

Kernels of Signature Operators Twisted by Almost Flat Bundles Are Not Homotopy Invariant

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September 30, 2026

Abstract

At an Oberwolfach problem session in 2006, T. Schick asked the following. Fix closed Riemannian manifolds and a homotopy equivalence f . Do the signature operators twisted by a bundle E and by f^*E have isomorphic kernels once the curvature of E is small enough? By a theorem of Hilsum and Skandalis, their indices agree. Sauer and Schick later asked the same for the twisted Betti numbers. We show that the answer is no, by an elementary example. On the flat square torus let $f(x, y) = (x + \frac{c}{2\pi} \sin 2\pi y, y)$, a diffeomorphism isotopic to the identity. Let E_ε be the trivial Hermitian line bundle with the connection $d + i\varepsilon \cos(2\pi y) dx$, whose curvature has norm $2\pi|\varepsilon|$. The twisted signature operator $D = d_\nabla + d_\nabla^*$ has a four-dimensional kernel for E_ε , for every ε , and a zero kernel for f^*E_ε whenever $\varepsilon c \notin 4\pi\mathbb{Z}$; for its chiral half D^+ the dimensions are 2 and 0. The pull-back moves the harmonic part of the connection form by the curvature flux $\varepsilon c/2$. Products give examples in every dimension ≥ 2 , for instance kernels of dimensions 16 and 0 on the 4-torus (8 and 0 for D^+). The answer stays negative for the underlying Euclidean rank-two bundle, for the spin Dirac operator, and for the natural readings of degree-wise twisted Betti numbers; for non-flat bundles the kernel is in general graded only by parity, not by degree. By contrast, the number of eigenvalues near zero is stable: if $r = |\varepsilon|(1 + c^2)^{1/2} < \pi$, both operators have exactly four eigenvalues in $[-r, r]$. This is an unrefereed note.

1 Introduction

Let $f: M' \rightarrow M$ be a homotopy equivalence of closed oriented Riemannian manifolds (M', g') and (M, g) . Hilsum and Skandalis [HS92] proved that there is a constant $\delta > 0$, depending only on (M, g) , (M', g') and f , with the following property: for every Euclidean vector bundle (E, ∇) over M with a metric connection whose curvature has norm less than δ , the signature operators of M twisted by E and of M' twisted by f^*E have the same index. Thus δ does not depend on (E, ∇) . (We use the statement as given in [SS08, Theorem 57.1]. Another proof, by Hanke and Schick, is mentioned there with a reference to [HaS06]; see also [HaS08], whose abstract states that it gives a new proof.) Twisting with almost flat bundles is one of the approaches to the Novikov conjecture [CGM90].

At the problem session of the Oberwolfach workshop *Analysis and Topology in Interaction* (March 2006), Schick asked whether this invariance extends from the index to the kernel: “Is the kernel a homotopy invariant as well?” [BGIS06, p. 798]. The notes make this precise as follows; we paraphrase.

Question 1.1 (Schick [BGIS06, p. 798]). *Fix closed Riemannian manifolds (M, g) and (M', g') and a homotopy equivalence $f: M' \rightarrow M$. Is there a constant $\delta > 0$, depending on these data, such that for every bundle (E, ∇) over M whose curvature is bounded by δ in sup-norm, the kernels of D_E and*

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D'_{f^*E} are isomorphic? Here D_E is the signature operator of (M, g) twisted by (E, ∇) , and D'_{f^*E} that of (M', g') twisted by $(f^*E, f^*\nabla)$.

The notes record that Goette and Braverman expected a positive answer to be unlikely. In *Guido's book of conjectures*, Sauer and Schick [SS08, Question 57.2] asked the same question, in the setting of the Hilsum–Skandalis theorem, for the *twisted Betti numbers* $b_k(E)$, the dimensions of the degree- k parts of the kernels. They note that $\sum_k (-1)^k b_k$ is an index and hence invariant, that the answer is yes for flat bundles by the de Rham theorem, and that analysts tend to doubt the general statement. The first remark holds for flat bundles, and in general for the difference $b_{\text{ev}} - b_{\text{odd}}$ of the parity numbers of Section 4; for the degree-wise numbers of a non-flat bundle it can fail (Proposition 4.4: -2 against the index 0), as Azzali and Wahl warned [AW19, Remark 8.3]. The question is record OWR-1195-004 of the UnsolvedMath corpus [ula26]. The statement there does not say which operator D_E is (the record's title speaks of Dirac kernels). We follow the source, where D_E is the signature operator, and we treat the spin Dirac operator as well.

We show that both answers are negative, already for $M = M' = \mathbb{T}^2$ with one flat metric and f isotopic to the identity.

Theorem 1.2. *Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ with the flat metric $g = dx^2 + dy^2$, let $c \in \mathbb{R}$, and put*

$$f(x, y) = \left(x + \frac{c}{2\pi} \sin 2\pi y, y \right).$$

For $\varepsilon \in \mathbb{R}$ let E_ε be the trivial Hermitian line bundle $\mathbb{T} \times \mathbb{C}$ with the unitary connection $\nabla_\varepsilon = d + i\varepsilon \cos(2\pi y) dx$. Then:

- (i) f is a diffeomorphism of $\mathbb{T}$ that is isotopic to the identity and preserves orientation and area;*
- (ii) the curvature of ∇_ε is $2\pi i \varepsilon \sin(2\pi y) dx \wedge dy$, and its norm is $2\pi|\varepsilon|$;*
- (iii) for every ε , the kernel of D_{E_ε} has dimension 4, and $\dim \ker D_{E_\varepsilon}^+ = \dim \ker D_{E_\varepsilon}^- = 2$;*
- (iv) the kernel of $D_{f^*E_\varepsilon}$ has dimension 4 if $\varepsilon c \in 4\pi\mathbb{Z}$, and it is zero otherwise.*

*Both operators are formed with the metric g . Consequently, if $c \neq 0$, $\delta > 0$ and $0 < |\varepsilon| < \min\{\delta/(2\pi), 4\pi/|c|\}$, then E_ε has curvature norm less than δ , while the kernels of D_{E_ε} and $D_{f^*E_\varepsilon}$ have dimensions 4 and 0. So the answer to Question 1.1 is negative.*

Here $D = d_\nabla + d_\nabla^*$ is the (ungraded) twisted signature operator and $D^\pm$ are its chiral halves (Section 2). Since $\ker D = \ker D^+ \oplus \ker D^-$, it does not matter whether “the kernel” means $\ker D$ (dimensions 4 and 0) or $\ker D^+$ (dimensions 2 and 0). Section 4 extends the example to all dimensions ≥ 2 (16 against 0 on $\mathbb{T}^4$, and 8 against 0 for D^+), to the underlying Euclidean rank-two bundle (8 against 0, and 4 against 0 for D^+), to the spin Dirac operator (2 against 0), and to [SS08, Question 57.2] under each reading of b_k that we know of. For non-flat bundles the kernel is in general graded by parity but not by degree (Proposition 4.4). The mechanism is a lattice condition on the harmonic part of the connection form (Lemma 2.2), which f^* moves by the small curvature flux $\varepsilon c/2$ (Remark 3.1); the small spectrum is stable (Proposition 5.1).

2 Line bundles on the flat torus

Let (M, g) be a closed oriented Riemannian manifold of dimension n , and (E, ∇) a Hermitian vector bundle with a unitary connection. Let d_∇ be the twisted exterior derivative on E -valued forms, and d_∇^* its formal adjoint. The operator $D_E = d_\nabla + d_\nabla^*$ is self-adjoint and elliptic, and it anticommutes

with the parity $(-1)^p$. We call D_E the (ungraded) *twisted signature operator*. If n is even, D_E also anticommutes with the involution $\tau = i^{p(p-1)+n/2}*$ on p -forms, and its *chiral half* is $D_E^+ : \Omega^+ \rightarrow \Omega^+$, the restriction of D_E to the $(+1)$ -eigenspace of τ [LM89]. The index of D_E^+ is the twisted signature; this is the index in the Hilsum–Skandalis theorem. Moreover $\ker D_E = \ker D_E^+ \oplus \ker D_E^-$. Kernel dimensions without a superscript $\pm$ are those of D_E . A Euclidean bundle is treated through its complexification. As in [SS08, Theorem 57.1], $\|F^\nabla\|$ is the supremum of the operator norms of $F^\nabla(v)$ over the unit vectors $v \in \Lambda^2 T_p M$, $p \in M$; other natural norms differ from it by constant factors.

Now let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ with the metric $dx^2 + dy^2$, the orientation $\text{vol} = dx \wedge dy$ and area 1. A smooth real 1-form $\alpha = a dx + b dy$ defines the unitary connection $\nabla = d + i\alpha$ on $\mathbb{T} \times \mathbb{C}$. Write $\nabla_x = \partial_x + ia$ and $\nabla_y = \partial_y + ib$; both are skew-adjoint on $L^2(\mathbb{T})$. The curvature is $F^\nabla = i(\partial_x b - \partial_y a) \text{vol}$, and $\|F^\nabla\| = \sup |\partial_x b - \partial_y a|$. For $\omega = u + v dx + w dy + k \text{vol}$ with complex functions u, v, w, k ,

$$D\omega = -(\nabla_x v + \nabla_y w) + (\nabla_x u + \nabla_y k) dx + (\nabla_y u - \nabla_x k) dy + (\nabla_x w - \nabla_y v) \text{vol}, \quad (1)$$

where $D = D_\nabla$. Here $\tau u = iu \text{vol}$, $\tau(k \text{vol}) = -ik$ and $\tau(v dx + w dy) = -iw dx + iv dy$. We use the twisted Cauchy–Riemann operators $\bar{\partial}_\nabla = \frac{1}{2}(\nabla_x + i\nabla_y)$ and $\partial_\nabla = \frac{1}{2}(\nabla_x - i\nabla_y)$ on $C^\infty(\mathbb{T})$; note that $(\bar{\partial}_\nabla)^* = -\partial_\nabla$.

Lemma 2.1. *Let α be any smooth real 1-form on $\mathbb{T}$. The maps $U_{\text{ev}}(u + k \text{vol}) = (\frac{u-ik}{\sqrt{2}}, \frac{u+ik}{\sqrt{2}})$ and $U_{\text{odd}}(v dx + w dy) = (\frac{v+iw}{\sqrt{2}}, \frac{v-iw}{\sqrt{2}})$ are unitary, and*

$$U_{\text{odd}} D|_{\Omega^{\text{ev}}} U_{\text{ev}}^{-1} = 2\bar{\partial}_\nabla \oplus 2\partial_\nabla.$$

The kernel of D is spanned by the forms $s(1 + i\text{vol})$ and $s(dx + i dy)$ with $s \in \ker \bar{\partial}_\nabla$, which lie in Ω^+ , and $t(1 - i\text{vol})$ and $t(dx - i dy)$ with $t \in \ker \partial_\nabla$, which lie in Ω^- . Hence $\dim \ker D^+ = 2 \dim \ker \bar{\partial}_\nabla$, $\dim \ker D^- = 2 \dim \ker \partial_\nabla$, and $\dim \ker D = 2(\dim \ker \bar{\partial}_\nabla + \dim \ker \partial_\nabla)$.

Proof. Unitarity follows from $|z - iw|^2 + |z + iw|^2 = 2(|z|^2 + |w|^2)$. By (1), $D(u + k \text{vol}) = v dx + w dy$ with $v = \nabla_x u + \nabla_y k$ and $w = \nabla_y u - \nabla_x k$, and expanding gives $v + iw = (\nabla_x + i\nabla_y)(u - ik)$ and $v - iw = (\nabla_x - i\nabla_y)(u + ik)$. This is the displayed identity. As D is self-adjoint and exchanges even and odd forms, $D|_{\Omega^{\text{odd}}}$ is the adjoint of $D|_{\Omega^{\text{ev}}}$, so it corresponds to $(2\bar{\partial}_\nabla)^* \oplus (2\partial_\nabla)^* = -2\partial_\nabla \oplus -2\bar{\partial}_\nabla$. Hence $\ker D|_{\Omega^{\text{ev}}} = U_{\text{ev}}^{-1}(\ker \bar{\partial}_\nabla \oplus \ker \partial_\nabla)$ and $\ker D|_{\Omega^{\text{odd}}} = U_{\text{odd}}^{-1}(\ker \partial_\nabla \oplus \ker \bar{\partial}_\nabla)$. Now $U_{\text{ev}}^{-1}(s, 0) = \frac{s}{\sqrt{2}}(1 + i\text{vol})$, $U_{\text{ev}}^{-1}(0, t) = \frac{t}{\sqrt{2}}(1 - i\text{vol})$, $U_{\text{odd}}^{-1}(0, s) = \frac{s}{\sqrt{2}}(dx + i dy)$ and $U_{\text{odd}}^{-1}(t, 0) = \frac{t}{\sqrt{2}}(dx - i dy)$. Finally, $\tau(1 \pm i\text{vol}) = \pm(1 \pm i\text{vol})$ and $\tau(dx \pm i dy) = \pm(dx \pm i dy)$. $\square$

Lemma 2.2. *Let $\alpha = a dx + b dy$ be a smooth real 1-form on $\mathbb{T}$, and let $A = \int_{\mathbb{T}} a$ and $B = \int_{\mathbb{T}} b$. If $(A, B) \in (2\pi\mathbb{Z})^2$, then $\ker \bar{\partial}_\nabla$ and $\ker \partial_\nabla$ are one-dimensional. Otherwise both are zero.*

Proof. Put $e_{mn}(x, y) = e^{2\pi i(mx+ny)}$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, so that $\partial_{\bar{z}} e_{mn} = \pi i(m + in) e_{mn}$. Then $\bar{\partial}_\nabla = \partial_{\bar{z}} + \rho$ with $\rho = \frac{i}{2}(a + ib)$. Let $\rho_0 = \int_{\mathbb{T}} \rho = \frac{i}{2}(A + iB)$. The function $\rho - \rho_0$ has mean zero, and $|\pi i(m + in)| \geq \pi$ for $(m, n) \neq (0, 0)$. So the Fourier series $h = \sum_{(m,n) \neq (0,0)} \frac{(\rho - \rho_0)_{mn}}{\pi i(m + in)} e_{mn}$ defines a smooth function with $\partial_{\bar{z}} h = \rho - \rho_0$. For every smooth s ,

$$\bar{\partial}_\nabla(e^{-h}s) = e^{-h}(\partial_{\bar{z}}s - (\partial_{\bar{z}}h)s + \rho s) = e^{-h}(\partial_{\bar{z}} + \rho_0)s,$$

so $\ker \bar{\partial}_\nabla = e^{-h} \ker(\partial_{\bar{z}} + \rho_0)$. Since $(\partial_{\bar{z}} + \rho_0)e_{mn} = (\pi i(m + in) + \rho_0)e_{mn}$, the kernel of $\partial_{\bar{z}} + \rho_0$ is spanned by the e_{mn} with $\pi i(m + in) = -\rho_0$. There is at most one such (m, n) , and it exists if and only if $-\rho_0 = \frac{1}{2}(B - iA)$ lies in $\pi i(\mathbb{Z} + i\mathbb{Z}) = \{-\pi n + \pi im\}$, that is, if and only if $(A, B) \in (2\pi\mathbb{Z})^2$. The operator $\partial_\nabla = \partial_z + \frac{i}{2}(a - ib)$ is treated in the same way, with $\partial_z e_{mn} = \pi i(m - in) e_{mn}$ and $\frac{i}{2}(A - iB)$ in place of ρ_0 . (The complex gauge transformation e^{-h} is the device of Aharonov and Casher [AC79] for zero modes in a planar magnetic field; here the flux is zero.) $\square$

Corollary 2.3. *For every unitary connection $d + i\alpha$ on $\mathbb{T} \times \mathbb{C}$, the kernel of D is 4-dimensional, with two dimensions in each of $\Omega^\pm$, if $(\int_{\mathbb{T}} a, \int_{\mathbb{T}} b) \in (2\pi\mathbb{Z})^2$, and it is zero otherwise.*

Remark 2.4. The form $A dx + B dy$ is the harmonic part of α . If ∇ is flat, then $\alpha - A dx - B dy$ is exact, ∇ is gauge equivalent to $d + i(A dx + B dy)$, with holonomies e^{-iA} and e^{-iB} around the two circles, and Corollary 2.3 is the familiar computation of the twisted de Rham cohomology of a flat line bundle. For non-flat ∇ the *dimension* of the kernel still depends only on the harmonic part; the kernel itself depends on α (see the functions $s_\pm$ in Section 3).

3 The example

Proof of Theorem 1.2. (i) The map f commutes with the translations by $\mathbb{Z}^2$, so it induces a smooth map of $\mathbb{T}$. Its inverse is $(x, y) \mapsto (x - \frac{c}{2\pi} \sin 2\pi y, y)$. Its Jacobian matrix $\begin{pmatrix} 1 & c \cos 2\pi y \\ 0 & 1 \end{pmatrix}$ has determinant 1. The maps $f_t(x, y) = (x + \frac{tc}{2\pi} \sin 2\pi y, y)$, $0 \leq t \leq 1$, form an isotopy from the identity to f .

(ii) With $\alpha_\varepsilon = \varepsilon \cos(2\pi y) dx$ we have $F^{\nabla_\varepsilon} = i d\alpha_\varepsilon = 2\pi i \varepsilon \sin(2\pi y) \text{ vol}$. The space $\Lambda^2 T_p \mathbb{T}$ is spanned by the unit vector $\partial_x \wedge \partial_y$, so $\|F^{\nabla_\varepsilon}\| = 2\pi|\varepsilon|$.

(iii) Here $a = \varepsilon \cos 2\pi y$ and $b = 0$, so $A = B = 0$, and Corollary 2.3 applies. Explicitly, $\ker \bar{\partial}_{\nabla_\varepsilon}$ and $\ker \partial_{\nabla_\varepsilon}$ are spanned by $s_\pm = \exp(\mp \frac{\varepsilon}{2\pi} \sin 2\pi y)$, respectively. So $\ker D_{E_\varepsilon}$ has the basis $s_\pm(1 \pm i \text{vol})$, $s_\pm(dx \pm i dy)$.

(iv) The pulled-back connection is $f^* \nabla_\varepsilon = d + i f^* \alpha_\varepsilon$, and

$$f^* \alpha_\varepsilon = \varepsilon \cos(2\pi y) d\left(x + \frac{c}{2\pi} \sin 2\pi y\right) = \varepsilon \cos(2\pi y) dx + \varepsilon c \cos^2(2\pi y) dy.$$

Hence $A = 0$ and $B = \varepsilon c \int_0^1 \cos^2(2\pi y) dy = \varepsilon c/2$. By Corollary 2.3 the kernel of $D_{f^* E_\varepsilon}$ is zero unless $\varepsilon c/2 \in 2\pi\mathbb{Z}$, and then it is 4-dimensional.

The final assertion follows from (ii)–(iv). $\square$

Remark 3.1 (Holonomy and flux). Let $\gamma(t) = (x_0, t)$. Then $\oint_\gamma \alpha_\varepsilon = 0$, while $\oint_{f \circ \gamma} \alpha_\varepsilon = \oint_\gamma f^* \alpha_\varepsilon = \varepsilon c/2$. The loops are homotopic through $H(s, t) = f_s(\gamma(t))$, and by Stokes' theorem the difference is the curvature flux $\int_{[0,1]^2} H^* d\alpha_\varepsilon = \int_0^1 \int_0^1 \varepsilon c \sin^2(2\pi t) ds dt = \varepsilon c/2$. For flat connections the flux vanishes, so maps homotopic to the identity preserve holonomies, and by Remark 2.4 also the dimension of the kernel. In general, if the isotopy f_t is generated by vector fields X_t , Cartan's formula gives $f^* \alpha - \alpha = \int_0^1 f_t^*(\iota_{X_t} d\alpha) dt$ up to an exact form. So small curvature moves the harmonic part only a little; but the lattice condition of Lemma 2.2 is not open.

Since f is a diffeomorphism, $D_{f^* E}$ on $(\mathbb{T}, g)$ is unitarily equivalent to D_E on $(\mathbb{T}, (f^{-1})^* g)$. So the kernel of D_{E_ε} also changes when g is replaced by the flat metric $(f^{-1})^* g$.

4 Variants

Proposition 4.1 (Products). *Let (N, h) be a closed oriented Riemannian manifold, $B(N) = \sum_q b_q(N)$, and $M = \mathbb{T} \times N$ with the product metric. Let $E = \text{pr}_1^* E_\varepsilon$. Then $\|F^E\| = 2\pi|\varepsilon|$, $(f \times \text{id}_N)^* E = \text{pr}_1^*(f^* E_\varepsilon)$, $\dim \ker D_E = 4B(N)$, and $\ker D_{(f \times \text{id})^* E} = 0$ if $\varepsilon c \notin 4\pi\mathbb{Z}$. For $N = \mathbb{T}$ this gives, on $\mathbb{T}^4$ with the flat metric, kernels of D of dimensions 16 and 0, and $\dim \ker D_E^\pm = 8$ against $\dim \ker D_{(f \times \text{id})^* E}^\pm = 0$.*

Proof. For a unit vector $v \in \Lambda^2 T_p M$, $F^E(v)$ is $2\pi i \varepsilon \sin(2\pi y)$ times the component of v along $\partial_x \wedge \partial_y$, which has absolute value at most 1; this gives $\|F^E\|$. On M the E -valued forms are the completed graded tensor product of the E_ε -valued forms on $\mathbb{T}$ and the forms on N . With the parity $(-1)^p$ of

the first factor, $D_E = D_1 \otimes 1 + (-1)^p \otimes D_N$, where $D_1 = D_{E_\varepsilon}$ and $D_N = d + d^*$ on N . The two terms anticommute, so $D_E^2 = D_1^2 \otimes 1 + 1 \otimes D_N^2$, a sum of commuting nonnegative operators. Hence $\ker D_E = \ker D_E^2 = \ker D_1 \otimes \ker D_N$, and $\dim \ker D_N = B(N)$ by Hodge theory. The same argument with f^*E_ε gives 0. On $\mathbb{T}^4$ the index of D_E^+ does not change when the connection is deformed to d , so it is the signature of $\mathbb{T}^4$, which is 0; hence $\dim \ker D_E^\pm = 8$. $\square$

Proposition 4.2 (Euclidean bundles). *Let J be the rotation of $\mathbb{R}^2$ by $\pi/2$, and let $E_\varepsilon^\mathbb{R}$ be the Euclidean bundle $\mathbb{T} \times \mathbb{R}^2$ with the orthogonal connection $d + \varepsilon \cos(2\pi y) dx \otimes J$. Its curvature has norm $2\pi|\varepsilon|$. The kernel of the twisted signature operator D has dimension 8 for $E_\varepsilon^\mathbb{R}$ and 0 for $f^*E_\varepsilon^\mathbb{R}$ if $\varepsilon c \notin 4\pi\mathbb{Z}$; for its chiral half D^+ the dimensions are 4 and 0. The real kernels of the real operator $d_\nabla + d_\nabla^*$ also have dimensions 8 and 0.*

Proof. The curvature is $2\pi\varepsilon \sin(2\pi y) \text{vol} \otimes J$, and $\|J\| = 1$. Since J is parallel, $E_\varepsilon^\mathbb{R} \otimes \mathbb{C}$ is the orthogonal sum of its $(\pm i)$ -eigenbundles. They are the trivial line bundles with the connections $d \pm i\varepsilon \cos(2\pi y) dx$. Both connections have harmonic part 0, while their pull-backs by f have harmonic parts $\pm(\varepsilon c/2) dy$. So Corollary 2.3 gives $4 + 4$ and $0 + 0$ for D , and $2 + 2$ and $0 + 0$ for D^+ . The real kernel of a real operator complexifies to its complex kernel. $\square$

Proposition 4.3 (Spin Dirac operator). *Give $\mathbb{T}$ the spin structure for which the spinor bundle $S = \mathbb{T} \times \mathbb{C}^2$ is trivialized by parallel spinors; f preserves it, being isotopic to the identity. Twisted by E_ε , the spin Dirac operator has a 2-dimensional kernel; twisted by f^*E_ε , it has none if $\varepsilon c \notin 4\pi\mathbb{Z}$.*

Proof. With Clifford multiplications $c(\partial_x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $c(\partial_y) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, the twisted Dirac operator is $c(\partial_x)\nabla_x + c(\partial_y)\nabla_y = \begin{pmatrix} 0 & -2\partial_\nabla \\ 2\partial_\nabla & 0 \end{pmatrix}$. Its kernel is $\ker \bar{\partial}_\nabla \oplus \ker \partial_\nabla$, and Lemma 2.2 applies. $\square$

Twisted Betti numbers. Let $\Delta_k = d_\nabla^* d_\nabla + d_\nabla d_\nabla^*$ on Ω^k . Then $D_E^2 = \bigoplus_k \Delta_k + F^\nabla \wedge + (F^\nabla \wedge)^*$, since $d_\nabla^2 = F^\nabla \wedge$. For a flat bundle, $\ker D_E = \bigoplus_k \ker \Delta_k$ is graded by degree, and $\ker \Delta_k \cong H^k(M; E)$. In general the last two terms change the degree by ± 2 , so $\ker D_E$ is graded only by the parity (and by τ), not by degree, and the numbers b_k of [SS08, Question 57.2] need a definition. Azzali and Wahl [AW19, Remark 8.3] observed that the alternating sum of degree-wise Betti numbers may then differ from the twisted Euler characteristic. Three readings are natural: $b_k^\cap(E) = \dim(\ker D_E \cap \Omega^k)$; $b_k^\pi(E) = \dim \pi_k(\ker D_E)$, where π_k is the projection onto Ω^k ; and the parity numbers $b_{\text{ev}}(E)$ and $b_{\text{odd}}(E)$, the dimensions of $\ker D_E$ on even and on odd forms. As $D_E \omega$ has parts of degrees $k \pm 1$ for $\omega \in \Omega^k$, $b_k^\cap(E) = \dim \ker \Delta_k$.

Proposition 4.4. *Let $\varepsilon \neq 0$ and $\varepsilon c \notin 4\pi\mathbb{Z}$. Then*

$$(b_0^\cap, b_1^\cap, b_2^\cap)(E_\varepsilon) = (0, 2, 0), \quad (b_0^\pi, b_1^\pi, b_2^\pi)(E_\varepsilon) = (2, 2, 2), \quad (b_{\text{ev}}, b_{\text{odd}})(E_\varepsilon) = (2, 2),$$

*and all these numbers vanish for f^*E_ε . Moreover $\sum_k (-1)^k b_k^\cap(E_\varepsilon) = -2$ and $\sum_k (-1)^k b_k^\pi(E_\varepsilon) = 2$, while the twisted Euler characteristic, the index of $D_{E_\varepsilon}|_{\Omega^{\text{ev}}}$, is $b_{\text{ev}} - b_{\text{odd}} = 0$.*

Proof. By Theorem 1.2(iii), the even part of $\ker D_{E_\varepsilon}$ consists of the forms $\lambda s_+ (1 + i \text{vol}) + \mu s_- (1 - i \text{vol})$. The Ω^0 -component is $\lambda s_+ + \mu s_-$, and the Ω^2 -component is $i(\lambda s_+ - \mu s_-) \text{vol}$. For $\varepsilon \neq 0$ the functions s_+ and s_- are linearly independent, since $s_+/s_- = \exp(-\frac{\varepsilon}{\pi} \sin 2\pi y)$ is not constant. Hence the even kernel meets neither Ω^0 nor Ω^2 , while its projections to both are 2-dimensional. On $\mathbb{T}$ the odd forms are the 1-forms, so $\ker D_{E_\varepsilon} \cap \Omega^1 = \pi_1(\ker D_{E_\varepsilon})$ is the 2-dimensional odd part. Finally, $\ker D_{f^*E_\varepsilon} = 0$. $\square$

So [SS08, Question 57.2] has a negative answer under each reading, also on $\mathbb{T}^4$, where the proof of Proposition 4.1 gives $\ker D_E \cap \Omega^k = (\ker D_{E_\varepsilon} \cap \Omega^1) \otimes H^{k-1}(\mathbb{T})$, of dimensions $0, 2, 4, 2, 0$, against 0.

Indeed, write an element of $\ker D_E = \ker D_{E_\varepsilon} \otimes \ker D_N$ as $\sum_j \kappa_j \otimes h_j$, where $\kappa_j \in \ker D_{E_\varepsilon}$ and (h_j) is a basis of $\ker D_N = H^*(\mathbb{T})$ by homogeneous forms; it lies in Ω^k if and only if each κ_j lies in $\Omega^{k-\deg h_j}$, and $\ker D_{E_\varepsilon}$ meets Ω^0 and Ω^2 only in 0. For $\varepsilon = 0$ the first two readings give the Betti numbers 1, 2, 1 of $\mathbb{T}$.

5 Small eigenvalues

Proposition 5.1. *Let $r = |\varepsilon|(1 + c^2)^{1/2} < \pi$ and $d = \text{dist}(\varepsilon c/2, 2\pi\mathbb{Z})$.*

- (a) *Each of D_{E_ε} and $D_{f^*E_\varepsilon}$ has exactly four eigenvalues in $[-r, r]$, counted with multiplicity, and no eigenvalue λ with $r < |\lambda| < 2\pi - r$.*
- (b) *The four eigenvalues of D_{E_ε} in $[-r, r]$ are 0. Those of $D_{f^*E_\varepsilon}$ are σ and $-\sigma$, each of multiplicity two, where $e^{-|\varepsilon|/\pi}d \leq \sigma \leq d$.*
- (c) *If moreover $r + |\varepsilon| < \pi$, then for both bundles the twisted Laplacians $\Delta_0, \Delta_1, \Delta_2$ of Section 4 have exactly 1, 2, 1 eigenvalues in $[0, r^2 + 2\pi|\varepsilon|]$, all of them in $[0, r^2]$, and $\bigoplus_k \Delta_k$ has no other eigenvalue below $(2\pi - r)^2 - 2\pi|\varepsilon|$.*

Proof. (a) Write the connection as $d + i\varepsilon\beta$, with $\beta = \cos(2\pi y)dx$ or $\beta = \cos(2\pi y)(dx + c\cos(2\pi y)dy)$. Then $D = D_0 + \varepsilon R$, where $D_0 = d + d^*$ and $R\omega = i(\beta \wedge \omega - \iota_\beta \omega)$. Here R is self-adjoint with $R^2 = |\beta|^2$ pointwise, so $\|R\| = \sup |\beta| \leq (1 + c^2)^{1/2}$. As D_0^2 has the eigenvalues $4\pi^2(m^2 + n^2)$, D_0 has the eigenvalue 0 with multiplicity 4 and no other eigenvalue in $(-2\pi, 2\pi)$. For $0 \leq t \leq 1$ the spectrum of $D_t = D_0 + t\varepsilon R$ lies within distance r of that of D_0 [Kat95, Theorem V.4.10]. So the circle $|z| = \pi$ stays at distance at least $\pi - r$ from the spectrum. The Riesz projections $\frac{1}{2\pi i} \oint_{|z|=\pi} (z - D_t)^{-1} dz$ depend continuously on t in norm, so they all have rank 4: if two projections P, Q satisfy $\|P - Q\| < 1$, then Q maps the range of P injectively into that of Q , and vice versa.

(b) For E_ε this follows from Theorem 1.2(iii). For f^*E_ε , $a = \varepsilon \cos 2\pi y$ and $b = \varepsilon c \cos^2 2\pi y$ depend only on y , so $\bar{\partial}_\nabla$ and ∂_∇ preserve the functions of y , where $2\bar{\partial}_\nabla = iL_+$ and $2\partial_\nabla = -iL_-$ with $L_\pm = \frac{d}{dy} \pm a + ib$. Let $W = \frac{\varepsilon}{2\pi} \sin 2\pi y$ and $\theta = \frac{\varepsilon c}{8\pi} \sin 4\pi y$, so that $W' = a$ and $\theta' = b - B$ with $B = \varepsilon c/2$. Substituting $\varphi = e^{\mp W - i\theta} \psi$ gives $L_\pm \varphi = e^{\mp W - i\theta} (\psi' + iB\psi)$. By Parseval, $\|\psi' + iB\psi\| \geq d\|\psi\|$, with equality for $\psi = e^{2\pi i n y}$ and $|2\pi n + B| = d$. Since $e^{\mp 2W}$ lies between $e^{-|\varepsilon|/\pi}$ and $e^{|\varepsilon|/\pi}$, the smallest singular value $\sigma_\pm$ of $L_\pm$ satisfies $e^{-|\varepsilon|/\pi}d \leq \sigma_\pm \leq d$. By Lemma 2.1, each singular pair of $L_\pm$ gives two eigenvectors of D , with eigenvalues $\pm\sigma_\pm$. These are four eigenvalues in $[-r, r]$, since $d \leq |\varepsilon c|/2 \leq r$, and by (a) they are all of them. The translation $y \mapsto y + \frac{1}{2}$ changes a to $-a$ and fixes b , so it conjugates L_+ to L_- , and $\sigma_+ = \sigma_-$.

(c) By Section 4, $\bigoplus_k \Delta_k = D^2 - K$ with $K = F^\nabla \wedge + (F^\nabla \wedge)^*$. For both bundles $F^\nabla = 2\pi i \varepsilon \sin(2\pi y) \text{vol}$, by Theorem 1.2(ii) and since f preserves y and vol . So K acts pointwise on $u + k \text{vol}$ by a Hermitian 2×2 matrix with eigenvalues $\pm 2\pi \varepsilon \sin 2\pi y$, and $\|K\| = 2\pi|\varepsilon|$. By (a), D^2 has exactly four eigenvalues in $[0, r^2]$, and all its other eigenvalues are at least $(2\pi - r)^2$. By the min-max principle, the fifth eigenvalue of $\bigoplus_k \Delta_k$ is at least $(2\pi - r)^2 - 2\pi|\varepsilon|$, and $(2\pi - r)^2 - 2\pi|\varepsilon| - (r^2 + 2\pi|\varepsilon|) = 4\pi(\pi - r - |\varepsilon|) > 0$. On the other hand, a form $\omega \in \Omega^k$ with constant coefficients satisfies $d_\nabla \omega = i\varepsilon\beta \wedge \omega$ and $d_\nabla^* \omega = -i\varepsilon \iota_\beta \omega$. Since $|\beta \wedge \omega|^2 + |\iota_\beta \omega|^2 = |\beta|^2 |\omega|^2$ pointwise, $\langle \Delta_k \omega, \omega \rangle = \varepsilon^2 \int_{\mathbb{T}} |\beta|^2 |\omega|^2 \leq r^2 \|\omega\|^2$. The forms with constant coefficients of degrees 0, 1, 2 form spaces of dimensions 1, 2, 1, so by the min-max principle $\Delta_0, \Delta_1, \Delta_2$ have at least 1, 2, 1 eigenvalues in $[0, r^2]$. This proves (c). $\square$

This is the expected stability of small eigenvalues. The number of eigenvalues in $[-r, r]$, where r is of the order of the curvature, is the same for D_{E_ε} and $D_{f^*E_\varepsilon}$, namely the total Betti number of $\mathbb{T}$. What is not stable is the exact kernel: the four small eigenvalues are 0 for E_ε and $\pm\sigma$, with $\sigma \approx |\varepsilon c|/2$, for f^*E_ε . Perturbation arguments control eigenvalues only up to errors of the size of the perturbation,

so they cannot tell 0 from $\sigma < r$; the index is locally constant, but the kernel dimension is only upper semicontinuous. For flat bundles the kernel is twisted cohomology, whose homotopy invariance comes from topology, via the de Rham theorem [SS08]. The twisted Laplacians show the same contrast: if $r + |\varepsilon| < \pi$, then by Proposition 5.1(c) $\Delta_0, \Delta_1, \Delta_2$ have 1, 2, 1 eigenvalues in $[0, r^2]$ for both bundles, while for $\varepsilon c \neq 0$ their kernels have dimensions 0, 2, 0 for E_ε and vanish for f^*E_ε (Proposition 4.4).

A modified question, counting small eigenvalues, remains open.

Question 5.2. Fix $(M, g), (M', g')$ and f as in Question 1.1, and let $N_D(\lambda)$ be the number of eigenvalues of D in $[-\lambda, \lambda]$. Are there constants $C, \delta > 0$ such that $N_{D_E}(\lambda) \leq N_{D'_{f^*E}}(C(\lambda + \|F^\nabla\|))$ and $N_{D'_{f^*E}}(\lambda) \leq N_{D_E}(C(\lambda + \|F^\nabla\|))$ for every bundle (E, ∇) with $\|F^\nabla\| \leq \delta$ and every $\lambda \in [0, \delta]$?

Our family is consistent with this for $C \geq |c|/(4\pi)$ and small δ : by Proposition 5.1, both counting functions equal 4 on $[\sigma, 2\pi - r)$, and $\sigma \leq |\varepsilon c|/2 = \frac{|c|}{4\pi}\|F\|$. For flat bundles it asks for a dilatation-type comparison of small spectra; compare the theorem of Gromov and Shubin [GS91] that, for Galois coverings, the spectral density functions near zero of the Laplacians on L^2 -forms are homotopy invariant up to dilatation. Question 5.2 belongs to a circle of ideas discussed by Gromov [Gro96, pp. 80 and 141]. He compares the numbers of eigenvalues of $d + d^*$ in $[-a, a]$ on bi-Lipschitz equivalent manifolds, up to a dilatation of a . He also relates the passage from flat to almost flat bundles in the Hilsum–Skandalis theorem to the homotopy invariance of the spectrum of $d + d^*$ near zero on coverings, in the sense of Novikov and Shubin [GS91]. We have not examined Question 5.2, or Question 1.1 under a uniform spectral gap assumption, beyond the example.

Verification. The proofs above are complete and do not depend on computation. The following numerical checks were made; they are checks, not proofs.

1. A program by the author, `check_example.py` (numpy, a few seconds), discretizes the operators in a Fourier basis. It confirms the kernel dimensions of Theorem 1.2 for six pairs (ε, c) with $|\varepsilon| \leq 1$ and for εc in and near $4\pi\mathbb{Z}$, those of Proposition 4.1 (for $N = \mathbb{T}$), Propositions 4.2 and 4.3 for one pair, and Lemma 2.2 for a connection depending on both variables (eight values of (A, B)). It also confirms the splittings by τ , parity and degree (Proposition 4.4), the bounds of Proposition 5.1(a),(b), the formulas of Section 3 (by quadrature), and the four kernel forms of Theorem 1.2(iii) (by spectral differentiation).
2. A second program by the author, `check_laplacians.py` (numpy, about a second), computes exact Rayleigh–Ritz compressions of $\Delta_0, \Delta_1, \Delta_2$ for two truncations. It confirms Proposition 5.1(c) for twelve pairs (ε, c) with $r + |\varepsilon| < \pi$.
3. An independent verification run, which was AI-assisted, re-derived the argument and checked it with separate code; its scripts were re-run for the release.
4. A second independent verification run, also AI-assisted, examined the first version of this note. It re-derived the proofs, wrote new code from the text, and reproduced every numerical claim of that version, among them the kernel dimensions for 14 pairs (ε, c) , the products $\mathbb{T}^3$ and $\mathbb{T}^4$, and Proposition 5.1(a),(b) for nine pairs; it also reproduced the outputs of that version’s release scripts exactly. It found no mathematical error. Its required corrections, which concern presentation, credit, the citation search and one numerical remark that was too broad (now replaced by Proposition 5.1(c)), have been applied. Its code and outputs are included in the release.

Scope and priority. Theorem 1.2 and Propositions 4.1 to 4.4 answer Question 1.1 and [SS08, Question 57.2] in the negative: for the signature operator (ungraded and chiral) and the spin Dirac operator, for Hermitian and Euclidean bundles, in every dimension ≥ 2 , and already for $M = M'$ with one flat metric and f isotopic to the identity. (If f is an isometry, the answer is trivially yes.) We read the report [BGIS06], and [SS08] in a web-archived copy of the e-periodica PDF. We quote [HS92, HaS06] from [SS08] and did not consult them; of [HaS08] we read the abstract, and of [Gro96] pp. 80 and 141. We cite [AC79, CGM90] for context and checked only their bibliographic data. Of [AW19] we read Section 8 in arXiv version 3, whose numbering we use. In September 2026 we searched arXiv, Crossref, DataCite, zbMATH Open, OpenAlex and Semantic Scholar, and made one web search; an independent verification run repeated the database searches and also queried OpenCitations. OpenAlex counts two works citing [SS08]: they are two records of [AW19], dated 2018 and 2019. Semantic Scholar lists [AW19] as the only work citing [SS08], and OpenCitations lists none. OpenAlex counts 72 works citing [HS92], and Semantic Scholar lists 92; judging by their titles, none of the latter treats the kernel question. Azzali and Wahl ask related questions for twists by 2-cocycles [AW19, Question 8.2], which we do not address. We found no earlier written answer to either question. The example is elementary and may be known to experts: Goette and Braverman expected a negative answer [BGIS06], and Sauer and Schick report doubts among analysts [SS08]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexample and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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