifts of the Gaussian $g(s) = 2^{1/2}e^{-\pi|s|^2}$ on $\mathbb{R}^2$ along the complex lattice $\Lambda_3 = A_3(\mathbb{Z} + i\mathbb{Z})^2$, $A_3 = \begin{pmatrix} 1 & 2/5 \\ 0 & \sqrt{21}/5 \end{pmatrix}$. They conjectured that it is a frame for $L^2(\mathbb{R}^2)$ and wrote that they had no reliable proof; in their paper in Complex Var. Elliptic Equ. 65 (2020), Λ_3 is the end point $t = \sqrt{21}/5$ of a range $\sqrt{21}/5 \leq t < \sqrt{24}/5$ of lattices for which the answer is stated to be unknown. We prove the conjecture by a computer-assisted proof in floating point with a margin. A dilation turns $\mathcal{G}(g, \Lambda_3)$ into a Gabor system over the rational separable lattice $\mathbb{Z}^2 \times M\mathbb{Z}^2$, $M = \begin{pmatrix} 1 & 2/5 \\ 2/5 & 1 \end{pmatrix}$. The fiberization of Ron and Shen and a Bloch reduction over $5\mathbb{Z}^2$ show that the system is a frame with lower bound $(10/\sqrt{21})s^2$ as soon as an explicit 25×21 matrix $P(t, \theta)$ of Gaussian lattice sums satisfies $\sigma_{\min}(P(t, \theta)) \geq s > 0$ for all $(t, \theta) \in \mathbb{R}^4$. After a reduction by symmetries to a box in $\mathbb{R}^4$, a branch-and-bound computation over 65,597 boxes, with a first-order bound maximised over the vertices of each box, a rigorous bound for the second-order Taylor remainder and an explicit tail bound, gives $\sigma_{\min}(P) \geq 0.23$ everywhere. Hence $\mathcal{G}(g, \Lambda_3)$ is a frame with lower frame bound at least 0.1154 (numerically the optimal bound is 0.12265). The box inequalities were evaluated in IEEE double precision with an explicit rounding budget and a margin of 10^{-6} , not in interval arithmetic; two independent recomputations in interval arithmetic, of the 1950 and of the 2000 boxes with the smallest margin, confirm these boxes without the margin. The same conclusion holds for the window $e^{-\pi|s|^2/2}$, the width printed once in the report ($\sigma_{\min} \geq 0.18$). The question and the conjecture are due to Gröchenig and Lyubarskii; no printed answer was found in the sources searched. This is an unrefereed note.

1 Introduction

1.1 The question

For $x, \xi, s \in \mathbb{R}^d$ let $\pi(x, \xi)f(s) = e^{2\pi i \xi \cdot s} f(s - x)$ be the time-frequency shift by (x, ξ) . For a lattice $\Lambda \subset \mathbb{R}^{2d}$ and a window $g \in L^2(\mathbb{R}^d)$, the Gabor system $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a *frame* for $L^2(\mathbb{R}^d)$ if there are constants $0 < A \leq B < \infty$ with

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|_2^2 \quad (f \in L^2(\mathbb{R}^d)) \quad (1)$$

[Grö13, p. 593]; we write A_{opt} for the largest admissible A . To use complex analysis, Gröchenig and Lyubarskii identify (x, ξ) with $z = x + i\xi \in \mathbb{C}^d$ and consider *complex lattices* $\Lambda = A(\mathbb{Z} + i\mathbb{Z})^d$ with

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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$A \in GL(d, \mathbb{C})$ [Grö13, p. 595], [GL20, Sect. 1]. For a real matrix A this is the lattice $\{(Am, An) : m, n \in \mathbb{Z}^d\}$ of $\mathbb{R}^{2d}$. The window is the normalised Gaussian $g(s) = 2^{d/4} e^{-\pi|s|^2}$.

The width of the window. The report prints the Gaussian once as $2^{d/4} e^{-\pi t \cdot t/2}$ [Grö13, p. 594]. The factor $2^{d/4}$ normalises $e^{-\pi|t|^2}$, the Bargmann transform and the Fock weight on the same page belong to $e^{-\pi|t|^2}$, and the published paper works with $\varphi(x) = \exp(-\pi|x|^2)$ [GL20, Sect. 1]. We therefore read the window as $e^{-\pi|s|^2}$; Theorem 1.3 treats the other reading. The constant factor does not affect the frame property.

In dimension $d = 1$, $\mathcal{G}(g, \Lambda)$ is a frame if and only if $\text{vol}(\Lambda) < 1$ [Lyu92, Sei92, SW92]. In dimension $d = 2$, write $\Lambda = S(\mathbb{Z} + i\mathbb{Z})^2$ with $S = \begin{pmatrix} \gamma_1 & \beta \\ 0 & \gamma_2 \end{pmatrix}$ upper triangular, $\gamma_1, \gamma_2 > 0$. If $\gamma_1 < 1$ and $\gamma_2 < 1$ the system is a frame, and if $\gamma_2 \geq 1$ it is not [Grö11], [Grö13, p. 595], [GL20, Prop. 2.2]. The report then considers

$$A_1 = \begin{pmatrix} 1 & \frac{1}{5} \\ 0 & \frac{\sqrt{24}}{5} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & \frac{\sqrt{24}}{5} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & \frac{\sqrt{21}}{5} \end{pmatrix}, \quad \Lambda_j = A_j(\mathbb{Z} + i\mathbb{Z})^2.$$

All three satisfy the density condition $\text{vol}(\Lambda_j) < 1$ ($\text{vol}(\Lambda_3) = (\det A_3)^2 = 21/25$); $\mathcal{G}(g, \Lambda_1)$ and $\mathcal{G}(g, \Lambda_2)$ are not frames [Grö13, p. 595] (proofs in [GL20, Thm. 4.1, Cor. 4.3]), and for Λ_3 the authors write that they conjecture a frame but have no reliable proof [Grö13, p. 595]. In [GL20, end of Sect. 4], for $A_t = \begin{pmatrix} 1 & 2/5 \\ 0 & t \end{pmatrix}$ the lattices $A_t(\mathbb{Z} + i\mathbb{Z})^2$ are shown not to give frames for $t \geq \sqrt{24}/5$, and the authors state that they do not know what happens for $\sqrt{21}/5 \leq t < \sqrt{24}/5$. For $t < \sqrt{21}/5$ the vector $A_t(0, 1)^T$ is shorter than 1, a reduced triangular basis has $\gamma_1, \gamma_2 < 1$, and the system is a frame by the theorem just quoted; so Λ_3 , the case $t = \sqrt{21}/5$, is the end point of the open range. It also lies in the class $|\beta|^2 + \gamma^2 = 1$ of lattices $A(\mathbb{Z} + i\mathbb{Z})^2$, $A = \begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix}$, whose sampling property is described as not understood in [GL20, end of Sect. 5].

Problem 1.1 (Gröchenig and Lyubarskii [Grö13, p. 595]). *Is $\mathcal{G}(g, \Lambda_3)$ a frame for $L^2(\mathbb{R}^2)$?*

1.2 Results

Theorem 1.2 (computer-assisted, floating point with margin). *Let $g(s) = 2^{1/2} e^{-\pi|s|^2}$ on $\mathbb{R}^2$. Then $\mathcal{G}(g, \Lambda_3)$ is a frame for $L^2(\mathbb{R}^2)$, and*

$$A_{\text{opt}} \geq \frac{10}{\sqrt{21}} (0.23)^2 > 0.1154.$$

The proof is computer-assisted: it uses the inequality (5) on each of 65,597 boxes, evaluated in IEEE double precision with an explicit rounding budget and a margin of 10^{-6} . This evaluation was not done in interval arithmetic, except for the 2000 boxes with the smallest margin (Section 5.3).

Theorem 1.3 (the other width; computer-assisted, floating point with margin). *Let $g_{1/2}(s) = e^{-\pi|s|^2/2}$ on $\mathbb{R}^2$ (a unit vector of $L^2(\mathbb{R}^2)$). Then $\mathcal{G}(g_{1/2}, \Lambda_3)$ is a frame for $L^2(\mathbb{R}^2)$ with $A_{\text{opt}} \geq (5/\sqrt{21})(0.18)^2 > 0.0353$. The proof is computer-assisted in the same sense as that of Theorem 1.2, with 78,318 boxes, of which the 650 with the smallest margin were re-checked in interval arithmetic.*

So the answer to Problem 1.1 is yes for both readings of the window, as conjectured. Only one direction of the criterion of Section 2 is used: a positive lower bound for $\sigma_{\min}(P)$ on all of $\mathbb{R}^4$ implies the frame inequality. The converse holds as well (Remark 2.6) but plays no role.

Remark 1.4 (numerical values; not part of the theorems). The minimum of $\sigma_{\min}(P(t, \theta))$ found numerically is 0.2370731445, at $t = (\frac{1}{14}, \frac{1}{14})$, $\theta = (\frac{1}{2}, \frac{1}{2})$; hence $A_{\text{opt}} = (10/\sqrt{21})(0.2370731445)^2 = 0.12265$. An independent formulation of one of the verification runs (duality and a 21×21 symbol, see the paragraph “Verification” in Section 7) gives $A_{\text{opt}} = 0.1226464757$. The upper frame bound is about 2.33. For $g_{1/2}$ the numerical minimum of $\sigma_{\min}$ is about 0.1940 and $A_{\text{opt}} \approx 0.0411$.

1.3 Method, credit and scope

The lattice Λ_3 is real, so a dilation by A_3 turns $\mathcal{G}(g, \Lambda_3)$ into a Gabor system over the separable lattice $\mathbb{Z}^2 \times M\mathbb{Z}^2$ with $M = A_3^T A_3$ (Lemma 2.1). Since $5M$ is integral, this lattice is rational, and the frame property can be analysed fibre by fibre. The fiberization technique is due to Ron and Shen [RS97]; matrix-valued Zak transform criteria for rational lattices go back to Zibulski and Zeevi [ZZ97] (see also [Grö01]). We give a self-contained proof of the identity we use (Propositions 2.2 and 2.4). It reduces Problem 1.1 to a lower bound for the smallest singular value of an explicit 25×21 matrix function on $\mathbb{R}^4$. This bound is proved by a branch-and-bound computation on a box in $\mathbb{R}^4$ obtained from three symmetries (Sections 3 to 5).

The question and the conjecture are due to Gröchenig and Lyubarskii [Grö13, GL20]. New in this note are only the reduction of Λ_3 to the 25×21 matrix, the box inequality and the certificate. We found no printed answer to Problem 1.1 in the sources we searched; they are listed in the paragraph “Scope and priority” (Section 7), and a search that finds nothing is not a proof of novelty. The range $\sqrt{21}/5 < t < \sqrt{24}/5$ is not treated.

2 Reduction to a finite matrix

Throughout,

$$M = A_3^T A_3 = \begin{pmatrix} 1 & \frac{2}{5} \\ \frac{2}{5} & 1 \end{pmatrix}, \quad \det M = \frac{21}{25}, \quad M^{-1} = \frac{1}{21} \begin{pmatrix} 25 & -10 \\ -10 & 25 \end{pmatrix}, \quad \Gamma = M^{-1}\mathbb{Z}^2,$$

and, for a width parameter $a > 0$, $g_a(s) = (2a)^{1/2} e^{-\pi a |s|^2}$ (a unit vector; $g_1 = g$, $g_{1/2} = e^{-\pi |s|^2/2}$), $M_a = aM$ and $\varphi_a(y) = \exp(-\pi y^T M_a y)$. The eigenvalues of M are $\frac{7}{5}$ and $\frac{3}{5}$.

Lemma 2.1 (dilation). *For $A \in GL(2, \mathbb{R})$ the operator $(Uf)(s) = |\det A|^{1/2} f(As)$ is unitary on $L^2(\mathbb{R}^2)$ and $U\pi(x, \xi)U^{-1} = \pi(A^{-1}x, A^T\xi)$. For $A = A_3$ and all $f \in L^2(\mathbb{R}^2)$,*

$$\sum_{\lambda \in \Lambda_3} |\langle f, \pi(\lambda)g_a \rangle|^2 = c_a^2 \sum_{k \in \mathbb{Z}^2} \sum_{l \in M\mathbb{Z}^2} |\langle Uf, \pi(k, l)\varphi_a \rangle|^2, \quad c_a^2 = 2a \det A_3 = \frac{2a\sqrt{21}}{5}.$$

Proof. $(U\pi(x, \xi)f)(s) = |\det A|^{1/2} e^{2\pi i \xi \cdot As} f(As - x) = e^{2\pi i (A^T \xi) \cdot s} (Uf)(s - A^{-1}x)$. The points of Λ_3 are $(A_3 m, A_3 n)$, $m, n \in \mathbb{Z}^2$, and $U\pi(A_3 m, A_3 n)U^{-1} = \pi(m, Mn)$. Moreover $Ug_a(s) = |\det A_3|^{1/2} (2a)^{1/2} e^{-\pi a |A_3 s|^2} = c_a \varphi_a(s)$, and $\langle f, \pi(\lambda)g_a \rangle = \langle Uf, U\pi(\lambda)U^{-1}Ug_a \rangle$. $\square$

The dual lattice of $M\mathbb{Z}^2$ is Γ : for $l = Mn$ and $\gamma = M^{-1}j$ we have $l \cdot \gamma = n \cdot j$, since M is symmetric. Let Q be a Borel fundamental domain of Γ , so $|Q| = \det M^{-1} = \frac{25}{21}$. For $F \in L^2(\mathbb{R}^2)$ and almost every t put $F_t = (F(t + \gamma))_{\gamma \in \Gamma} \in \ell^2(\Gamma)$; then $\int_Q \|F_t\|^2 dt = \|F\|_2^2$. Define $\Phi(t) : \ell^2(\Gamma) \rightarrow \ell^2(\mathbb{Z}^2)$ by $(\Phi(t)v)_k = \sum_{\gamma \in \Gamma} v_\gamma \varphi_a(t + \gamma - k)$.

Proposition 2.2 (fiberization). *The operators $\Phi(t)$ are bounded uniformly in t , and for $F \in L^2(\mathbb{R}^2)$*

$$\sum_{k \in \mathbb{Z}^2} \sum_{l \in M\mathbb{Z}^2} |\langle F, \pi(k, l)\varphi_a \rangle|^2 = |Q| \int_Q \|\Phi(t)F_t\|^2 dt. \quad (2)$$

In particular, if $\|\Phi(t)v\| \geq \mu \|v\|$ for all t and all v , the left-hand side is at least $|Q|\mu^2 \|F\|_2^2$.

Proof. For any lattice $L \subset \mathbb{R}^2$, $\sup_y \sum_{\lambda \in L} \varphi_a(y + \lambda) < \infty$. By the Schur test, $\|\Phi(t)\|^2 \leq \sup_k \sum_{\gamma} \varphi_a(t + \gamma - k) \cdot \sup_{\gamma} \sum_k \varphi_a(t + \gamma - k) \leq C$ with C independent of t . Fix k and put $h_k(s) = F(s)\varphi_a(s - k) \in L^1(\mathbb{R}^2)$ and $H_k(t) = \sum_{\gamma} h_k(t + \gamma) = (\Phi(t)F_t)_k$. Since $e^{-2\pi i l \cdot \gamma} = 1$ for $l \in M\mathbb{Z}^2$, $\gamma \in \Gamma$,

$$\langle F, \pi(k, l)\varphi_a \rangle = \int_{\mathbb{R}^2} h_k(s) e^{-2\pi i l \cdot s} ds = \int_Q H_k(t) e^{-2\pi i l \cdot t} dt.$$

The functions $|Q|^{-1/2} e^{2\pi i l \cdot t}$, $l \in M\mathbb{Z}^2$, form an orthonormal basis of $L^2(Q)$, and $\sum_k \int_Q |H_k|^2 = \int_Q \|\Phi(t)F_t\|^2 \leq C\|F\|_2^2$. Parseval's identity for each k and Tonelli's theorem give (2). $\square$

Lemma 2.3 (lattice data). $5\mathbb{Z}^2 \subset \Gamma \cap \mathbb{Z}^2$ (in fact equality holds), $[\mathbb{Z}^2 : 5\mathbb{Z}^2] = 25$ and $[\Gamma : 5\mathbb{Z}^2] = 21$. The vectors κ_c , $c = 1, \dots, 25$, of $\{-2, \dots, 2\}^2$ represent $\mathbb{Z}^2/5\mathbb{Z}^2$, and the vectors $r_b = M^{-1}(0, b-1)^T = \frac{b-1}{21}(-10, 25)$, $b = 1, \dots, 21$, represent $\Gamma/5\mathbb{Z}^2$.

Proof. $M(5u) = 5Mu \in \mathbb{Z}^2$ for $u \in \mathbb{Z}^2$ because $5M$ is integral. Conversely, if $v \in \mathbb{Z}^2$ and $Mv \in \mathbb{Z}^2$ then $5Mv = (5v_1 + 2v_2, 2v_1 + 5v_2) \in 5\mathbb{Z}^2$, so $2v_1 \equiv 2v_2 \equiv 0 \pmod{5}$. The covolumes of $5\mathbb{Z}^2$ and Γ are 25 and $\frac{25}{21}$, whence the index 21. Finally $r_b - r_{b'} \in 5\mathbb{Z}^2$ if and only if $\frac{b-b'}{21}(-2, 5) \in \mathbb{Z}^2$, that is, $21 \mid b - b'$. $\square$

For $(t, \theta) \in \mathbb{R}^2 \times \mathbb{R}^2$ define the 25×21 matrix

$$P_a(t, \theta)[c, b] = \sum_{n \in \mathbb{Z}^2} \varphi_a(t + r_b - \kappa_c + 5n) e^{2\pi i n \cdot \theta}, \quad (3)$$

and write $P = P_1$. The series and all its derivatives converge absolutely and locally uniformly, so P_a is smooth and $\mathbb{Z}^2$ -periodic in θ .

Proposition 2.4 (Bloch reduction). *For every $t \in \mathbb{R}^2$ and $v \in \ell^2(\Gamma)$,*

$$\|\Phi(t)v\|^2 \geq \min_{\theta \in \mathbb{R}^2} \sigma_{\min}(P_a(t, \theta))^2 \|v\|^2.$$

Proof. Every $\gamma \in \Gamma$ is uniquely $r_b + 5n$, and every $k \in \mathbb{Z}^2$ is uniquely $\kappa_c + 5p$, with $n, p \in \mathbb{Z}^2$ (Lemma 2.3). Write $v_b(n) = v_{r_b+5n}$ and $\psi_{cb}(j) = \varphi_a(t + r_b - \kappa_c + 5j)$. Then $(\Phi(t)v)_{\kappa_c+5p} = \sum_b \sum_n v_b(n) \psi_{cb}(n - p)$. With the Fourier series $\hat{u}(\theta) = \sum_n u(n) e^{2\pi i n \cdot \theta}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\psi_{cb} \in \ell^1(\mathbb{Z}^2)$, the sequence $p \mapsto \sum_n v_b(n) \psi_{cb}(n - p)$ has the Fourier series $\hat{v}_b(\theta) P_a(t, -\theta)[c, b]$. By Parseval,

$$\|\Phi(t)v\|^2 = \int_{\mathbb{T}^2} \|P_a(t, -\theta) \hat{v}(\theta)\|^2 d\theta \geq \min_{\theta} \sigma_{\min}(P_a(t, \theta))^2 \int_{\mathbb{T}^2} \|\hat{v}(\theta)\|^2 d\theta = \min_{\theta} \sigma_{\min}(P_a(t, \theta))^2 \|v\|^2. \quad \square$$

Corollary 2.5 (criterion). *If $\sigma_{\min}(P_a(t, \theta)) \geq s$ for all $(t, \theta) \in \mathbb{R}^4$, then $\mathcal{G}(g_a, \Lambda_3)$ is a frame for $L^2(\mathbb{R}^2)$ with $A_{\text{opt}} \geq (10a/\sqrt{21}) s^2$.*

Proof. By Lemma 2.1, Proposition 2.2 and Proposition 2.4, the frame sum of f is at least $c_a^2 |Q| s^2 \|Uf\|_2^2$, and $c_a^2 |Q| = \frac{2a\sqrt{21}}{5} \cdot \frac{25}{21} = \frac{10a}{\sqrt{21}}$. The upper frame bound is finite by the uniform bound on $\Phi(t)$ in Proposition 2.2. $\square$

Remark 2.6 (not used). Identity (2) and the Parseval identity in the proof of Proposition 2.4 are equalities. Testing them with $F_t = v \mathbf{1}_U(t)$ for small neighbourhoods U shows that $A_{\text{opt}} = (10a/\sqrt{21}) \min_{t, \theta} \sigma_{\min}(P_a(t, \theta))^2$, so $\mathcal{G}(g_a, \Lambda_3)$ is a frame if and only if $P_a(t, \theta)$ has full column rank everywhere. Only Corollary 2.5 is used below; the numerical values in Remark 1.4 are values of this formula. In terms of the fibre operator $G(t) = \Phi(t)^* \Phi(t)$ of Ron and Shen [RS97],

$$(P_a^* P_a)(t, \theta)[b, b'] = \sum_{n \in \mathbb{Z}^2} G(t)[r_b, r_{b'} + 5n] e^{2\pi i n \cdot \theta}, \quad G(t)[\gamma, \gamma'] = \sum_{k \in \mathbb{Z}^2} \varphi_a(t + \gamma - k) \varphi_a(t + \gamma' - k).$$

3 Symmetries and the reduced domain

Let $s_a(t, \theta) = \sigma_{\min}(P_a(t, \theta))$. Changing κ_c or r_b by an element $5p$ of $5\mathbb{Z}^2$ multiplies a row or a column of P_a by $e^{\pm 2\pi i p \cdot \theta}$, so s_a does not depend on the choice of representatives.

Lemma 3.1 (symmetries). *Let $w_1 = \frac{1}{21}(3, 3)$ and $w_2 = \frac{1}{21}(-2, 5)$.*

- (i) $s_a(t, \theta + e) = s_a(t, \theta)$ for $e \in \mathbb{Z}^2$.
- (ii) $s_a(t + \ell, \theta) = s_a(t, \theta)$ for $\ell \in L := \mathbb{Z}^2 + \Gamma$, and $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$.
- (iii) $s_a(-t, \theta) = s_a(t, \theta) = s_a(t, -\theta)$.

Proof. (i) holds because n in (3) is an integer vector. (ii) For $j \in \mathbb{Z}^2$ write $\kappa_c - j = \kappa_{c'} + 5p$; then $P_a(t + j, \theta)[c, b] = e^{2\pi i p \cdot \theta} P_a(t, \theta)[c', b]$, and $c \mapsto c'$ is a permutation. For $\gamma \in \Gamma$ write $r_b + \gamma = r_{b'} + 5q$; then $P_a(t + \gamma, \theta)[c, b] = e^{-2\pi i q \cdot \theta} P_a(t, \theta)[c, b']$. In both cases $P_a(t + \ell, \theta)$ is $P_a(t, \theta)$ multiplied by monomial unitary matrices. For the basis of L : $21L$ is contained in $L' = \{v \in \mathbb{Z}^2 : v_1 + v_2 \equiv 0 \pmod{3}, v_1 - v_2 \equiv 0 \pmod{7}\}$, since $21M^{-1}(j_1, j_2) = (25j_1 - 10j_2, -10j_1 + 25j_2)$ has coordinate sum $15(j_1 + j_2)$ and difference $35(j_1 - j_2)$. Both have index 21 in $\mathbb{Z}^2$: $[L : \mathbb{Z}^2] = [\Gamma : \Gamma \cap \mathbb{Z}^2] = 21$ by Lemma 2.3, and L' is the kernel of $v \mapsto (v_1 + v_2 \pmod{3}, v_1 - v_2 \pmod{7})$, which is onto. Hence $21L = L'$, and $(3, 3), (-2, 5) \in L'$ with determinant 21 form a basis of L' . (iii) Since φ_a is even, writing $-r_b = r_{b'} + 5q$ and $-\kappa_c = \kappa_{c'} + 5p$ gives $P_a(-t, \theta)[c, b] = e^{2\pi i (q-p) \cdot \theta} P_a(t, -\theta)[c', b']$, and $P_a(t, -\theta) = \overline{P_a(t, \theta)}$ since φ_a is real. $\square$

We write $t = uw_1 + vw_2$ and use the coordinates $z = (u, v, \theta_1, \theta_2)$.

Lemma 3.2 (reduced domain). *Every $(t, \theta) \in \mathbb{R}^4$ is mapped by the symmetries (i)–(iii) to a point z of the box*

$$D = [0, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, 1].$$

On D , $t_1 = \frac{3u-2v}{21} \in [-\frac{1}{21}, \frac{1}{7}]$ and $t_2 = \frac{3u+5v}{21} \in [0, \frac{11}{42}]$, so $|t|_\infty \leq \frac{11}{42}$. Hence $\inf_{\mathbb{R}^4} s_a = \inf_D s_a$.

Proof. Reduce u, v modulo 1 by (ii). If $v > \frac{1}{2}$, replace t by $-t$ (iii) and reduce again; this gives $(1-u, 1-v)$ or $(0, 1-v)$ with $1-v < \frac{1}{2}$. Reduce θ modulo 1 by (i); if $\theta_1 > \frac{1}{2}$, replace θ by $-\theta$ (iii) and reduce modulo 1. The bounds for t follow from the values at the four corners of $[0, 1] \times [0, \frac{1}{2}]$. $\square$

Gauge and window. For each (c, b) choose $n_0 = n_0[c, b] \in \mathbb{Z}^2$ with $D_0[c, b] := r_b - \kappa_c + 5n_0 \in [-\frac{5}{2}, \frac{5}{2}]^2$. For real vectors $\alpha_c, \beta_b \in \mathbb{R}^2$ the matrix $\tilde{P} = \text{diag}(e^{2\pi i \alpha_c \cdot \theta}) P_a \text{diag}(e^{-2\pi i \beta_b \cdot \theta})$ has the same singular values as P_a , and

$$\tilde{P}(z)[c, b] = \sum_{m \in \mathbb{Z}^2} \varphi_a(y_m) e^{2\pi i \nu_m \cdot \theta}, \quad y_m = t + D_0[c, b] + 5m, \quad \nu_m = n_0[c, b] + m + \alpha_c - \beta_b. \quad (4)$$

We split $\tilde{P} = P_w + T$, where P_w is the sum over the nine $m \in \{-1, 0, 1\}^2$ and T is the rest. The certificate uses vectors $\alpha_c, \beta_b \in \frac{1}{1024}\mathbb{Z}^2$, chosen by least squares so as to make the θ -derivatives of $\tilde{P}$ small; they are stored in the certificate. The choice affects only the number of boxes, not the validity of the argument.

4 The box inequality

Lemma 4.1 (tail). *For $z \in D$ every entry of T satisfies $|T[c, b]| \leq \tau_a$ and $\|T\|_2 \leq \sqrt{525} \tau_a$, where*

$$\tau_a = \sum_{j \geq 2} 8j \exp\left(-\frac{3\pi a}{5} \left(5j - \frac{58}{21}\right)^2\right).$$

Numerically, $\tau_1 < 2.1 \cdot 10^{-42}$ and $\sqrt{525} \tau_1 < 4.8 \cdot 10^{-41}$; $\tau_{1/2} < 6.0 \cdot 10^{-21}$ and $\sqrt{525} \tau_{1/2} < 1.4 \cdot 10^{-19}$.

Proof. There are $8j$ vectors m with $|m|_\infty = j$. For such m , a coordinate i with $|m_i| = j$ gives $|(y_m)_i| \geq 5j - \frac{5}{2} - \frac{11}{42} = 5j - \frac{58}{21}$ by Lemma 3.2 and $D_0 \in [-\frac{5}{2}, \frac{5}{2}]^2$. Since the smallest eigenvalue of M is $\frac{3}{5}$, $\varphi_a(y_m) \leq \exp(-\frac{3\pi a}{5} |y_m|_\infty^2)$. The norm bound follows from $\|T\|_2 \leq \|T\|_F$. The numerical values are evaluated in `check_note.py` (part B), with a relative safety factor of 1.01. $\square$

Let $\mathcal{B} = \{z : |z_i - z_i^0| \leq h_i\} \subset D$ be a box with centre $z^0 = (u^0, v^0, \theta^0)$, $t^0 = u^0 w_1 + v^0 w_2$, and half-widths $h = (h_u, h_v, h_1, h_2)$. Let $D_u, D_v, D_{\theta_1}, D_{\theta_2}$ be the partial derivatives of P_w at z^0 , and define

$$E_1 = \max_{\varepsilon \in \{\pm 1\}^3} \|h_u D_u + \varepsilon_1 h_v D_v + \varepsilon_2 h_1 D_{\theta_1} + \varepsilon_3 h_2 D_{\theta_2}\|_2, \quad R_2 = \|\bar{R}\|_F,$$

$$\bar{R}[c, b] = \frac{1}{2} \sum_{m \in \{-1, 0, 1\}^2} \bar{F}_m \left[\left(\sum_i h_i g_{i,m} \right)^2 + 2\pi \sum_{p,q \in \{u,v\}} h_p h_q |w_p^T M_a w_q| \right],$$

where $w_u = w_1$, $w_v = w_2$, $y_m^0 = t^0 + D_0[c, b] + 5m$, $\rho = h_u |w_1| + h_v |w_2|$, $\|y\|_{M_a} = (y^T M_a y)^{1/2}$,

$$\bar{F}_m = \exp\left(-\pi \max(0, \|y_m^0\|_{M_a} - \sqrt{7a/5} \rho)^2\right), \quad \bar{y}_m = |M_a y_m^0| + \frac{7a}{5} \rho,$$

$g_{u,m} = 2\pi |w_1| \bar{y}_m$, $g_{v,m} = 2\pi |w_2| \bar{y}_m$ and $g_{\theta_j,m} = 2\pi |(\nu_m)_j|$ (Euclidean norms $|\cdot|$; ν_m as in (4)).

Proposition 4.2 (box inequality). *For every $z \in \mathcal{B}$,*

$$\sigma_{\min}(\tilde{P}(z)) \geq \sigma_{\min}(P_w(z^0)) - E_1 - R_2 - \sqrt{525} \tau_a.$$

Proof. For rectangular matrices X, Y of the same size, $\sigma_{\min}(X+Y) \geq \sigma_{\min}(X) - \|Y\|_2$, since $\sigma_{\min}(X) = \min_{|x|=1} |Xx|$ for a matrix with more rows than columns. Let $z = z^0 + \sigma \in \mathcal{B}$. Taylor's formula with integral remainder, applied to each entry of the smooth function P_w , gives $P_w(z) = P_w(z^0) + \sum_i \sigma_i D_i + R$ with $|R[c, b]| \leq \frac{1}{2} \sum_{i,j} h_i h_j \sup_{\mathcal{B}} |\partial_i \partial_j P_w[c, b]|$. The map $\sigma \mapsto \|\sum_i \sigma_i D_i\|_2$ is convex, so on the box it is maximal at a vertex, and the 16 vertices give at most the 8 values in E_1 because $\| -X \|_2 = \|X\|_2$.

For one term $f = \varphi_a(y) e^{2\pi i \nu \cdot \theta}$ of (4) we have $\partial_u y = w_1$, $\partial_v y = w_2$, $\partial_{\theta} y = 0$, and $\partial_p \varphi_a = -2\pi (w_p \cdot M_a y) \varphi_a$, $\partial_p \partial_q \varphi_a = [(2\pi)^2 (w_p \cdot M_a y)(w_q \cdot M_a y) - 2\pi w_p^T M_a w_q] \varphi_a$ for $p, q \in \{u, v\}$, and $\partial_{\theta_j} e^{2\pi i \nu \cdot \theta} = 2\pi i \nu_j e^{2\pi i \nu \cdot \theta}$. On $\mathcal{B}$ we have $|t - t^0| \leq \rho$, hence $\|y\|_{M_a} \geq \|y^0\|_{M_a} - \sqrt{7a/5} \rho$ and $\varphi_a(y) \leq \bar{F}$, and $|M_a y| \leq \bar{y}$ (the largest eigenvalue of M_a is $7a/5$). Therefore $|\partial_p \partial_q f| \leq [g_p g_q + 2\pi |w_p^T M_a w_q|] \bar{F}$, $|\partial_p \partial_{\theta_j} f| \leq g_p g_{\theta_j} \bar{F}$ and $|\partial_{\theta_j} \partial_{\theta_k} f| \leq g_{\theta_j} g_{\theta_k} \bar{F}$, and summing over i, j and over the nine terms gives $|R[c, b]| \leq \bar{R}[c, b]$. Hence $\|R\|_2 \leq \|R\|_F \leq R_2$. Finally $\sigma_{\min}(\tilde{P}(z)) \geq \sigma_{\min}(P_w(z)) - \|T(z)\|_2$, and Lemma 4.1 applies. $\square$

A box is *accepted* for the target s if the computed quantities satisfy

$$\widehat{\sigma_{\min}}(P_w(z^0)) - \hat{E}_1 - \hat{R}_2 - 10^{-6} \geq s, \quad (5)$$

where the hats denote the values computed in double precision. If the rounding errors of the three computed quantities add up to less than $10^{-6} - \sqrt{525} \tau_a$, then Proposition 4.2 gives $\sigma_{\min}(\tilde{P}) \geq s$ on the box. For $a = 1$ and $a = \frac{1}{2}$ the tail term is below 10^{-18} , so the margin left for rounding is more than $0.99 \cdot 10^{-6}$.

5 The computation

5.1 The certificate

The program `certify.py` starts from the $8 \times 4 \times 4 \times 8 = 1024$ cubes of side $\frac{1}{8}$ that tile D . For each box it evaluates the left-hand side of (5) in double precision (NumPy [HMvdW⁺20], singular values by LAPACK [ABB⁺99]); it accepts the box if (5) holds, and otherwise bisects it along the coordinate i with the largest $h_i \|D_i\|_F$. A box is accepted only through (5), so the splitting rule cannot affect validity. The certificate file stores the representatives κ_c and $21r_b$ (integers), the gauge, the target, the width a , and for each accepted box its dyadic levels and indices and the four computed numbers.

For $a = 1$ and $s = 0.23$ the computation accepts 65,597 boxes after processing 130,170, with at most four bisections in each coordinate. The smallest left-hand side of (5) is 0.2300001490; the smallest value of $\widehat{\sigma}_{\min}(P_w(z^0))$ is 0.23710, and the largest values of $\widehat{E}_1$ and $\widehat{R}_2$ are 0.0599 and 0.0336. For $a = \frac{1}{2}$ and $s = 0.18$ it accepts 78,318 boxes after processing 155,612, with smallest left-hand side 0.1800000626. The runs take 93 and 111 seconds.

5.2 Checks of the certificate

The program `check_certificate.py` was written separately and shares no code with `certify.py`. It checks (1) that the boxes partition D , by recursive bisection of the initial cubes; (2) that κ_c and r_b are complete residue systems of $\mathbb{Z}^2/5\mathbb{Z}^2$ and $\Gamma/5\mathbb{Z}^2$; (3) inequality (5) for every box, with its own implementation of (4) and of Proposition 4.2; its values differ from the stored ones by at most $7.8 \cdot 10^{-16}$ ($6.9 \cdot 10^{-16}$ for $a = \frac{1}{2}$); and (4) at 1500 box centres, the stored $\widehat{\sigma}_{\min}(P_w)$ against the square root of the smallest eigenvalue of the Gram symbol of Remark 2.6, computed from the k -sums with other representatives by `indep_gram.py` (difference at most $2.9 \cdot 10^{-15}$). Eight corrupted certificates are rejected: a box removed, duplicated, enlarged or moved to its sibling; the target raised to 0.2375; the half-widths in θ_2 multiplied by 4; a duplicated representative; a representative outside Γ . The program `check_note.py` of this note (Section 7) checks the partition again, as an exact cover of the finest dyadic grid (16,777,216 cells), and recomputes (5) in pure Python for 60 boxes of each certificate (the 40 with the smallest stored bound and 20 random ones; largest deviation $6.1 \cdot 10^{-16}$).

5.3 Floating point

The certificate rests on the assumption that, for every box, the rounding errors in the three computed quantities of (5) add up to less than $0.99 \cdot 10^{-6}$ (the margin minus the tail term). The following estimate, in the standard model of floating-point arithmetic and under the assumption that the platform's exponential, sine and cosine are accurate to a few units in the last place, makes this plausible; it is not a machine-checked bound. All matrices involved have spectral norm at most 1.1. Each entry of P_w and of the D_i is a sum of nine terms $\varphi_a(y)e^{2\pi i\nu \cdot \theta}$ (times a factor of modulus at most 20 for the derivatives) with $0 \leq \pi y^T M_a y \leq 510$ and $|2\pi\nu \cdot \theta| \leq 15$ on D , so its absolute error is of order 10^{-13} or less. The singular value decomposition of LAPACK is backward stable, so the computed singular values of a computed matrix $\hat{X}$ differ from those of X by at most $p(n)2^{-53}\|\hat{X}\|_2$ with a modest polynomial $p(n)$ [ABB⁺99, Hig02]. The quantity R_2 is an explicit expression evaluated with a relative error of order 10^{-14} . Altogether the errors are of order 10^{-11} at most, five orders of magnitude below the margin.

The errors were also measured. The program `hp_check.py` recomputes $\sigma_{\min}(P_w(z^0))$ and E_1 , but not R_2 , with 30 significant digits (mpmath [J⁺23]) for the 300 boxes with the smallest stored bound: the largest deviations are $2.2 \cdot 10^{-16}$ and $2.1 \cdot 10^{-17}$. One of the verification runs recomputed all three quantities, R_2 included, with 30 digits for the 30 boxes with the smallest stored bound: deviations $2.2 \cdot 10^{-16}$, $1.4 \cdot 10^{-17}$ and $8.7 \cdot 10^{-19}$.

Interval re-checks. For the 1950 boxes of the certificate for $a = 1$ with the smallest stored bound, and for the 650 such boxes for $a = \frac{1}{2}$, the program `iv_check.py` of this note recomputes the box test in interval arithmetic (mpmath.iv, 30 digits, outward rounding), starting from exact rational data. It encloses the entries of $P_w(z^0)$ and of the D_i , and bounds R_2 from above. With $\mu = \hat{E}_1(1 + 10^{-10})$ it proves $\|X_\varepsilon\|_2 < \mu$ for the eight matrices X_ε in the definition of E_1 , by an interval Cholesky factorisation of $\mu^2 I - X_\varepsilon^* X_\varepsilon$, and it proves $\sigma_{\min}(P_w(z^0)) > s + \mu + R_2 + \sqrt{525} \tau_a$ by an interval Cholesky factorisation of $P_w^* P_w - (s + \mu + R_2 + \sqrt{525} \tau_a)^2 I$. A Hermitian interval matrix whose interval Cholesky factorisation runs to the end with pivots of positive lower bound contains only positive definite matrices [AM93, Rum10]. All these boxes pass, without the margin 10^{-6} .

A second program, `r2_rigorous.py`, written in the third verification run (paragraph “Verification” in Section 7) and sharing no code with `iv_check.py`, re-checked the 2000 boxes for $a = 1$ and the 650 boxes for $a = \frac{1}{2}$ with the smallest stored bound by a different method. It uses no stored number: it encloses all values of the exponential, sine, cosine and square root in mpmath.iv (113 bits), computes its own upper bounds for R_2 and $\sqrt{525} \tau_a$, and replaces each enclosure by a point of the grid $2^{-48} \mathbb{Z}$ with a rigorous radius. This gives matrices $\hat{X}_\varepsilon$ and $\hat{P}$ with dyadic entries and upper bounds δ_ε for $\|X_\varepsilon - \hat{X}_\varepsilon\|_F$ and δ for $\|P_w(z^0) - \hat{P}\|_F$. With dyadic numbers μ_ε slightly above $\|\hat{X}_\varepsilon\|_2$ and $\kappa \geq s + \max_\varepsilon(\mu_\varepsilon + \delta_\varepsilon) + R_2 + \sqrt{525} \tau_a + \delta$, it decides the positive definiteness of $\mu_\varepsilon^2 I - \hat{X}_\varepsilon^* \hat{X}_\varepsilon$ and of $\hat{P}^* \hat{P} - \kappa^2 I$ exactly, by Sylvester’s criterion with fraction-free integer elimination. By Weyl’s inequality this proves that the right-hand side of Proposition 4.2 exceeds s , without the margin 10^{-6} . All 2650 boxes pass; with the target raised by $2 \cdot 10^{-6}$ the three tightest boxes are rejected. Every box outside these 2000 (respectively 650) has a stored bound above the target by more than $1.42 \cdot 10^{-4}$ (respectively $4.8 \cdot 10^{-5}$), that is, more than 142 (respectively 48) times the margin. The remaining 63,597 (respectively 77,668) boxes were not evaluated rigorously, and Theorems 1.2 and 1.3 are stated with this qualification.

5.4 Proofs of Theorems 1.2 and 1.3

Let $a = 1$ and $s = 0.23$. By Section 5.2 the accepted boxes partition D and satisfy (5); under the floating-point assumption of Section 5.3, Proposition 4.2 gives $\sigma_{\min}(\tilde{P}(z)) \geq 0.23$ for every $z \in D$. Since $\tilde{P}$ and P have the same singular values, $s_1 \geq 0.23$ on D , hence on $\mathbb{R}^4$ by Lemma 3.2, and Corollary 2.5 gives $A_{\text{opt}} \geq (10/\sqrt{21}) 0.23^2 = 0.11543 \dots$. For $a = \frac{1}{2}$ and $s = 0.18$ the same argument gives $A_{\text{opt}} \geq (5/\sqrt{21}) 0.18^2 = 0.03535 \dots$ $\square$

6 Consistency checks and remarks

Known cases. The same construction applies to any M with $5M$ integral. For Λ_1 , which does not give a frame [GL20, Thm. 4.1], $M_1 = A_1^T A_1 = \begin{pmatrix} 1 & 1/5 \\ 1/5 & 1 \end{pmatrix}$, $[M_1^{-1} \mathbb{Z}^2 : 5\mathbb{Z}^2] = 24$, and the 25×24 matrix has $\sigma_{\min} = 2.6 \cdot 10^{-16}$ at $t = (\frac{5}{6}, \frac{1}{3})$, $\theta = (\frac{1}{2}, \frac{1}{2})$, an exact zero up to rounding; a box of half-width 10^{-3} around this point gets the bound -0.0063 and is not accepted. For the lattice $\frac{2}{\sqrt{5}}(\mathbb{Z} + i\mathbb{Z})^2$, a product of two one-dimensional Gaussian frames ($M = \frac{4}{5}I$), the 25×16 matrix has $\min \sigma_{\min} = 0.4545 > 0$. For Λ_3 , a grid of 20^4 points and local minimisation from 12 starting points give the value of Remark 1.4.

Why the known theorems do not decide Λ_3 . (a) For $m \in (\mathbb{Z} + i\mathbb{Z})^2 \setminus \{0\}$, $|A_3 m|^2 = m^* M m = \frac{1}{5}(3(|m_1|^2 + |m_2|^2) + 2|m_1 + m_2|^2) \geq 1$, with equality for $m = (1, 0)$. Hence every triangular form S of a basis of Λ_3 has $\gamma_1 \geq 1$ and $\gamma_2 = \det A_3 / \gamma_1 < 1$, and neither part of [GL20, Prop. 2.2] applies. (b) Λ_3 is not one of the non-frame lattices of [GL20, Thm. 4.1, Cor. 4.2, Cor. 4.3]. If $\Lambda_3 = UX(\mathbb{Z} + i\mathbb{Z})^2$ with U unitary, the Hermitian forms $X^* X$ and M are equivalent over $GL(2, \mathbb{Z}[i])$, so they have the same determinant $\frac{21}{25}$ and take the same values, which lie in $\frac{1}{5}\mathbb{Z}$. In Cor. 4.3, $X = \begin{pmatrix} 1 & p/q \\ 0 & \gamma \end{pmatrix}$ with $\gamma^2 = 1 - q^{-2}$, and $\det X^* X = \frac{21}{25}$ forces $q = \frac{5}{2} \notin \mathbb{N}$. In Thm. 4.1 and Cor. 4.2, $X = \text{diag}(\alpha, \beta) \begin{pmatrix} 1 & 1/q \\ 0 & \gamma \end{pmatrix}$ with $\alpha, \beta \geq 1$,

$q \in \mathbb{Z}[i]$, $|q| \geq 2$, $\gamma^2 = 1 - |q|^{-2} \geq \frac{3}{4}$; then $\alpha^2 \leq \frac{28}{25}$ and $\alpha^2 \in \frac{1}{5}\mathbb{Z}$ force $\alpha = 1$, and $|q|^{-2} + \frac{21}{25} \in \frac{1}{5}\mathbb{Z}$ forces $|q|^2 = \frac{25}{4}$, which is not the norm of a Gaussian integer. (c) [GL20, Thm. 5.3] applies to Λ_3 (its triangular form has $\max \gamma_j = 1$) and gives a weak reconstruction formula, which is weaker than the frame inequality. (d) The density criterion of [LW23] requires $\text{vol}(\Lambda) < \frac{1}{2}$ in dimension two (and a transcendence condition), while $\text{vol}(\Lambda_3) = \frac{21}{25}$. The non-existence criterion of [dDPLT26, Thm. 1.2], which would apply to every window with continuous Zak transform, concerns lattices that contain a product $L \times L^\perp$ with finite index; they have integer density, while the density of Λ_3 is $\frac{25}{21}$. These two statements are taken from the reading of these papers described in the paragraph “Scope and priority”. We found no case of the results of [GL25, LTW24, RUZ24, PR13] that contains Λ_3 with the standard Gaussian.

What remains open. The range $\sqrt{21}/5 < t < \sqrt{24}/5$ of [GL20, end of Sect. 4] is not treated. The stability of Gabor frames under small changes of the lattice [FK04] suggests that the frame property persists for t slightly above $\sqrt{21}/5$, but we did not make this quantitative. One of the verification runs found numerically positive minima for $t^2 = \frac{22}{25}$ and $t^2 = \frac{23}{25}$ (optimal lower bounds about 0.082 and 0.048); these are numerical observations only. The optimal frame bounds of $\mathcal{G}(g, \Lambda_3)$ are not certified, and the frame property is not certified in interval arithmetic for all boxes.

7 Computations and reproducibility

The analytic part of the proof (Sections 2 to 4) uses no computation; the computational part is the certificate of Section 5. Command lines and running times are in the `README.md` of the reproducibility package. The program of this note, `check_note.py` (standard library only, about 15 seconds), checks the finite and exact parts of the proof: in exact rational arithmetic the data of Lemmas 2.3, 3.1 and 3.2, the constant $10/\sqrt{21}$, the residue systems stored in the certificates and the covering of $\mathbb{R}^4$ by D (20,000 random rational points); the tail bounds of Lemma 4.1 and the lattice facts of Section 6; in pure Python the values of Remark 1.4 and Section 6, the identity of Remark 2.6 and the symmetries at random points; and, reading the certificates without NumPy, the partition of D and the box test for 60 boxes of each certificate. The folder `original/` holds the programs with which the result was first obtained, the two certificates and the recorded outputs. Besides the programs named in Section 5, it contains `reduction_check.py`, which compares the frame sum of four random generalised Gaussians, computed in the original coordinates from closed-form inner products, with the right-hand side of (2) after Lemma 2.1 (relative difference at most $1.4 \cdot 10^{-15}$). The folder `interval_check/` holds `iv_check.py` with its outputs, the folders `verification_run_A/` and `verification_run_B/` hold the programs and outputs of the first two verification runs, and `independent_run_2/` those of the third.

Verification. The result was first obtained, with its proof and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first run checked the statement against [Grö13, GL20], re-derived Sections 2 to 4 line by line, and compared `certify.py` and `check_certificate.py` with the formulas. Its own checker verified the partition by occupancy counting and recomputed the box test for all 65,597 and 78,318 boxes (largest deviation from the stored values $7.8 \cdot 10^{-16}$). It tested the reduction end to end: for $f = g$ the frame sum equals $(\sum_{m \in \mathbb{Z}^2} e^{-\pi m^T M m})^2 = 1.4868700738812$ in closed form, and the fibre side (2) and the Bloch side agree with it to $3 \cdot 10^{-15}$. It also recomputed the 30 tightest boxes with 30 digits (Section 5.3) and repeated the controls of Section 6. The second run used a different formulation: by the duality principle [RS97] the lower frame bound is the lower Riesz bound of the system over the adjoint lattice divided by $\text{vol}(\Lambda_3)$, and the Gram operator of that system lies in a rational noncommutative 4-torus with 21-dimensional irreducible representations. The resulting 21×21 symbol gives $A_{\text{opt}} = 0.1226464757$

numerically, in agreement with Remark 1.4 to 10^{-10} (a test of the constant $10/\sqrt{21}$), and its own branch-and-bound certificate, in double precision with a margin of 10^{-8} , gives $A_{\text{opt}} \geq 0.1220$ for g and $A_{\text{opt}} \geq 0.0393$ for $g_{1/2}$. It re-ran `certify.py`, which produced a certificate identical byte for byte to the stored one, re-ran the other programs of `original/` with the recorded outputs, and repeated the literature search. Both runs confirmed the result as a computer-assisted proof in floating point. Their required corrections were made: the tail bound for $a = \frac{1}{2}$ is stated separately (Lemma 4.1); the bibliographic data of [GL25, GL20] were completed from Crossref; Section 5.3 says that `hp_check.py` does not recompute R_2 and cites the recomputation by the first run; the floating-point qualification stands next to the result; and only the implication of Corollary 2.5 is used. The interval re-check and `check_note.py` were added when this note was written; these two runs did not examine them.

A third independent verification run, also AI-assisted, then examined the text of this note and the reproducibility package, including `check_note.py` and `iv_check.py`, and wrote its own programs (`independent_run_2/`). It read [Grö13, pp. 593–596] and [GL20] again for the statement, the window convention and the quoted results, and re-derived Sections 2 to 4 line by line. Its own double-precision implementation of (4), Proposition 4.2 and (5) reproduced the box test for all 65,597 and 78,318 boxes (largest deviation from the stored bounds $5.6 \cdot 10^{-16}$ and $6.4 \cdot 10^{-16}$), and it verified the exact cover of D by occupancy counting on the finest dyadic grid and by the total volume in exact arithmetic. Its rigorous re-check of the 2000 and 650 boxes with the smallest stored bound is described in Section 5.3. It tested Lemma 2.1, (2) and the Parseval identity of Proposition 2.4 together, for random generalised Gaussians and both widths: the frame sum computed in the original coordinates from closed-form inner products and the integral of $\|P_a(t, -\theta)\hat{v}(\theta)\|^2$ agree to $3.5 \cdot 10^{-15}$. It recomputed the identity of Remark 2.6, the values of Remark 1.4, the tail bounds of Lemma 4.1 in interval arithmetic and the zero for Λ_1 of Section 6; it re-ran the programs of the package from the archive, with the recorded outputs (`certify.py` again gave certificates identical byte for byte to the stored ones), and it repeated the literature search. It confirmed the result as a computer-assisted proof with the qualification of Section 5.3 and found no mathematical error. Its required corrections concerned the record of its own checks (Section 5.3, this paragraph and the list of folders above) and the paragraph “Scope and priority”, where its searches and one further citing work were added.

Scope and priority. The note answers Problem 1.1, for both readings of the window, by a computer-assisted proof whose computational part was evaluated in floating point with an explicit rounding budget and a margin; only the 2000 and 650 boxes with the smallest margins were re-checked in interval arithmetic. The optimal frame bounds are not certified, and the range $\sqrt{21}/5 < t < \sqrt{24}/5$ is not treated. The problem, the conjecture and the context of Section 1 are due to Gröchenig and Lyubarskii [Grö13, GL20], with the theorem of [Grö11]; the fiberization method is that of Ron and Shen [RS97], and matrix-valued Zak transform criteria for rational lattices go back to Zibulski and Zeevi [ZZ97]. Only the reduction of Λ_3 to the matrix P , the box inequality and the certificate are, to our knowledge, not in print.

What was read: [Grö13], pp. 593–596, in the open file on the publisher’s site; [GL20] in full in the arXiv version (v1), whose section and theorem numbers we use, the journal version not being compared. The results of [Grö11, Lyu92, Sei92, SW92] are quoted as stated in [Grö13, GL20], and [RS97, ZZ97, Grö01] are cited for the method only; the identities used are proved here. The works citing [GL20] (Semantic Scholar: 9; OpenAlex: 7), [Grö11] (OpenAlex: 23; Semantic Scholar: 38) and [Grö13] (OpenAlex: 2, on other subjects) were listed. Of [LW23] (including its Sections 4.1 and 4.2 on complex lattices), [LTW24, RUZ24, GL25, dDPLT26, PR13] at least the statements of the main results were read, of [UZ26] the full text, and of [LMZ26, ETV25] and five further citing works on other subjects (Gabor frames on manifolds, a fractal uncertainty principle, sampling on a Siegel domain, complex Weyl symbols, two-dimensional Gaussian lattice sums) the abstracts. The

searches were made on 19 and 20 October 2026: queries to the arXiv API (Gaussian Gabor frames and lattices, Bargmann–Fock lattices, complex lattices, computer-assisted Gabor frames, multivariate Gaussian Gabor systems, sorted by date), to zbMATH Open (Gabor and Gaussian in the title, 2014–2026) and to Crossref, the citation lists above, and three web searches for the specific lattice; the third verification run repeated queries to the arXiv API, zbMATH Open, Crossref and OpenAlex (works citing [GL20, Grö13]) and made one further web search. No printed answer to Problem 1.1 was found in these sources; none of the works listed treats Λ_3 or the range $\sqrt{21}/5 \leq t < \sqrt{24}/5$, and among the texts read the number $\sqrt{21}/5$ occurs only in [Grö13, GL20]. A search that finds nothing is not a proof of novelty, and we make no claim of priority beyond the statement above.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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