

The Minimum Distance of Unshifted Spherical Fibonacci Lattices

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Abstract

For the unshifted spherical Fibonacci lattice with $F_n \geq 2$ points, we prove that the least Euclidean distance is $2/\sqrt{F_n}$. The unique closest pair consists of the north pole and the next point in the defining order. This answers the separation question stated in Brauchart's contribution to Oberwolfach Report 40/2012. The proof combines a global inequality for the Lambert equal-area map with Zaremba's classical formula for the figure of merit of a Fibonacci lattice. More generally, a rational lattice with N points and Zaremba index at least $N/4$ has exactly the same minimum distance after the unshifted Lambert lift. No limiting argument or numerical hypothesis is used.

1 The point set and the result

Put $F_0 = 0$, $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. For $N \geq 2$ and an integer a coprime to N , define

$$z_k = \left(2\sqrt{\frac{k}{N} \left(1 - \frac{k}{N}\right)} \cos \frac{2\pi ak}{N}, 2\sqrt{\frac{k}{N} \left(1 - \frac{k}{N}\right)} \sin \frac{2\pi ak}{N}, 1 - \frac{2k}{N} \right), \quad 0 \leq k < N. \quad (1)$$

Write $Z(N, a) = \{z_0, \dots, z_{N-1}\} \subset \mathbb{S}^2$ and

$$\delta(Z) = \min_{\substack{x, y \in Z \\ x \neq y}} |x - y|,$$

where $|\cdot|$ is Euclidean distance in $\mathbb{R}^3$. For $N = F_n$ and $a = F_{n-1}$, this is the spherical Fibonacci lattice of Aistleitner, Brauchart and Dick [1, Section 5.2].

Brauchart [2, p. 2438] conjectured that the pole-to-next-point distance is the minimum for this construction. The following theorem establishes the stated formula.

Theorem 1. *For every $n \geq 3$,*

$$\delta(Z(F_n, F_{n-1})) = \frac{2}{\sqrt{F_n}}.$$

The unique unordered pair attaining the minimum is $\{z_0, z_1\}$.

The index range in (1) is the one used in [1, Section 5.2] and in the defining formula in [2, p. 2438]. The latter page subsequently writes the set with indices $1, \dots, F_n$. Those two endpoint conventions give isometric sets: the orthogonal map $(x, y, z) \mapsto (x, -y, -z)$ takes z_k to z_{N-k} . With the second convention, the closest pair is $\{z_{N-1}, z_N\}$. Neither convention is the half-step-shifted irrational spiral studied in [4, Section 2]; no assertion about its exact minimum is intended here.

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2 A global distance inequality

The relevant geometry can be treated without any approximation by tangent planes.

Lemma 2. *Let $x, y \in \mathbb{S}^2$ have third coordinates t_1, t_2 and longitude difference θ . Set $h = |t_1 - t_2|$ and $c = \cos \theta$. Then*

$$|x - y|^2 \geq \begin{cases} 2h, & c \leq 0, \\ 2h\sqrt{1 - c^2}, & c \geq 0. \end{cases}$$

Longitudes at a pole may be assigned arbitrarily.

Proof. Put

$$v = 1 - t_1 t_2, \quad r = \sqrt{(1 - t_1^2)(1 - t_2^2)}.$$

The scalar-product formula and an elementary identity give

$$|x - y|^2 = 2(v - cr), \quad v^2 = r^2 + h^2, \quad v \geq 0. \quad (2)$$

If $c \leq 0$, then $v - cr \geq v \geq h$. If $c \geq 0$, Cauchy–Schwarz gives

$$cr + \sqrt{1 - c^2} h \leq \sqrt{c^2 + (1 - c^2)} \sqrt{r^2 + h^2} = v.$$

Substitution in (2) proves the result. At a pole $r = 0$, so the same calculation applies. $\square$

Write $\|u\| = \text{dist}(u, \mathbb{Z})$. For two points in (1), put $q = j - i$ with $i < j$ and $d = \|aq/N\|$. Their height difference is $2q/N$. If $d \geq 1/4$, the cosine of the longitude difference is nonpositive. If $0 \leq d \leq 1/4$, concavity of sine on $[0, \pi/2]$ gives $\sin(2\pi d) \geq 4d$. Consequently Lemma 2 yields

$$|z_i - z_j|^2 \geq \begin{cases} 4q/N, & d \geq 1/4, \\ 16qd/N, & d \leq 1/4. \end{cases} \quad (3)$$

Both inequalities are consistent at $d = 1/4$.

3 The arithmetic criterion

The dual lattice and its Zaremba index are

$$L^*(N, a) = \{(u, v) \in \mathbb{Z}^2 : u + av \equiv 0 \pmod{N}\},$$

$$\rho(N, a) = \min_{\substack{(u, v) \in L^*(N, a) \\ (u, v) \neq (0, 0)}} \max(1, |u|) \max(1, |v|).$$

This is the classical figure of merit for the rank-one rule with generating vector $(1, a)$; see [5, 3].

Theorem 3. *For $N \geq 2$ and $\gcd(a, N) = 1$,*

$$\delta(Z(N, a))^2 \geq \min \left\{ \frac{4}{N}, \frac{16\rho(N, a)}{N^2} \right\}.$$

If $\rho(N, a) \geq N/4$, then $\delta(Z(N, a)) = 2/\sqrt{N}$. If $\rho(N, a) < N/4$, its unique closest pair is $\{z_0, z_1\}$.

Proof. For $1 \leq q < N$, choose an integer p nearest to aq/N and put $u = pN - aq$. Since a and N are coprime, $u \neq 0$. The vector (u, q) belongs to $L^*(N, a)$, hence

$$q \left\| \frac{aq}{N} \right\| = \frac{q|u|}{N} \geq \frac{\rho(N, a)}{N}. \quad (4)$$

Apply (3) to obtain the asserted lower bound. The point $z_0 = (0, 0, 1)$ satisfies

$$|z_0 - z_1|^2 = 2 \left(1 - \left(1 - \frac{2}{N} \right) \right) = \frac{4}{N}. \quad (5)$$

This proves equality under the stated index threshold.

Suppose now that the threshold is strict. The second case of (3), together with (4), is strictly larger than $4/N$. In the first case equality forces $q = 1$. It also forces equality in $v - cr \geq v \geq h$ from Lemma 2, so $v = h$ and $r = 0$. One of the two points must therefore be a pole. The only pole in $Z(N, a)$ is z_0 , giving $\{z_0, z_1\}$. $\square$

Proof of Theorem 1. Zaremba's classical formula [5, p. 186] is

$$\rho(F_n, F_{n-1}) = F_{n-2} \quad (n \geq 3). \quad (6)$$

The same formula and its comparison with $F_n/3$ are recalled in [3, p. 7]. For $n \geq 4$, the recurrence gives

$$F_n = 2F_{n-2} + F_{n-3} \leq 3F_{n-2};$$

for $n = 3$, the ratio is $F_1/F_3 = 1/2$. Thus in all cases $F_{n-2} \geq F_n/3 > F_n/4$. Apply Theorem 3. $\square$

4 Scope and checks

The arithmetic input (6) is an established theorem, not a new claim of this paper. The argument above supplies the geometric implication needed for the spherical separation question. It concerns the exact rational, unshifted construction in (1), not discrepancy, optimal packing among all N -point sets, or the exact separation of shifted irrational Fibonacci configurations.

A common height shift $k \mapsto k + \eta$, $0 \leq \eta \leq 1$, in the latitude coordinates preserves (3) and the lower bound in Theorem 3, because height differences are unchanged. For $0 < \eta < 1$, pole attainment (5) is unavailable, so equality is not asserted.

The arithmetic condition cannot simply be omitted. For $N = 12, a = 1$, the pair z_5, z_6 has

$$|z_5 - z_6|^2 = 2 - \frac{\sqrt{105}}{6} < \frac{1}{3} = \frac{4}{N}, \quad \rho(12, 1) = 1.$$

The accompanying reproducibility script checks the polynomial identity in Lemma 2 symbolically and checks the classical arithmetic formula for all $1 \leq q < F_n$, $3 \leq n \leq 30$, using integers. It also computes all pair distances for $3 \leq n \leq 17$ in floating point as a diagnostic. These finite checks are not used to prove either theorem.

This unrefereed note was prepared with AI assistance in mathematical exploration, source retrieval, drafting and computational checks. Its proof has undergone an originating-system self-audit; no independent review or proof-assistant verification is claimed. The bounded literature search did not locate the same geometric resolution of the cited question; this does not certify absolute priority.

References

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