

Elliptic Values of Word Maps on $\mathrm{SL}(2, \mathbb{R})$ and a Question of Calegari and Walker on Rotation Numbers

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October 17, 2026

Abstract

In an Oberwolfach report of 2012, D. Calegari, reporting on joint work with A. Walker, asked whether there is a nontrivial word w in the free group $F = \langle a, b \rangle$ such that the lifted rotation number $\widetilde{\mathrm{rot}}(\rho(w))$ vanishes for every homomorphism ρ from F to the universal central extension $\mathrm{Homeo}^+(S^1)^\sim$ of the group of orientation-preserving homeomorphisms of the circle. We show that there is no such word. The main step is elementary: for every nontrivial w in the commutator subgroup of F there are $A, B \in \mathrm{SL}(2, \mathbb{R})$ with $|\mathrm{tr} w(A, B)| < 2$. The proof combines the classical trace polynomials of Fricke and Vogt with a comparison of signs on the two sides of the surface of reducible characters. Lifting such a pair to the universal cover of $\mathrm{PSL}(2, \mathbb{R})$ gives w a rotation number that is not an integer; words outside the commutator subgroup are treated with rigid rotations. All these representations are real-analytic. For $w = [a, b]$ the negative answer is already contained in an example of Calegari and Walker (2011). For the word w_0 of length 36 printed in the report we give rational matrices A and B , both hyperbolic, with $\mathrm{tr} w_0(A, B) = 0$; with the lifts that have fixed points, $\widetilde{\mathrm{rot}}(\rho(w_0)) = -\frac{1}{2}$, and the extremal value $R(w_0, 0, 0)$ equals 1. We also recall, after Calegari and Walker, why $\mathrm{rot}(\rho(w_0)) = 0$ whenever $\rho(a)$ or $\rho(b)$ is elliptic. The second question of the report, on real-analytic representations that realise the extremal rotation number $R(w, r, s)$, is not treated. No printed answer to the first question was found in the sources we searched; this is not a claim of priority. This is an unrefereed note.

1 Introduction

1.1 The question

Let F be the free group with basis a, b . Let $\mathrm{Homeo}^+(S^1)^\sim$ be the group of homeomorphisms f of $\mathbb{R}$ that commute with the translation $T_1(x) = x + 1$. These are the lifts of the orientation-preserving homeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$; the group $\mathrm{Homeo}^+(S^1)^\sim$ is the universal central extension of $\mathrm{Homeo}^+(S^1)$ [CW11, §1.4], and the kernel of the projection $\mathrm{Homeo}^+(S^1)^\sim \rightarrow \mathrm{Homeo}^+(S^1)$ is the central subgroup $\{T_k : k \in \mathbb{Z}\}$ of integer translations [CW11, §2]. The rotation number of $f \in \mathrm{Homeo}^+(S^1)^\sim$ is

$$\widetilde{\mathrm{rot}}(f) = \lim_{n \rightarrow \infty} \frac{f^n(x) - x}{n};$$

the limit exists and does not depend on x (Section 2). For $w \in F$ and $r, s \in \mathbb{R}$ let $X(w, r, s)$ be the set of the values $\widetilde{\mathrm{rot}}(\rho(w))$ over all homomorphisms $\rho : F \rightarrow \mathrm{Homeo}^+(S^1)^\sim$ with $\widetilde{\mathrm{rot}}(\rho(a)) = r$ and $\widetilde{\mathrm{rot}}(\rho(b)) = s$, and let $R(w, r, s) = \sup X(w, r, s)$ [Cal12, p. 2173], [CW11, §2.2]. The function R is the subject of Calegari and Walker's paper [CW11].

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At the end of his report [Cal12] on this joint work with A. Walker, given at the Oberwolfach workshop *Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry* in July 2012, Calegari states two open questions [Cal12, p. 2174], which we paraphrase:

- (Q1) Does some nontrivial word $w \in F$ satisfy $\widetilde{\text{rot}}(\rho(w)) = 0$ for every homomorphism $\rho : F \rightarrow \text{Homeo}^+(S^1)^\sim$?
- (Q2) For an arbitrary $w \in F$, does some real-analytic representation ρ attain the extremal value, $\widetilde{\text{rot}}(\rho(w)) = R(w, r, s)$?

In (Q2) the representations have $\widetilde{\text{rot}}(\rho(a)) = r$ and $\widetilde{\text{rot}}(\rho(b)) = s$; the report notes that every interior point of $X(w, r, s)$ is realised by a real-analytic representation, and so is $R(w, r, s)$ when w is positive. We do not treat (Q2). The questions form record OWR-12007-010 of the UnsolvedMath collection [ula26], where both are listed as open.

Question (Q1) is motivated in [Cal12] by representations of F into $\text{PSL}(2, \mathbb{R})$, which acts on the circle $\mathbb{RP}^1$ by projective transformations, and by the word

$$w_0 = \text{ababaabbaBBAABABabababaabbABBAABABAB} \quad (1)$$

of length 36, in which capital letters denote inverses: $A = a^{-1}$ and $B = b^{-1}$. The report states that $\text{rot}(\rho(w_0)) = 0$ for all representations ρ of F into $\text{PSL}(2, \mathbb{R})$, and that this can be proved with trace identities and numerical methods. The report does not restate the convention on capital letters; it is the convention of Calegari's work on stable commutator length [Cal09, §4], whereas [CW11] writes a^{-1} and b^{-1} . In this reading w_0 is the commutator $[uau^{-1}, b]$ with $u = \text{ababaabb}$, where $[x, y] = xyx^{-1}y^{-1}$; the other natural readings of the string give words with the same trace polynomial (Remark 5.1).

1.2 Results

For $w \in F$ and $A, B \in \text{SL}(2, \mathbb{R})$ let $w(A, B)$ denote the image of w under the homomorphism $F \rightarrow \text{SL}(2, \mathbb{R})$ with $a \mapsto A$ and $b \mapsto B$. An element of $\text{SL}(2, \mathbb{R})$ is *elliptic* if its trace lies in $(-2, 2)$, *hyperbolic* if $|\text{tr}| > 2$; the same words are used for its image in $\text{PSL}(2, \mathbb{R})$.

Theorem A. *For every $w \neq 1$ in the commutator subgroup $[F, F]$ there are $A, B \in \text{SL}(2, \mathbb{R})$ with $|\text{tr } w(A, B)| < 2$.*

The proof (Section 4) uses the classical trace polynomials of Fricke and Vogt, $\text{tr } w(A, B) = t_w(\text{tr } A, \text{tr } B, \text{tr } AB)$. For $w \in [F, F]$, $t_w - 2$ vanishes on the surface $\kappa = 2$ of the characters of reducible representations, where $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$ is the trace polynomial of the commutator, and we compare the signs of $t_w - 2$ on the two sides of this surface near its compact part and near an unbounded sheet. The lifts of elliptic elements to the universal cover $\widetilde{\text{PSL}}(2, \mathbb{R}) \subset \text{Homeo}^+(S^1)^\sim$ of $\text{PSL}(2, \mathbb{R})$ have rotation numbers outside $\mathbb{Z}$ (Lemma 2.3). This gives the answer to (Q1).

Corollary B. *For every $w \in F$ with $w \neq 1$ there is a homomorphism $\rho : F \rightarrow \widetilde{\text{PSL}}(2, \mathbb{R}) \subset \text{Homeo}^+(S^1)^\sim$ with $\widetilde{\text{rot}}(\rho(w)) \notin \mathbb{Z}$. In particular, Question (Q1) has a negative answer.*

As $\widetilde{\text{PSL}}(2, \mathbb{R})$ acts by real-analytic diffeomorphisms, the answer stays negative for lifts of real-analytic or smooth diffeomorphisms, and, as $\widetilde{\text{rot}}(\rho(w)) \notin \mathbb{Z}$, also for rotation numbers in $\mathbb{R}/\mathbb{Z}$. For $w = [a, b]$ the negative answer is contained in [CW11, Example 1.1]: deforming the holonomy of a complete hyperbolic once-punctured torus through cone structures makes the image of $[a, b]$ elliptic, with nonzero rotation number. Kim, Koberda and Mj [KKM19, Theorem 4.21] map a multicurve of peripheral or splitting curves on a hyperbolic orbifold, by faithful representations into $\text{PSL}(2, \mathbb{R})$, to

elliptic elements with one and the same nonzero rotation number; for F , as the fundamental group of a once-punctured torus or of a pair of pants, this covers the words that represent peripheral curves. We found no such statement for arbitrary words; Remark 4.2 compares Theorem A with results on word maps on $\mathrm{PSL}(2, \mathbb{R})$ [GKP18, HLS15].

For the word of the report we obtain the following.

Proposition C. *Let*

$$A = \begin{pmatrix} 3/2 & 1 \\ 1/2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -3/2 & 5/2 \end{pmatrix}. \quad \text{Then} \quad w_0(A, B) = \begin{pmatrix} -8 & 10 \\ -13/2 & 8 \end{pmatrix},$$

so that $\mathrm{tr} w_0(A, B) = 0$; here $A, B \in \mathrm{SL}(2, \mathbb{Q})$ are hyperbolic, with $\mathrm{tr} A = \frac{5}{2}$ and $\mathrm{tr} B = \frac{7}{2}$. For all lifts $\rho(a), \rho(b) \in \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of the images of A and B in $\mathrm{PSL}(2, \mathbb{R})$, the number $\widetilde{\mathrm{rot}}(\rho(w_0))$ lies in $\frac{1}{2} + \mathbb{Z}$. For the lifts that have fixed points, $\widetilde{\mathrm{rot}}(\rho(a)) = \widetilde{\mathrm{rot}}(\rho(b)) = 0$ and $\widetilde{\mathrm{rot}}(\rho(w_0)) = -\frac{1}{2}$.

Thus the word of the report admits a representation into $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, with both generators hyperbolic and with rotation numbers $(r, s) = (0, 0)$, for which $\widetilde{\mathrm{rot}}(\rho(w_0)) = -\frac{1}{2}$. In Section 5 we also show that $X(w_0, 0, 0) = [-1, 1]$, so that $R(w_0, 0, 0) = 1$ (Corollary 5.2), and we recall the following statement, which is essentially Example 4.8 of [CW11]; see also the proof of [Man18, Lemma 6.18(i)].

Proposition D. *Let $\rho : F \rightarrow \mathrm{Homeo}^+(S^1)^\sim$ be a homomorphism such that $\rho(a)$ or $\rho(b)$ is conjugate in $\mathrm{Homeo}^+(S^1)^\sim$ to a translation $x \mapsto x + t$. Then $\widetilde{\mathrm{rot}}(\rho(w_0)) = 0$. In particular, $\widetilde{\mathrm{rot}}(\rho(w_0)) = 0$ for every homomorphism $\rho : F \rightarrow \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ for which $\rho(a)$ or $\rho(b)$ is elliptic.*

Remark 5.3 relates Propositions C and D to the statement in the report. Finally, the cyclic covering construction of [CW11, Example 4.9] transports these examples to other rotation numbers (Proposition 5.4); it shows that the answer to (Q1) remains negative if ρ is restricted to representations with $(\widetilde{\mathrm{rot}}\rho(a), \widetilde{\mathrm{rot}}\rho(b)) \notin \mathbb{Z}^2$ (Corollary 5.5).

Readings of the question. We read (Q1) literally: all homomorphisms $F \rightarrow \mathrm{Homeo}^+(S^1)^\sim$. Restricted versions can have the opposite answer for trivial reasons. If $\rho(a)$ and $\rho(b)$ must be conjugate to translations (the setting of $R(w, r-, s-)$ in [Cal12, §§2–3] and [CW11, Definition 4.3]), then $w = [a, b]$ has $\widetilde{\mathrm{rot}}(\rho(w)) = 0$ for all such ρ , by the argument of Proposition D; and $X([a, b], r, s) = \{0\}$ if r or s is irrational [CW11, Example 4.9]. Theorem A says nothing about such versions; but the answer is negative for the family of all ρ with $(\widetilde{\mathrm{rot}}\rho(a), \widetilde{\mathrm{rot}}\rho(b)) \notin \mathbb{Z}^2$ (Corollary 5.5).

What is new. The contribution of this note is Theorem A for every word of the commutator subgroup, hence the complete negative answer to (Q1), and the computations for the word of the report (Proposition C and Corollary 5.2). The trace polynomials are classical [Gol09, Hor72]; Lemma 3.4 below reproves the part of Goldman’s description of real characters [Gol03, §2.6] that we need; Proposition D is due to Calegari and Walker, and so is the construction behind Proposition 5.4. The literature search is described in the paragraph “Scope and priority” at the end of the note.

2 Rotation numbers and the universal cover of $\mathrm{PSL}(2, \mathbb{R})$

2.1 Rotation numbers

For $f \in \mathrm{Homeo}^+(S^1)^\sim$ the function $\varphi_f(x) = f(x) - x$ is continuous and has period 1.

Lemma 2.1. *Let $f, g \in \text{Homeo}^+(S^1)^\sim$. (a) The limit $\widetilde{\text{rot}}(f)$ exists, $\min \varphi_f \leq \widetilde{\text{rot}}(f) \leq \max \varphi_f$ and $\max \varphi_f - \min \varphi_f < 1$; so $|\widetilde{\text{rot}}(f) - (f(x) - x)| < 1$ for all x , and $f(p) = \underline{p} + \widetilde{\text{rot}}(f)$ for some p . (b) $\widetilde{\text{rot}}(f) = k \in \mathbb{Z}$ if and only if $f(p) = \underline{p} + k$ for some $p \in \mathbb{R}$. (c) $\widetilde{\text{rot}}(gfg^{-1}) = \widetilde{\text{rot}}(f)$, $\widetilde{\text{rot}}(f^n) = n\widetilde{\text{rot}}(f)$ for $n \in \mathbb{Z}$, and $\widetilde{\text{rot}}(fg) = \widetilde{\text{rot}}(f) + \widetilde{\text{rot}}(g)$ if $fg = gf$. With $\iota(x) = -x$ one has $\iota f \iota \in \text{Homeo}^+(S^1)^\sim$ and $\widetilde{\text{rot}}(\iota f \iota) = -\widetilde{\text{rot}}(f)$. (d) $|\widetilde{\text{rot}}(fg) - \widetilde{\text{rot}}(f) - \widetilde{\text{rot}}(g)| \leq 1$; consequently $|\widetilde{\text{rot}}(fgf^{-1}g^{-1})| \leq 1$.*

Proof. (a) If $x < y < x + 1$, then $0 \leq f(y) - f(x) \leq 1$ and $0 < y - x < 1$, so $|\varphi_f(y) - \varphi_f(x)| < 1$; by periodicity the oscillation of φ_f , and likewise of φ_{f^n} , is less than 1. So $M_n = \max \varphi_{f^n}$ and $m_n = \min \varphi_{f^n}$ satisfy $M_{n+k} \leq M_n + M_k$, $m_{n+k} \geq m_n + m_k$ and $0 \leq M_n - m_n < 1$; hence M_n/n and m_n/n converge to a common limit, which is $\widetilde{\text{rot}}(f)$ and lies in $[m_n/n, M_n/n]$ for every n . For $n = 1$ this is the inequality, and the intermediate value theorem gives p . (b) If $f(p) = \underline{p} + k$, then $f^n(p) = \underline{p} + nk$; the converse is part of (a). (c) The first three statements are in [CW11, Proposition 2.1]; the last follows from $(\iota f \iota)^n(x) - x = -(f^n(-x) + x)$. (d) This is the statement that $\widetilde{\text{rot}}$ is a homogeneous quasimorphism of defect at most 1, a form of the Milnor–Wood inequality [CW11, proof of Lemma 2.11 and Definition 3.23]. Applied to fgf^{-1} and g^{-1} , whose rotation numbers are $\widetilde{\text{rot}}(g)$ and $-\widetilde{\text{rot}}(g)$ by (c), it gives the bound for the commutator. $\square$

2.2 The universal cover of $\text{PSL}(2, \mathbb{R})$

Let $e(\theta) = (\cos \theta, \sin \theta) \in \mathbb{R}^2$. For $M \in \text{SL}(2, \mathbb{R})$, a *lift* of M is an increasing homeomorphism $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $Me(\theta)$ is a positive multiple of $e(\Phi(\theta))$ for every $\theta \in \mathbb{R}$.

Lemma 2.2. *(a) Every $M \in \text{SL}(2, \mathbb{R})$ has lifts, two lifts of M differ by a translation $T_{2\pi k}$ with $k \in \mathbb{Z}$, and every lift satisfies $\Phi(\theta + \pi) = \Phi(\theta) + \pi$. (b) If Φ and Ψ are lifts of M and N , then $\Phi \circ \Psi$ is a lift of MN and Φ^{-1} is a lift of M^{-1} . The lifts of $\pm I$ are the translations $T_{k\pi}$ (k even for I , odd for $-I$), and the lifts of $-M$ are the maps $T_\pi \circ \Phi$. (c) If M has a positive eigenvalue, then M has a lift with a fixed point.*

Proof. (a) The map $\theta \mapsto Me(\theta)/|Me(\theta)|$ induces a homeomorphism of the circle $\mathbb{R}/2\pi\mathbb{Z}$ of directions, of degree 1 because M lies in the connected group $\text{GL}^+(2, \mathbb{R})$. Its lifts to $\mathbb{R}$ are increasing homeomorphisms, and two of them differ by some $T_{2\pi k}$. Since $Me(\theta + \pi) = -Me(\theta)$, the continuous function $\Phi(\theta + \pi) - \Phi(\theta) - \pi$ takes values in $2\pi\mathbb{Z}$, so it is a constant $2\pi m$; applying this twice gives $\Phi(\theta + 2\pi) = \Phi(\theta) + 2\pi + 4\pi m$, so $m = 0$. (b) follows from the definition and (a). (c) If $Mv = \lambda v$ with $\lambda > 0$ and $v = e(\theta_0)$, then every lift maps θ_0 into $\theta_0 + 2\pi\mathbb{Z}$, and a suitable $T_{2\pi k} \circ \Phi$ fixes θ_0 . $\square$

Let $\widetilde{\text{PSL}}(2, \mathbb{R})$ be the set of all lifts of all elements of $\text{SL}(2, \mathbb{R})$. By Lemma 2.2 it is a group of real-analytic diffeomorphisms of $\mathbb{R}$ that commute with T_π ; each of its elements is a lift of exactly one matrix M (if Φ is a lift of M and of N , then every $e(\theta)$ is an eigenvector of $M^{-1}N$ with a positive eigenvalue, so $N = M$), and $\Phi \mapsto \pm M$ is a surjective homomorphism onto $\text{PSL}(2, \mathbb{R})$, whose kernel is the central subgroup $\{T_{k\pi} : k \in \mathbb{Z}\}$. An element Φ of $\widetilde{\text{PSL}}(2, \mathbb{R})$ covers the action of $\pm M$ on the circle $\mathbb{RP}^1 = \mathbb{R}/\pi\mathbb{Z}$ of lines through the origin. Conjugation by $x \mapsto \pi x$ turns Φ into the element $x \mapsto \Phi(\pi x)/\pi$ of $\text{Homeo}^+(S^1)^\sim$ and T_π into T_1 ; we identify $\widetilde{\text{PSL}}(2, \mathbb{R})$ with its image in $\text{Homeo}^+(S^1)^\sim$, so that

$$\widetilde{\text{rot}}(\Phi) = \lim_{n \rightarrow \infty} \frac{\Phi^n(\theta) - \theta}{n\pi}.$$

(The group $\widetilde{\text{PSL}}(2, \mathbb{R})$ is the universal covering group of $\text{PSL}(2, \mathbb{R})$; we shall not need this.) Every homomorphism $F \rightarrow \text{PSL}(2, \mathbb{R})$ is covered by homomorphisms $\rho : F \rightarrow \widetilde{\text{PSL}}(2, \mathbb{R})$, one for each choice of lifts of the images of a and b .

Lemma 2.3. *Let $\Phi \in \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ be a lift of $M \in \mathrm{SL}(2, \mathbb{R})$. (a) If $|\mathrm{tr} M| \geq 2$, then $\widetilde{\mathrm{rot}}(\Phi) \in \mathbb{Z}$. (b) If $|\mathrm{tr} M| < 2$, then $\widetilde{\mathrm{rot}}(\Phi) \notin \mathbb{Z}$. (c) If $\mathrm{tr} M = 0$, then $\widetilde{\mathrm{rot}}(\Phi) \in \frac{1}{2} + \mathbb{Z}$.*

Proof. (a) M has a real eigenvector $e(\theta_0)$. Then $\Phi(\theta_0) = \theta_0 + k\pi$ for some $k \in \mathbb{Z}$, hence $\Phi^n(\theta_0) = \theta_0 + nk\pi$, because Φ commutes with T_π ; so $\widetilde{\mathrm{rot}}(\Phi) = k$. (b) If $\widetilde{\mathrm{rot}}(\Phi) = k \in \mathbb{Z}$, then by Lemma 2.1(b) there is θ_0 with $\Phi(\theta_0) = \theta_0 + k\pi$. Then $Me(\theta_0)$ is a multiple of $e(\theta_0)$, so M has a real eigenvalue λ , the other eigenvalue is $1/\lambda$, and $|\mathrm{tr} M| = |\lambda + 1/\lambda| \geq 2$. (c) By the Cayley–Hamilton theorem $M^2 = (\mathrm{tr} M)M - I = -I$. So Φ^2 is a lift of $-I$, that is $\Phi^2 = T_{(2m+1)\pi}$ for some $m \in \mathbb{Z}$, and $\widetilde{\mathrm{rot}}(\Phi) = m + \frac{1}{2}$. $\square$

For $w \in F$ let $e_a(w)$ and $e_b(w)$ be its exponent sums in a and b ; $w \in [F, F]$ if and only if both vanish.

Lemma 2.4. *Let $w \in [F, F]$ and $A, B \in \mathrm{SL}(2, \mathbb{R})$, and put $W = w(A, B)$. Let $\rho : F \rightarrow \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ be a homomorphism such that $\rho(a)$ is a lift of A or of $-A$, and $\rho(b)$ is a lift of B or of $-B$. Then $\rho(w)$ is the same for all such ρ , and it is a lift of W . If $\mathrm{tr} W \geq 2$, then $\widetilde{\mathrm{rot}}(\rho(w))$ is an even integer; if $\mathrm{tr} W \leq -2$, it is an odd integer.*

Proof. By Lemma 2.2, the possible values of $\rho(a)$ are the maps $T_{k\pi}\Phi_A$ with $k \in \mathbb{Z}$, where Φ_A is one lift of A , and similarly for $\rho(b)$. Since T_π is central in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, replacing $\rho(a)$ by $T_{k\pi}\rho(a)$ and $\rho(b)$ by $T_{l\pi}\rho(b)$ replaces $\rho(w)$ by $T_{(ke_a(w)+le_b(w))\pi}\rho(w) = \rho(w)$. If $\rho(a)$ and $\rho(b)$ are lifts of A and B , then $\rho(w)$ is a lift of W by Lemma 2.2(b). If $\mathrm{tr} W \geq 2$, the eigenvalues of W are real (as $(\mathrm{tr} W)^2 \geq 4$), have product 1 and positive sum, so they are positive: $We(\theta_0) = \lambda e(\theta_0)$ with $\lambda > 0$ for some θ_0 . Then $\rho(w)(\theta_0) = \theta_0 + 2\pi k$ for some $k \in \mathbb{Z}$, so $\rho(w)^n(\theta_0) = \theta_0 + 2\pi kn$ and $\widetilde{\mathrm{rot}}(\rho(w)) = 2k$. If $\mathrm{tr} W \leq -2$, the eigenvalues are negative, $We(\theta_0)$ is a positive multiple of $e(\theta_0 + \pi)$, $\rho(w)(\theta_0) = \theta_0 + \pi + 2\pi k$, and $\widetilde{\mathrm{rot}}(\rho(w)) = 2k + 1$. $\square$

In particular, for $w \in [F, F]$ and a homomorphism $\bar{\rho} : F \rightarrow \mathrm{PSL}(2, \mathbb{R})$ the number $\widetilde{\mathrm{rot}}(\rho(w))$ does not depend on the choice of the lift ρ of $\bar{\rho}$; this is how the statement of the report about w_0 is to be read.

3 Trace polynomials and reducible characters

For $A, B \in \mathrm{SL}(2, \mathbb{C})$ the triple $(\mathrm{tr} A, \mathrm{tr} B, \mathrm{tr} AB)$ is the *character* of the pair (A, B) . Let

$$\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2.$$

By a classical theorem of Vogt and Fricke [Gol09, Theorem A], for every $w \in F$ there is a polynomial $t_w \in \mathbb{C}[x, y, z]$ with

$$\mathrm{tr} w(A, B) = t_w(\mathrm{tr} A, \mathrm{tr} B, \mathrm{tr} AB) \quad \text{for all } A, B \in \mathrm{SL}(2, \mathbb{C}),$$

and $t_{[a,b]} = \kappa$ [Gol09, (2.2.8)]. Since every point of $\mathbb{C}^3$ is a character [Gol09, Theorem A], t_w is unique. The algorithm of [Gol09, §2.2], based on the identity $\mathrm{tr}(XY) + \mathrm{tr}(XY^{-1}) = \mathrm{tr} X \mathrm{tr} Y$, shows that t_w has integer coefficients (see also [Hor72]); we only use that its coefficients are real, which also follows from Lemma 3.4(a) below, since t_w is real on a nonempty open subset of $\mathbb{R}^3$.

Let $V = \{(x, y, z) \in \mathbb{C}^3 : \kappa(x, y, z) = 2\}$ and $V(\mathbb{R}) = V \cap \mathbb{R}^3$. For $\lambda, \mu \in \mathbb{C}^*$ put

$$P(\lambda, \mu) = (\lambda + \lambda^{-1}, \mu + \mu^{-1}, \lambda\mu + (\lambda\mu)^{-1}).$$

Lemma 3.1. (i) $\kappa(P(\lambda, \mu)) = 2$ for all $\lambda, \mu \in \mathbb{C}^*$. (ii) $V = P((\mathbb{C}^*)^2)$. (iii) $P(\lambda, \mu)$ is the character of the pair $A = \mathrm{diag}(\lambda, \lambda^{-1})$, $B = \begin{pmatrix} \mu & 1 \\ 0 & \mu^{-1} \end{pmatrix}$. (iv) If $w \in [F, F]$, then $t_w = 2$ on V .

Proof. Let $x = \lambda + \lambda^{-1}$, $y = \mu + \mu^{-1}$, $z_1 = \lambda\mu + (\lambda\mu)^{-1}$ and $z_2 = \lambda\mu^{-1} + \lambda^{-1}\mu$. Then $z_1 + z_2 = xy$ and $z_1 z_2 = \lambda^2 + \lambda^{-2} + \mu^2 + \mu^{-2} = x^2 + y^2 - 4$, so z_1 and z_2 are the roots of $Z^2 - xyZ + x^2 + y^2 - 4 = \kappa(x, y, Z) - 2$. This gives (i). (ii) For $(x, y, z) \in V$ choose $\lambda, \mu \in \mathbb{C}^*$ with $\lambda + \lambda^{-1} = x$ and $\mu + \mu^{-1} = y$. Then z is a root of $\kappa(x, y, Z) - 2$, so either $z = z_1$ and $(x, y, z) = P(\lambda, \mu)$, or $z = z_2$ and $(x, y, z) = P(\lambda, \mu^{-1})$. (iii) Here $AB = \begin{pmatrix} \lambda\mu & \lambda \\ 0 & (\lambda\mu)^{-1} \end{pmatrix}$. (iv) The upper triangular matrices form a group, and taking the diagonal is a homomorphism from it to an abelian group. So for the pair of (iii), $w(A, B)$ is upper triangular with diagonal entries $\lambda^{\pm e_a(w)} \mu^{\pm e_b(w)}$, which equal 1 if $w \in [F, F]$. Thus $t_w(P(\lambda, \mu)) = 2$, and (ii) gives the claim. $\square$

Lemma 3.2. *If $g \in \mathbb{C}[x, y, z]$ vanishes on V , then $\kappa - 2$ divides g ; if g has real coefficients, so has the quotient.*

Proof. As a polynomial in z , $\kappa - 2 = z^2 - xyz + x^2 + y^2 - 4$ is monic. Division with remainder gives $g = (\kappa - 2)Q + R_1(x, y)z + R_0(x, y)$, where Q, R_1, R_0 have real coefficients if g has. If $(x^2 - 4)(y^2 - 4) \neq 0$, the discriminant $(xy)^2 - 4(x^2 + y^2 - 4) = (x^2 - 4)(y^2 - 4)$ of $\kappa(x, y, Z) - 2$ is nonzero, so this polynomial has two distinct roots $z \neq z'$, and $R_1 z + R_0 = R_1 z' + R_0 = 0$ gives $R_1(x, y) = R_0(x, y) = 0$. Such (x, y) form a dense subset of $\mathbb{C}^2$, so $R_1 = R_0 = 0$. $\square$

Lemma 3.3. *Let $g \in \mathbb{C}[x, y, z]$ and $G = g \circ P$, a Laurent polynomial in λ, μ . If G vanishes on a nonempty open subset of the torus $\{|\lambda| = |\mu| = 1\}$ or of $(\mathbb{R} \setminus \{0\})^2$, then $\kappa - 2$ divides g .*

Proof. The function $(\vartheta, \phi) \mapsto G(e^{i\vartheta}, e^{i\phi})$ is a trigonometric polynomial, hence real-analytic on $\mathbb{R}^2$; if it vanishes on a nonempty open set, it vanishes identically, and so do its Fourier coefficients, which are the coefficients of G . In the second case $(\lambda\mu)^N G$ is, for large N , a polynomial that vanishes on a nonempty open subset of $\mathbb{R}^2$, hence zero. In both cases $G = 0$, so g vanishes on $P((\mathbb{C}^*)^2) = V$ (Lemma 3.1(ii)), and Lemma 3.2 applies. $\square$

The next lemma reproves the part of Goldman's description of real characters [Gol03, §2.6] that we need.

Lemma 3.4. *Let $(x, y, z) \in \mathbb{R}^3$. (a) If $|x| > 2$, then (x, y, z) is the character of a pair in $\mathrm{SL}(2, \mathbb{R})$. (b) If $|x| < 2$ and $\kappa(x, y, z) \geq 2$, then (x, y, z) is the character of a pair in $\mathrm{SL}(2, \mathbb{R})$. (c) If $|x| < 2$ and $\kappa(x, y, z) \leq 2$, then (x, y, z) is the character of a pair in $\mathrm{SU}(2)$; consequently $t_w(x, y, z) \in [-2, 2]$ for every $w \in F$.*

Proof. (a) Let $\lambda \in \mathbb{R}$ with $|\lambda| > 1$ and $\lambda + \lambda^{-1} = x$, and put $A = \mathrm{diag}(\lambda, \lambda^{-1})$, $p = (\lambda z - y)/(\lambda^2 - 1)$, $q = y - p$ and $B = \begin{pmatrix} p & 1 \\ pq - 1 & q \end{pmatrix}$. Then $\det B = 1$, $\mathrm{tr} B = y$, and $\mathrm{tr} AB = \lambda p + \lambda^{-1} q = z$, because $\lambda^2(\lambda z - y) + (\lambda^2 - 1)y - (\lambda z - y) = \lambda z(\lambda^2 - 1)$.

For (b) and (c) write $x = 2 \cos \vartheta$ with $\vartheta \in (0, \pi)$, and define $d \in \mathbb{R}$ by $z = y \cos \vartheta + d \sin \vartheta$. Expanding gives

$$\kappa(x, y, z) - 2 = \sin^2 \vartheta (d^2 + y^2 - 4). \quad (2)$$

(b) Let $A = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}$ and $B = \begin{pmatrix} y/2 & r + d \\ r & y/2 \end{pmatrix}$. Then $\mathrm{tr} B = y$ and $\mathrm{tr} AB = y \cos \vartheta + d \sin \vartheta = z$, and $\det B = 1$ means $(r + d/2)^2 = (d^2 + y^2 - 4)/4$, which has a real solution r because $d^2 + y^2 \geq 4$ by (2).

(c) Let $A = \mathrm{diag}(e^{i\vartheta}, e^{-i\vartheta})$ and $B = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $\alpha = (y - id)/2$ and $\beta = (1 - |\alpha|^2)^{1/2}$, which is real because $|\alpha|^2 = (y^2 + d^2)/4 \leq 1$ by (2). Then $A, B \in \mathrm{SU}(2)$, $\mathrm{tr} B = 2 \mathrm{Re} \alpha = y$ and $\mathrm{tr} AB = 2 \mathrm{Re}(e^{i\vartheta} \alpha) = y \cos \vartheta + d \sin \vartheta = z$. Since $w(A, B) \in \mathrm{SU}(2)$ has eigenvalues $e^{\pm i\psi}$ for some ψ , $t_w(x, y, z) = 2 \cos \psi \in [-2, 2]$. $\square$

Lemma 3.5. *Let $A_0 = \text{diag}(10, \frac{1}{10})$, $C = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$ and $B_0 = CA_0C^{-1}$. Then $|\text{tr } w(A_0, B_0)| > 2$ for every $w \in F \setminus \{1\}$. Consequently $t_w - 2$ is not the zero polynomial for $w \neq 1$.*

Proof. We use the action of $\text{SL}(2, \mathbb{R})$ on $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ by $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$. The matrix A_0 acts by $t \mapsto 100t$, and C by $t \mapsto (t - 2)/(t - 1)$; as $C^2 = -I$, C is an involution of $\mathbb{RP}^1$. Let $I_0 = [-\frac{1}{4}, \frac{1}{4}]$ and $I_\infty = \{|t| \geq 4\} \cup \{\infty\}$. Since $C(-\frac{1}{4}) = \frac{9}{5}$, $C(\frac{1}{4}) = \frac{7}{3}$, $C(0) = 2$, $C(4) = \frac{2}{3}$, $C(-4) = \frac{6}{5}$ and $C(\infty) = 1$, and the pole $t = 1$ of C lies in neither arc, we have $C(I_0) = [\frac{9}{5}, \frac{7}{3}]$ and $C(I_\infty) = [\frac{2}{3}, \frac{6}{5}]$. The four closed arcs $X(a) = I_\infty$, $X(a^{-1}) = I_0$, $X(b) = C(I_\infty)$ and $X(b^{-1}) = C(I_0)$ are pairwise disjoint, because $-\frac{1}{4} < \frac{1}{4} < \frac{2}{3} < \frac{6}{5} < \frac{9}{5} < \frac{7}{3} < 4$. For a letter $s \in \{a^{\pm 1}, b^{\pm 1}\}$ put $Y(s) = X(s^{-1})$. Each letter s maps $\mathbb{RP}^1 \setminus Y(s)$ into $X(s)$: for a this is the inclusion $100 \cdot \{|t| > \frac{1}{4}\} \subset \{|t| > 25\}$ (and $\infty \mapsto \infty$), for a^{-1} the inclusion $\frac{1}{100}\{|t| < 4\} \subset \{|t| < \frac{1}{25}\}$, and for $b^{\pm 1}$ it follows by conjugation with C .

Every $w \neq 1$ is conjugate to a cyclically reduced word $s_1 s_2 \cdots s_n$ with $n \geq 1$, and traces are invariant under conjugation. The element $w(A_0, B_0)$ acts as $s_1 \circ \cdots \circ s_n$. As $s_{i+1} \neq s_i^{-1}$, the arcs $X(s_{i+1})$ and $Y(s_i)$ are disjoint, so by induction w maps $\mathbb{RP}^1 \setminus Y(s_n)$ into $X(s_1)$. Since also $s_1 \neq s_n^{-1}$, the arc $X(s_1)$ lies in $\mathbb{RP}^1 \setminus Y(s_n)$; so w maps the closed arc $X(s_1)$ into itself and has a fixed point there. The same argument for the cyclically reduced word $w^{-1} = s_n^{-1} \cdots s_1^{-1}$ gives a fixed point of w in $X(s_n^{-1}) = Y(s_n)$, which is disjoint from $X(s_1)$. Moreover w is not the identity of $\mathbb{RP}^1$, since it maps $\mathbb{RP}^1 \setminus Y(s_n)$, which contains two of the four arcs other than $X(s_1)$, into $X(s_1)$. So $w(A_0, B_0) \neq \pm I$ has two independent real eigenvectors; its eigenvalues are real and distinct, $\lambda \neq \lambda^{-1}$, and $|\text{tr } w(A_0, B_0)| = |\lambda + \lambda^{-1}| > 2$. Finally $t_w(\text{tr } A_0, \text{tr } B_0, \text{tr } A_0 B_0) = \text{tr } w(A_0, B_0) \neq 2$. $\square$

Lemma 3.6. *The points of V at which the gradient of κ vanishes are $(2, 2, 2)$, $(2, -2, -2)$, $(-2, 2, -2)$ and $(-2, -2, 2)$. Consequently, if $p \in V(\mathbb{R})$ has first coordinate different from ± 2 , then every ball $U \subset \mathbb{R}^3$ centred at p meets both $\{\kappa > 2\}$ and $\{\kappa < 2\}$.*

Proof. The gradient is $(2x - yz, 2y - xz, 2z - xy)$. If it vanishes, then $2x^2 = 2y^2 = 2z^2 = xyz$. If this common value is 0, then $x = y = z = 0$ and $\kappa = -2 \neq 2$. Otherwise $x^2 = y^2 = z^2 = t \neq 0$ and $xyz = 2t$, so $t^3 = (xyz)^2 = 4t^2$ and $t = 4$: the coordinates are ± 2 with $xyz = 8$, which gives the four points, and they lie on V . At any other point p of $V(\mathbb{R})$ the gradient $\nabla \kappa(p)$ is a nonzero real vector, and $\kappa(p + \tau \nabla \kappa(p)) - 2 = \tau |\nabla \kappa(p)|^2 + O(\tau^2)$ takes both signs near $\tau = 0$. $\square$

4 Proof of Theorem A and Corollary B

Proof of Theorem A. Let $w \in [F, F]$, $w \neq 1$, and suppose that $|\text{tr } w(A, B)| \geq 2$ for all $A, B \in \text{SL}(2, \mathbb{R})$. The map $(A, B) \mapsto \text{tr } w(A, B)$ is continuous on the connected space $\text{SL}(2, \mathbb{R})^2$, takes the value 2 at (I, I) and, by assumption, avoids $(-2, 2)$. So its image lies in $[2, \infty)$:

$$t_w(c) \geq 2 \quad \text{for every character } c \in \mathbb{R}^3 \text{ of a pair in } \text{SL}(2, \mathbb{R})^2. \quad (3)$$

By Lemma 3.1(iv) the polynomial $t_w - 2$ vanishes on V , and by Lemma 3.5 it is not zero. Applying Lemma 3.2 repeatedly, we can write

$$t_w - 2 = (\kappa - 2)^m h, \quad m \geq 1, \quad h \in \mathbb{R}[x, y, z], \quad \kappa - 2 \nmid h.$$

By Lemma 3.3, $h \circ P$ vanishes neither on the nonempty open set $\{(e^{i\vartheta}, e^{i\phi}) : \vartheta, \phi, \vartheta + \phi \notin \pi\mathbb{Z}\}$ of the torus nor on $\{(\lambda, \mu) : \lambda, \mu > 1\}$. So there are points

$$p_1 = P(e^{i\vartheta_0}, e^{i\phi_0}) = (2 \cos \vartheta_0, 2 \cos \phi_0, 2 \cos(\vartheta_0 + \phi_0)) \in (-2, 2)^3 \quad \text{and} \quad p_2 = P(\lambda_0, \mu_0), \quad \lambda_0, \mu_0 > 1,$$

of $V(\mathbb{R})$ with $h(p_1) \neq 0$ and $h(p_2) \neq 0$; the first coordinate of p_2 is $\lambda_0 + \lambda_0^{-1} > 2$. Choose a ball U_1 centred at p_1 inside $\{|x| < 2\}$ and a ball U_2 centred at p_2 inside $\{x > 2\}$ such that h has no zero on

$U_1 \cup U_2$; then h has constant sign on each U_i . Put $U_i^\pm = U_i \cap \{\pm(\kappa - 2) > 0\}$; these sets are nonempty by Lemma 3.6, and $t_w - 2 = (\kappa - 2)^m h \neq 0$ on them.

On U_1^- we have $t_w \leq 2$ by Lemma 3.4(c), so $t_w - 2 < 0$. On U_1^+ we have $t_w \geq 2$ by Lemma 3.4(b) and (3), so $t_w - 2 > 0$. Comparing with the sign of $(\kappa - 2)^m h$, the second fact gives $h > 0$ on U_1 , and then the first gives $(-1)^m = -1$: m is odd.

Every point of U_2 is the character of a pair in $\mathrm{SL}(2, \mathbb{R})^2$ by Lemma 3.4(a), so $t_w - 2 > 0$ on $U_2^+ \cup U_2^-$ by (3). But m is odd and h has constant sign on U_2 , so $(\kappa - 2)^m h$ has opposite signs on U_2^+ and U_2^- . This contradiction proves Theorem A. $\square$

Proof of Corollary B. Let $w \neq 1$. Suppose first that $w \notin [F, F]$, say $e_a(w) \neq 0$ (otherwise exchange the roles of a and b). Choose $t \in \mathbb{R}$ with $e_a(w)t \notin \mathbb{Z}$ and let $\rho(a) = T_t$ and $\rho(b) = \mathrm{id}$. Then $\rho(w) = T_{e_a(w)t}$ and $\mathrm{rot}(\rho(w)) = e_a(w)t \notin \mathbb{Z}$. Here $T_t \in \widetilde{\mathrm{PSL}}(2, \mathbb{R})$: in the coordinate θ it is the translation by πt , a lift of the rotation matrix of angle πt . (The central translation T_1 would give $\mathrm{rot}(\rho(w)) = e_a(w)$, a nonzero integer.) Now let $w \in [F, F]$. Theorem A gives $A, B \in \mathrm{SL}(2, \mathbb{R})$ with $|\mathrm{tr} w(A, B)| < 2$. Let $\rho(a), \rho(b) \in \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ be lifts of A and B . By Lemma 2.4, $\rho(w)$ is a lift of $w(A, B)$, and by Lemma 2.3(b), $\mathrm{rot}(\rho(w)) \notin \mathbb{Z}$. $\square$

Theorem A can also be read as a statement about the image of word maps.

Corollary 4.1. *Let $w \in [F, F]$, $w \neq 1$. There is $\delta \in (0, 4)$ such that every element of $\mathrm{SL}(2, \mathbb{R})$ with trace in $(2 - \delta, 2)$ is of the form $w(A, B)$ with $A, B \in \mathrm{SL}(2, \mathbb{R})$. If $w \in F \setminus [F, F]$, every elliptic element of $\mathrm{SL}(2, \mathbb{R})$ is of this form.*

Proof. The set of values $\mathrm{tr} w(A, B)$ is connected, contains 2 and, by Theorem A, a number $t_0 \in (-2, 2)$; so it contains $[t_0, 2]$, and we put $\delta = 2 - t_0$. The set of the elements $w(A, B)$ is invariant under conjugation by $\mathrm{GL}(2, \mathbb{R})$, because $gw(A, B)g^{-1} = w(gAg^{-1}, gBg^{-1})$. Two elements of $\mathrm{SL}(2, \mathbb{R})$ other than $\pm I$ with the same trace t are conjugate in $\mathrm{GL}(2, \mathbb{R})$: in a basis v, Mv , where v is not an eigenvector, each of them has the matrix $\begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}$. This proves the first statement, since matrices with trace in $(-2, 2)$ are not $\pm I$. If $e_a(w) \neq 0$, say, then $w(R_\phi, I) = R_{e_a(w)\phi}$ for the rotation matrices R_ϕ , so every rotation matrix is of the form $w(A, B)$, and every elliptic element is conjugate in $\mathrm{GL}(2, \mathbb{R})$ to a rotation matrix. $\square$

Remark 4.2. An elliptic element of $\mathrm{PSL}(2, \mathbb{R})$ is conjugate to the image of a rotation matrix R_ϕ with $\phi \in (0, \pi)$; it acts on $\mathbb{RP}^1 = \mathbb{R}/\pi\mathbb{Z}$ with rotation number ϕ/π modulo 1, and its trace is $\pm 2 \cos \phi$. So, by Corollary 4.1, for every $w \in [F, F] \setminus \{1\}$ the image of the word map $w : \mathrm{PSL}(2, \mathbb{R})^2 \rightarrow \mathrm{PSL}(2, \mathbb{R})$ contains all elliptic elements whose rotation number modulo 1 lies in $(-\eta, \eta)$, for some $\eta > 0$, and for $w \in F \setminus [F, F]$ it contains all elliptic elements. Gordeev, Kunyavskii and Plotkin [GKP18, Proposition 2.6] prove that for every nontrivial word w in a free group of finite rank the image of the word map on $\mathrm{PSL}(2, \mathbb{R})$ contains all split semisimple, that is hyperbolic, elements, generalising a result of Hui, Larsen and Shalev [HLS15] (as stated in [GKP18]), and that it contains all semisimple elements if it contains an involution. For words in F , Corollary 4.1 adds the elliptic elements of small rotation number without any assumption. Theorem A does not answer Question 2.5 of [GKP18], not even for words in F : that question asks whether some word that is not a proper power induces a non-surjective word map on a non-compact simple group such as $\mathrm{PSL}(2, \mathbb{R})$, and neither [GKP18, Proposition 2.6] nor Theorem A decides whether the image contains the elliptic elements of every rotation number and the parabolic elements. For the word of the report, Proposition C exhibits an involution in the image of w_0 (the image of a matrix with trace 0), so by the second statement of [GKP18, Proposition 2.6] the image of w_0 on $\mathrm{PSL}(2, \mathbb{R})$ contains every semisimple element.

5 The word of the report

With capital letters read as inverses, the string (1) is the concatenation of **ababaabb**, a , **BBAABABA** = u^{-1} , b , u , a^{-1} , u^{-1} and b^{-1} , without cancellation; so

$$w_0 = [c, b], \quad c = u a u^{-1}, \quad u = \mathbf{ababaabb}.$$

The word is freely and cyclically reduced, and its exponent sums in a and in b are 0.

Remark 5.1. Reading the string backwards, inverting w_0 , or exchanging upper and lower case gives words w_0^{rev} , w_0^{-1} , w_0^{sw} with $w_0^{\text{rev}}(A, B) = w_0(A^\top, B^\top)^\top$, $\text{tr } w_0^{-1}(A, B) = \text{tr } w_0(A, B)$ and $w_0^{\text{sw}}(A, B) = w_0(A^{-1}, B^{-1})$. As $(A^\top, B^\top)$ and (A^{-1}, B^{-1}) have the same character as (A, B) , the four words have the same trace polynomial. Each is a commutator of a conjugate of $a^{\pm 1}$ with $b^{\pm 1}$, so Proposition D applies to all of them, and at the pair of Proposition C all four have trace 0 and rotation number $\pm \frac{1}{2}$.

Proof of Proposition C. We have $\det A = \det B = 1$ and $\text{tr } A, \text{tr } B > 2$, so $A^{\pm 1}$ and $B^{\pm 1}$ are hyperbolic with positive eigenvalues. The value of $w_0(A, B)$, a product of 36 rational matrices, was computed in exact arithmetic by the program of this note and, independently of it, by the programs of the first stage and of each verification run (Section 6); through the commutator structure, $C = u(A, B)Au(A, B)^{-1}$ has $\text{tr } C = \frac{5}{2}$ and $\text{tr } CB = \frac{11}{4}$, and indeed $\kappa(\frac{5}{2}, \frac{7}{2}, \frac{11}{4}) = 0$. By Lemma 2.4, $\rho(w_0)$ does not depend on the lifts and is a lift of $W = w_0(A, B)$; as $\text{tr } W = 0$, Lemma 2.3(c) and Lemma 2.1(d) give $\text{rot}(\rho(w_0)) = \pm \frac{1}{2}$.

By Lemma 2.2(c) and Lemma 2.1(b), A and B have lifts with fixed points, of rotation number 0. For a matrix S with positive eigenvalues, its lift Φ_S with a fixed point fixes the directions of the eigenvectors of S and of their negatives, and maps each of the four arcs between them, all shorter than π , onto itself; so $\Phi_S(\theta) - \theta$ is the signed angle in $(-\pi, \pi)$ from $e(\theta)$ to $Se(\theta)$. Follow a rational vector $v_0 \neq 0$ of direction θ_0 through the 72 letters of w_0^2 , the last letter first, obtaining $v_1, \dots, v_{72}$ with lifted angles θ_k , and let $h_k = \lfloor (\theta_k - \theta_0)/\pi \rfloor$. The parity of h_k is given by the sign of $\det(v_0, v_k)$ (or of $v_0 \cdot v_k$ if this determinant is 0), and $h_{k+1} - h_k$ lies in $\{0, 1\}$ or in $\{-1, 0\}$ according to the sign of $\det(v_k, v_{k+1})$; so the h_k are determined by signs of determinants of rational vectors. As $W^2 = -I$, we have $v_{72} = -v_0$ and $\theta_{72} - \theta_0 = h_{72}\pi$. The exact program of this note finds $h_{72} = -1$ for five choices of v_0 ; so $\rho(w_0)^2 = T_{-\pi}$ and $\widetilde{\text{rot}}(\rho(w_0)) = -\frac{1}{2}$. $\square$

Conjugating this representation by ι (Lemma 2.1(c)), or equivalently replacing A, B by their conjugates under $\text{diag}(1, -1)$, gives a representation with rotation numbers $(0, 0)$ and $\widetilde{\text{rot}}(\rho(w_0)) = +\frac{1}{2}$. So $\pm \frac{1}{2} \in X(w_0, 0, 0)$; the following corollary shows more. We use that $X(w, r, s)$ is a compact interval with maximum $R(w, r, s)$ and minimum $-R(w, -r, -s)$ [CW11, Proposition 2.16], [Cal12, p. 2173].

Corollary 5.2. $X(w_0, 0, 0) = [-1, 1]$; in particular $R(w_0, 0, 0) = 1$.

Proof. Since w_0 is a commutator, $X(w_0, 0, 0) \subseteq [-1, 1]$ by Lemma 2.1(d). Let

$$A_2 = \begin{pmatrix} 6 & -5 \\ 5/2 & -23/12 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 5/3 & 5/2 \\ 5/3 & 31/10 \end{pmatrix}.$$

Then $A_2, B_2 \in \text{SL}(2, \mathbb{Q})$, $\text{tr } A_2 = \frac{49}{12}$ and $\text{tr } B_2 = \frac{143}{30}$, so they have positive eigenvalues and lifts with fixed points, of rotation number 0. An exact computation gives

$$\text{tr } w_0(A_2, B_2) = -\frac{1172932637776500498744940813}{20615843020800000000000000} \approx -56.89 < -2.$$

By Lemma 2.4, $\widetilde{\text{rot}}(\rho(w_0))$ is odd for this representation, so it is ± 1 ; the exact program of this note finds $+1$. Conjugation by ι gives a representation with rotation numbers $(0, 0)$ and the opposite value. So $\pm 1 \in X(w_0, 0, 0)$, and $X(w_0, 0, 0) = [-1, 1]$ because it is an interval. $\square$

Proof of Proposition D. We recall the argument of [CW11, Example 4.8]. Suppose first that $\rho(a) = gT_tg^{-1}$ with $g \in \text{Homeo}^+(S^1)^\sim$. Then $f = \rho(c) = \rho(u)\rho(a)\rho(u)^{-1}$ equals kT_tk^{-1} with $k = \rho(u)g$, and with $\beta = k^{-1}\rho(b)k$ we get

$$\rho(w_0) = f\rho(b)f^{-1}\rho(b)^{-1} = k(T_t\beta T_{-t}\beta^{-1})k^{-1}.$$

The element $\beta T_{-t}\beta^{-1}$ has rotation number $-t$ (Lemma 2.1(c)), so by Lemma 2.1(a) there is a point p with $\beta T_{-t}\beta^{-1}(p) = p - t$. Then $T_t\beta T_{-t}\beta^{-1}$ fixes p , so its rotation number is 0 by Lemma 2.1(b), and so is that of its conjugate $\rho(w_0)$. If $\rho(b)$ is conjugate to a translation, the same argument applied to $\rho(w_0)^{-1} = \rho(b)f\rho(b)^{-1}f^{-1}$ gives $\widetilde{\text{rot}}(\rho(w_0)^{-1}) = 0$, hence $\widetilde{\text{rot}}(\rho(w_0)) = 0$. Finally, let $\rho : F \rightarrow \text{PSL}(2, \mathbb{R})$ and let $\rho(a)$ be a lift of an elliptic matrix M . Then $M = gR_\phi g^{-1}$ with $g \in \text{SL}(2, \mathbb{R})$ and a rotation matrix R_ϕ (the Möbius map of M fixes a point of the upper half-plane, and the stabiliser of i is $\text{SO}(2)$). By Lemma 2.2, $\rho(a) = G\Phi G^{-1}$ for a lift G of g and a lift Φ of R_ϕ , and Φ is a translation of $\mathbb{R}$. The case of $\rho(b)$ is the same. $\square$

Remark 5.3. The report states that $\widetilde{\text{rot}}(\rho(w_0)) = 0$ for all representations ρ of F into $\text{PSL}(2, \mathbb{R})$ [Cal12, p. 2174]; by Lemma 2.4 the choice of lifts plays no role. By Proposition D the statement holds whenever $\rho(a)$ or $\rho(b)$ is elliptic, that is, by Lemma 2.3, whenever $(\widetilde{\text{rot}}\rho(a), \widetilde{\text{rot}}\rho(b)) \notin \mathbb{Z}^2$; and by Proposition C it fails as literally printed when both generators are hyperbolic with integer rotation numbers, though not for every such representation: for $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $B_3 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, whose character is $(3, 3, 3)$ and whose commutator has trace -2 , one finds $\text{tr } w_0(A_3, B_3) = 247165558$, so that $\widetilde{\text{rot}}(\rho(w_0)) = 0$ by Lemma 2.4 and Lemma 2.1(d). The argument of Proposition D shows that $\widetilde{\text{rot}}(\rho([c, k])) = 0$ whenever $\rho(c)$ or $\rho(k)$ is conjugate to a translation. So Proposition D holds, with the same proof, for every commutator $[c, k]$ in which c is conjugate to $a^{\pm 1}$ and k to $b^{\pm 1}$, or vice versa, for instance for $[a, b]$; it does not explain the choice of the word w_0 . In the literal reading no nontrivial word of $[F, F]$ could take the place of w_0 , by Theorem A and Lemma 2.3. The representation of Proposition C is not faithful, since it maps w_0 to an involution of $\mathbb{RP}^1$. We do not address the reading in which ρ ranges over the discrete faithful representations. For these, $\rho(w_0)$ is neither elliptic (elliptic elements of discrete subgroups of $\text{PSL}(2, \mathbb{R})$ have finite order) nor trivial, so by Lemma 2.4 and Lemma 2.1(d), $\widetilde{\text{rot}}(\rho(w_0)) \in \{-1, 0, 1\}$, with $\widetilde{\text{rot}}(\rho(w_0)) = 0$ if and only if $\text{tr } w_0(A, B) \geq 2$. One of the verification runs (Section 6) found $\text{tr } w_0 > 2$ at each of the 10,630 marked Schottky representations that it tested in exact arithmetic; we do not know whether this holds for all discrete faithful representations.

The cyclic covering construction of [CW11, Example 4.9] moves these examples to other rotation numbers.

Proposition 5.4. *Let $w \in [F, F]$, $q \geq 1$ and $p, p' \in \mathbb{Z}$. Then $\frac{1}{q}X(w, 0, 0) \subseteq X(w, \frac{p}{q}, \frac{p'}{q})$.*

Proof. For $f \in \text{Homeo}^+(S^1)^\sim$ let $f_q(x) = f(qx)/q$. Then $f_q \in \text{Homeo}^+(S^1)^\sim$, the map $f \mapsto f_q$ is a homomorphism (conjugation by $x \mapsto qx$), f_q commutes with $T_{1/q}$ because $f_q(x + \frac{1}{q}) = f(qx + 1)/q = f_q(x) + \frac{1}{q}$, and $\widetilde{\text{rot}}(f_q) = \widetilde{\text{rot}}(f)/q$ because $f_q^n(x) - x = (f^n(qx) - qx)/q$. Let ρ_0 be a representation with rotation numbers $(0, 0)$, and define ρ by $\rho(a) = T_{p/q}\rho_0(a)_q$ and $\rho(b) = T_{p'/q}\rho_0(b)_q$. Since $T_{1/q}$ commutes with every $\rho_0(g)_q$, we get $\rho(g) = T_{(pe_a(g)+p'e_b(g))/q}\rho_0(g)_q$ for all $g \in F$; so $\rho(w) = \rho_0(w)_q$ and $\widetilde{\text{rot}}(\rho(w)) = \widetilde{\text{rot}}(\rho_0(w))/q$, while $\widetilde{\text{rot}}(\rho(a)) = \frac{p}{q}$ and $\widetilde{\text{rot}}(\rho(b)) = \frac{p'}{q}$ by Lemma 2.1(c). $\square$

For ρ_0 with values in $\widetilde{\text{PSL}}(2, \mathbb{R})$ and $q = 2$, the representation ρ is an action on the circle $(\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}$ of rays: $(\Phi)_2$ is the lift Φ read on this circle, and $T_{1/2}$ is the antipodal map. So if $\rho_0(a)$ and $\rho_0(b)$ are lifts of A and B and $p = p' = 1$, the generators of ρ act on rays as $-A$ and $-B$.

Corollary 5.5. (a) *For every $w \in F \setminus \{1\}$ there is $\rho : F \rightarrow \text{Homeo}^+(S^1)^\sim$ with $(\widetilde{\text{rot}}\rho(a), \widetilde{\text{rot}}\rho(b)) \notin \mathbb{Z}^2$ and $\widetilde{\text{rot}}(\rho(w)) \notin \mathbb{Z}$. (b) $[-\frac{1}{2}, \frac{1}{2}] \subseteq X(w_0, \frac{1}{2}, \frac{1}{2})$. In particular, with A, B as in Proposition C, the action of $-A$ and $-B$ on the circle of rays has rotation numbers $(\frac{1}{2}, \frac{1}{2})$ and $\widetilde{\text{rot}}(\rho(w_0)) = -\frac{1}{4}$.*

Proof. (a) If $w \notin [F, F]$, let $\rho(a) = T_t$ and $\rho(b) = T_{t'}$ with $t, t' \notin \mathbb{Z}$ and $e_a(w)t + e_b(w)t' \notin \mathbb{Z}$ (for instance $t' = \frac{1}{2}$ and t irrational if $e_a(w) \neq 0$). If $w \in [F, F]$, let ρ_0 be the representation constructed in the proof of Corollary B. If $\rho_0(a)$ or $\rho_0(b)$ is elliptic, $\rho = \rho_0$ will do, by Lemma 2.3(b). Otherwise $\widetilde{\text{rot}}\rho_0(a) = k$ and $\widetilde{\text{rot}}\rho_0(b) = l$ are integers, and replacing $\rho_0(a)$ and $\rho_0(b)$ by $T_{-k}\rho_0(a)$ and $T_{-l}\rho_0(b)$ leaves $\rho_0(w)$ unchanged and gives rotation numbers $(0, 0)$. So $\widetilde{\text{rot}}(\rho_0(w)) \in X(w, 0, 0) \setminus \mathbb{Z}$, and Proposition 5.4 with $q = 2$ and $p = p' = 1$ gives a representation with rotation numbers $(\frac{1}{2}, \frac{1}{2})$ and $\widetilde{\text{rot}}(\rho(w)) = \widetilde{\text{rot}}(\rho_0(w))/2 \notin \mathbb{Z}$. (b) follows from Corollary 5.2, Proposition C and Proposition 5.4 with $q = 2$. $\square$

In (b), $-A$ has rotation number $\frac{1}{2}$ on the circle of rays but is not conjugate to a rotation: its square acts as A^2 , which is not the identity of that circle, whereas the square of a rotation by a half turn is. So Proposition D really needs generators conjugate to rotations, not merely generators with rotation numbers outside $\mathbb{Z}$. The value $-\frac{1}{4}$ is consistent with the bound $\widetilde{\text{rot}}[f, g] \leq \frac{1}{q}$ when $\widetilde{\text{rot}}(f) = \frac{p}{q}$ [CW11, Example 4.9], [Man18, Lemma 6.18].

6 Computations and reproducibility

All statements are proved in the text; computation enters only as exact rational arithmetic (the matrices $w_0(A, B)$, $w_0(A_2, B_2)$, $w_0(A_3, B_3)$ and the integers h_k of the proof of Proposition C). The reproducibility package contains the program of this note, `check_note.py` (Python 3, standard library, under one second), with its recorded output. It checks the structure of w_0 ; the certificate of Proposition C, also through C , the readings of Remark 5.1 and $w_0(-A, -B) = w_0(A, B)$; the integers h_k for five start vectors, with the results $-\frac{1}{2}$ and, after conjugation by $\text{diag}(1, -1)$, $+\frac{1}{2}$; the pairs (A_2, B_2) and (A_3, B_3) ; the polynomial identities in the proofs of Lemmas 3.1, 3.2 and 3.4 and the singular points of Lemma 3.6; the data of Lemma 3.5 and $|\text{tr } w(A_0, B_0)| > 2$ for all 29,540 cyclically reduced words of length at most 9; and, as tests that are not part of any proof, it finds an exact pair with $|\text{tr } w(A, B)| < 2$ for each of 14 words of $[F, F]$, among them w_0 , and checks Proposition D on 180 random exact pairs with an elliptic generator. The package also contains the programs with which the results were first obtained (among them a symbolic computation of t_{w_0} : degree 24, 249 monomials, and $\kappa - 2$ divides $t_{w_0} - 2$ exactly once) and the programs of the three verification runs, with their recorded outputs (those of the run on the final text are in `independent_run_2/`); its `README.md` gives command lines, running times and what each program checks.

Verification. The results were first obtained with proofs and with the programs just mentioned. Two independent verification runs, both AI-assisted, followed; they examined the first written version of the results, and each wrote its own programs before looking at those of the first stage. The first re-derived every lemma and the proofs of Theorem A, Corollary B and Proposition D step by step and compared the statements used with the texts of [Cal12, CW11, Man18, Gol09, Gol03]; it recomputed the certificate under the four readings of Remark 5.1, computed t_{w_0} by another method (with the same result) and the rotation number of Proposition C to 50 digits, found the pair (A_2, B_2) and the argument of Corollary 5.2, tested Theorem A on 43 words, and tested 10,630 marked Schottky representations in exact arithmetic (Remark 5.3). The second re-derived the proofs, computed t_{w_0} with its own polynomial arithmetic, obtained $-\frac{1}{2}$ by an exact winding count, tested Theorem A on 68 words and Proposition D on 800 exact pairs, re-ran the programs of the first stage (identical outputs), searched the literature, where it found the relevant parts of [CW11, KKM19, GKP18], and proposed Proposition 5.4. Both runs confirmed Theorem A, Corollary B and Propositions C and D and found no gap. All corrections they required were made: the qualified description of the example of the report (Remark 5.3); the removal of an incorrect remark about the sign of the cofactor h on the

unbounded sheets of $V(\mathbb{R})$, which no proof used; rotations instead of central translations for words outside $[F, F]$; the credits to [CW11, Examples 1.1 and 4.8] and [Man18, Lemma 6.18]; the comparison with [GKP18, HLS15, KKM19]; an author list (arXiv:1106.1619 is by Bandman and Garion [BG12]); the convention on capital letters; and the record of the database queries. The program `check_note.py` was written together with this text.

When this note was written, the facts on lifts (Section 2) and the proof of Proposition D were written anew (the first version proved Proposition D with a trace identity and the lifting of paths), and Corollary 4.1, Remark 4.2, Proposition 5.4 and Corollary 5.5 were added. A third independent verification run, also AI-assisted, examined the final text, and these parts first. It read the sources again (the word of [Cal12] on the rendered page; [CW11] in the journal version; [Cal09, GKP18, KKM19, Man18, Gol09, Gol03]), re-derived every proof, and wrote its own programs before opening those of the package: the certificate of Proposition C and the matrices of Corollary 5.2 and Remark 5.3; the rotation numbers of Proposition C and Corollary 5.2 exactly, by counting the signed crossings of a fixed ray (a method different from the integers h_k), and these and that of Remark 5.3 again by iterating the lifts with 60 digits; t_{w_0} by the recursion $\text{tr}(XgYg) = \text{tr}(Xg)\text{tr}(Yg) - \text{tr}(XY^{-1})$, with the same degree, number of monomials and multiplicity of $\kappa - 2$; Lemma 3.5 for all 88,592 cyclically reduced words of length at most 10; Proposition D on 600 exact pairs; and exact elliptic values for 40 words of $[F, F]$, obtained by the mechanism of the proof of Theorem A (next to a sheet of $V(\mathbb{R})$ when m is odd, and next to the compact part when m is even, as for $[[a, b], [a, b^{-1}]]$, where $m = 2$). It ran all programs of the package again from the archive, with identical outputs, and repeated part of the literature search. It found no error in a theorem, proposition, corollary or lemma. It required that a sentence of Remark 5.3, which stated Proposition D for every commutator with one entry conjugate to a generator, be restricted as it now reads (for $[a, bab^{-1}]$ with $\rho(b)$ elliptic the conclusion fails), and corrections of two citations, of the description of the computations in the proof of Proposition C, of this paragraph and of the record of the searches; they were made.

Scope and priority. The note answers (Q1) of [Cal12] negatively, in its literal reading; (Q2) is not treated and remains open as far as we know, so the record OWR-12007-010 [ula26] is answered only in part. The questions are due to Calegari and Walker. The trace polynomials are classical (Vogt, Fricke; see [Gol09, Hor72]), and Lemma 3.4 is part of Goldman’s description of real characters [Gol03, §2.6]. For $w = [a, b]$ the answer to (Q1) is contained in [CW11, Example 1.1], and [KKM19, Theorem 4.21] gives elliptic images with nonzero rotation number for words that represent peripheral or splitting curves. Proposition D is essentially [CW11, Example 4.8] (see also the proof of [Man18, Lemma 6.18(i)]), and Proposition 5.4 is the cyclic covering construction of [CW11, Example 4.9]. Corollary 4.1 complements [GKP18, Proposition 2.6] and does not answer [GKP18, Question 2.5]. What this note adds is Theorem A for every word of the commutator subgroup, hence the complete negative answer to (Q1), and the computations for the word of the report. The reading of that example over discrete faithful representations is not addressed.

No printed answer to (Q1) was found in the sources searched on 17 October 2026: [Cal12] with the problem session of the same report; [CW11, Man18, KKM19, GKP18, Cho17, Cho18, Gor15, Mat15] and the problem list [NY25]; the works citing [CW11] listed by OpenAlex (29) and by Semantic Scholar (37; titles and available abstracts); zbMATH Open (the query “ziggurat rotation number” returns 3 documents, [CW11, Gor15, Cho17]; the plain query “ziggurat” returns 88, mostly on a sampling algorithm; further queries on word maps, trace polynomials and rotation numbers of words); queries to the arXiv API and to Crossref; and ten web searches, the last of them by the third verification run, which also repeated the queries to zbMATH Open, Crossref and OpenAlex. Word maps on $SL(2)$ over finite fields [BG12] and over algebraically closed fields (see [GKP18]) concern other questions. The arXiv API, OpenAlex and Semantic Scholar refused part of the requests, and

two meeting abstracts found by a web search refused automated access and were not read. A search that finds nothing is not a proof of novelty, and no priority is claimed; Theorem A is elementary and may be known. What was read: pages 2173–2175 of [Cal12] (text layer; the word also on the rendered page); [CW11] in the journal version, whose numbering we use (Sections 1, 2 and 4.2); [Man18], arXiv version 1, Section 6.2 with Lemma 6.18, whose numbering we use; [GKP18], arXiv version 1, Section 2 with the proof of Proposition 2.6, whose numbering we use; [KKM19], arXiv version 4, Section 4.4, whose numbering we use; [Gol09], Theorem A and Section 2.2; [Gol03], Section 2.6; [Cal09], Section 4. Of [Cho17, Gor15, Mat15] the abstracts were read, and the texts of [Cho18, NY25] were searched; [HLS15] is cited as stated in [GKP18], and [Hor72] for background only.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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